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SUBLINEARITY AND CONVEXITY ON THE GRASSMANN CONE G_2^4

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1. Introduction.

Let K be a convex body in E^n and denote by $\hat{P}(K,R)$ the r-dimensional volume of the orthogonal projection of K on the r-flat $\mathscr{R}$ through the origin parallel to the simple r-vector R. Minkowski [4] proved that the function $f(R)=|R|\,\hat{P}(K,R)$ is a convex function of R when $r=1$ or $n-1$. To study the analoguous problem for the general case, when $1<r<n-1$, the very concept of convexity needs clarification since in this case the space of simple r-vectors is not a linear space. This important topic is studied by H. Busemann, in conjunction with others, in a series of papers; see [2] for a survey of the results. The most important problems of the theory involve the relationships among the various concepts of convexity of functions defined on non-convex sets such as the Grassmann cone of simple r-vectors.

To formulate precisely the result of this note we denote by G_r^n the Grassmann cone of simple r-vectors in the rth exterior power V_r^n of the n-dimensional real linear space A^n. A real valued function f defined on G_r^n is said to be positive homogeneous if $f(\lambda R)=\lambda f(R)$ for $\lambda \geqq 0$ and f is said to be sublinear of order k if $f(R) \leqq \sum_1^k \lambda_i f(R_i)$ whenever $R=\sum_1^k \lambda_i R_i$, $\lambda_i \geqq 0, \sum \lambda_i = 1$. (We denote the simple r-vectors by $R,S,\ldots$ and arbitrary r-vectors by $\tilde{R},\tilde{S},\ldots$.) If f can be extended to a convex function defined on the entire linear space V_r^n we simply say that f is convex. Following H. Busemann (see [3] p. 100) define o_r^n to be the smallest integer such that sublinearity of order o_r^n implies convexity of f. Busemann proved, as an application of his more general theory of sublinear functions defined on general nonconvex sets, that

$$o_r^n \leqq g_r^n = \binom{n}{r} - \max(r, n-r)$$

and he conjectured that $o_r^n < g_r^n$, see corollary 4, theorem 4 and page 100 of [3]. In this note we show that

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$$o_2^{\,4} = 3 < 4 = g_2^{\,4}\,.$$

Our proof that $o_2^{\,4}=3$ makes use of the Plücker relation for the simplicity of vectors in $V_2^{\,4}$. Since in the general case the number of Plücker relations is greater than one, an analoguous analysis in the general case would be quite complicated. The interest in the case when $r=2$ and $n=4$ is due to the relation of this result to the important open problem referred to above, namely, whether the weak convexity implies the polyhedron inequality or the polyhedron inequality implies the convexity; see [2] for the definition of these convexity conditions. Our result reduces the second half of this problem, for two dimensional polyhedra in the four dimensional space, to the following: Does the polyhedron inequality imply sublinearity of order 3?

2. The proof.

The results in [3, theorem 4, corollary 4, p. 100], and [1, p. 21[3]], imply that $2 < o_2^{\,4} \leqq 4$. Thus to show that $o_2^{\,4}=3$ it suffices to prove the following theorem.

THEOREM. *If a positive homogeneous function f defined on $G_2^{\,4}$ is sublinear of order* 3, *then it is convex.*

The proof of this theorem is based on the following two lemmas.

LEMMA 1. *Let $\sigma_{12}, \sigma_{13}, \sigma_{23}, \sigma_{24}, \sigma_{14}, \sigma_{34}$ be six nonzero real numbers such that their sum σ, say, is zero. Then there exist distinct i,j,k such that*

$$0 < -\sigma_{ij}/(\sigma_{ik}+\sigma_{jk}) \leqq 1\,.$$

PROOF. Since $\sigma=0$, not all the sums $\alpha_{ijk}=\sigma_{ij}+\sigma_{jk}+\sigma_{ik}$ can be negative and hence we may assume without loss of generality that $\alpha_{123}=\sigma_{12}+\sigma_{23}+\sigma_{13}\geqq 0$. If at least one of the numbers $\sigma_{12}, \sigma_{23}, \sigma_{13}$, say σ_{ij}, is <0, then we are done; because in that case we have

