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# INVARIANT STATES OF VON NEUMANN ALGEBRAS

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In a recent paper Yeadon [10] has shown that the existence of a trace in a finite von Neumann algebra is a consequence of the Ryll–Nardzewski fixed point theorem. In the present note we shall formulate his ideas in the more general setting of groups of *-automorphisms of a von Neumann algebra, and then study the problem of existence of normal invariant states. The main result states that there are "enough" normal invariant states (viz. the von Neumann algebra is $G$-finite) if and only if the automorphism group is relatively compact in a topology described below.

If $\mathscr{R}$ is a von Neumann algebra we denote by $\mathscr{L}(\mathscr{R})$ the space of bounded linear maps of $\mathscr{R}$ into itself. $\mathscr{L}_*(\mathscr{R})$ denotes the ultraweakly continuous maps in $\mathscr{L}(\mathscr{R})$. We give $\mathscr{L}(\mathscr{R})$ the weak-operator topology defined by Kadison [7]. This topology is defined as the point-open topology, where $\mathscr{R}$ is given the weak topology. We then let $\mathscr{L}_*(\mathscr{R})$ have the relative topology from $\mathscr{L}(\mathscr{R})$. As a consequence of the Tychonoff theorem it was shown in [7] that the unit ball in $\mathscr{L}(\mathscr{R})$ is weak-operator compact. If $G$ is a group of *-automorphisms of $\mathscr{R}$ then $G \subset \mathscr{L}_*(\mathscr{R})$ [2, p. 57]. A von Neumann algebra $\mathscr{R}$ is said to be $G$-finite if the normal $G$-invariant states separate $\mathscr{R}^+$, that is, if $A$ is a non zero positive operator in $\mathscr{R}$ then there is a normal $G$-invariant state $\varrho$ such that $\varrho(A) \neq 0$. We denote by $\mathscr{R}^G$ the fixed points of $G$ in $\mathscr{R}$. It should be remarked that the results in this note can easily be extended to groups of identity preserving isometries of $\mathscr{R}$.

LEMMA. *Let $\mathscr{R}$ be a von Neumann algebra and $G$ a group of *-automorphisms of $\mathscr{R}$. Let $\omega$ be a normal state of $\mathscr{R}$. Then there exists a normal $G$-invariant state $\varrho$ of $\mathscr{R}$ such that $\varrho \mid \mathscr{R}^G = \omega \mid \mathscr{R}^G$ if and only if for each infinite orthogonal sequence $\{P_n\}$ of projections in $\mathscr{R}$ we have $\lim_n \omega(g(P_n)) = 0$ uniformly for $g \in G$.*

PROOF. Suppose there is a normal $G$-invariant state $\varrho$ of $\mathscr{R}$ such that $\varrho \mid \mathscr{R}^G = \omega \mid \mathscr{R}^G$. Let $\{P_n\}$ be an orthogonal sequence of projections in $\mathscr{R}$.

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If $\lim_n \omega(g(P_n))$ is not uniformly zero for $g$ in $G$, we can find $\varepsilon > 0$, a sequence $\{g_m\}$ in $G$ and a subsequence $\{P_m\}$ of $\{P_n\}$ such that $\omega(g_m(P_m)) \geqq \varepsilon$ for $m = 1, 2, \ldots$. Now

$$\Sigma\varrho(P_m) = \varrho(\Sigma P_m) \leqq \varrho(I) = 1 ,$$

so

$$\lim_m \varrho(g_m(P_m)) = \lim_m \varrho(P_m) = 0 .$$

Since $\varrho$ is $G$-invariant, its support $E$ lies in $\mathscr{R}^G$. Therefore $\omega(E) = \varrho(E) = 1$, so $E \geqq$ support $\omega$. Hence $\lim_m \omega(g_m(P_m)) = 0$ by [2, p. 62], a contradiction.

Conversely, suppose $\lim_n \omega(g(P_n)) = 0$ uniformly in $G$ for each infinite orthogonal sequence $\{P_n\}$ of projections in $\mathscr{R}$. Let $K = \{\omega \circ g : g \in G\}$. By a theorem of Akemann [1, Thm. II,2] $K$ is a weakly relatively compact subset of the predual $\mathscr{R}_*$ of $\mathscr{R}$. Let $Q$ be the norm closed convex hull in $\mathscr{R}_*$ of $K$. Since $Q$ is convex, it is weakly closed, hence weakly compact [4, V.6.4]. By the Ryll–Nardzewski theorem [9], or [5], applied to the Banach space $\mathscr{R}_*$, the weakly compact convex set $Q$, and the group $\tau \to \tau \circ g$ of affine isometries of $Q$, there is a fixed point $\varrho$ of $G$ in $Q$. Note that if $A \in \mathscr{R}^G$ and $\omega' = \sum_{i=1}^n \lambda_i \omega \circ g_i \in Q$, then $\omega'(A) = \omega(A)$, hence by continuity, $\omega'(A) = \omega(A)$ for all $\omega' \in Q$. In particular $\varrho \mid \mathscr{R}^G = \omega \mid \mathscr{R}^G$. The proof is complete.

At this point we pause to prove a result, which when both $\mathscr{R}$ and $G$ are abelian, is a well-known result in ergodic theory due to Markov, see [6, 4.8.7].

Corollary. *Let $\mathscr{R}$ be a von Neumann algebra and $G$ a group of *-automorphisms. Suppose $\omega$ is a faithful normal state of $\mathscr{R}$. Then there exists a faithful normal $G$-invariant state of $\mathscr{R}$ if and only if given $\varepsilon > 0$ we can find $\delta > 0$ such that if $E$ is a projection in $\mathscr{R}$ with $\omega(E) < \delta$, then $\omega(g(E)) < \varepsilon$ for all $g \in G$.*

Proof. Suppose $\varrho$ is a faithful normal $G$-invariant state of $\mathscr{R}$. If the conditions in the corollary are not satisfied we can find $\varepsilon > 0$, a sequence $\{E_n\}$ of projections in $\mathscr{R}$ such that $\