$$0< -\sigma_{ij}/(\sigma_{ik}+\sigma_{jk}) \leqq 1\,,$$

with $\{i,j,k\}=\{1,2,3\}$ and $k \neq i,j$. Consequently assume that

$$\sigma_{12}, \sigma_{13}, \sigma_{23} > 0\,.$$

Since $\sigma=0$, we conclude that $\sigma_{14}+\sigma_{24}+\sigma_{34}<0$. If $i\in\{1,2,3\}$ and $\sigma_{i4}>0$, then we have $0<-\sigma_{jk}/(\sigma_{j4}+\sigma_{k4})<1$, with $i\neq j,k$ and $j,k\in\{1,2,3\}$. Thus it suffices to consider the case when $\sigma_{12},\sigma_{23},\sigma_{13}>0$ and $\sigma_{14},\sigma_{24},\sigma_{34}<0$. In this case write $\beta_{ij}=\sigma_{ij}+\sigma_{i4}+\sigma_{j4}$, $i,j=1,2,3$. Since

$$\beta = \beta_{12}+\beta_{13}+\beta_{23} = \sigma+\sigma_{14}+\sigma_{24}+\sigma_{34} < 0\,,$$

not all the β_{ij} can be nonnegative. Hence for a pair i,j we have $\beta_{ij}<0$. Then $0<-\sigma_{ij}/(\sigma_{i4}+\sigma_{j4})<1$, since $\sigma_{ij}>0$ and $\sigma_{i4},\sigma_{j4}<0$. Thus lemma 1 is proved in all the cases.

We denote the co-ordinates of 2-vectors $\tilde{R}$ by

$$\tilde{R} = (p_{ij}) = (p_{34}, p_{42}, p_{23}, p_{12}, p_{13}, p_{14}) .$$

For $\tilde{R}=(p_{ij})$ and $\tilde{S}=(q_{ij})$ we write

$$\sigma(\tilde{R},\tilde{S}) = p_{12}q_{34}+p_{13}q_{42}+p_{23}q_{14}+p_{14}q_{23}+p_{34}q_{12}+p_{42}q_{13} .$$

The necessary and sufficient condition that $\tilde{U}$ is simple is that $\sigma(\tilde{U},\tilde{U})=0$, see [5, pp. 245, 247]. With this notation we state lemma 2.

LEMMA 2. *If R_i, $i=1,2,3$, are simple vectors, then $R_1+R_2+\alpha R_3$ is simple if $\sigma_{13}+\sigma_{23}\neq 0$ and $\alpha=-\sigma_{12}/(\sigma_{13}+\sigma_{23})$ where $\sigma_{ij}=\sigma(R_i,R_j)$.*

PROOF. Since R_i are simple, $\sigma_{ii}=0$. Hence with $\tilde{R}=R_1+R_2+\alpha R_3$ we have $\sigma(\tilde{R},\tilde{R})=2[\alpha(\sigma_{13}+\sigma_{23})+\sigma_{12}]$. Hence when $\sigma_{13}+\sigma_{23}\neq 0$ and $\alpha=-\sigma_{12}/(\sigma_{13}+\sigma_{23})$ we have $\sigma(\tilde{R},\tilde{R})=0$. This implies that $\tilde{R}$ is simple as observed above.

We return to the proof of the theorem stated above. In view of Busemann's result [3, Theorem 4, Corollary 4], it suffices to show that $R=R_1+R_2+R_3+R_4$, $R_i+R_j+R_k$ nonsimple unless $i=j=k$, implies $f(R)\leqq\sum_{i=1}^4 f(R_i)$. (We use here the fact that f is positive homogeneous.) Since R is simple, we have

$$\sigma = \sigma(R,R) = \textstyle\sum_{i,j=1}^4 \sigma_{ij} = 0 ,$$

where $\sigma_{ij}=\sigma(R_i,R_j)\neq 0$. Hence we can use lemmas 1 and 2 and taking $i,j,k=1,2,3$ in lemma 1, we can write

$$R = (R_1+R_2+\theta R_3)+(1-\theta)R_3+R_4$$

such that $0<\theta<1$, and $R_1+R_2+\theta R_3$ is simple. Consequently

$$f(R) \leqq \textstyle\sum_1^4 f(R_i) ,$$

using the sublinearity of order 3 and the equation

$$f(R_3) = f(\theta R_3)+f([1-\theta]R_3)$$

obtained from the positive homogeneity of f and the fact that $1>\theta>0$. This proves the theorem and hence $o_2{}^4=3$.

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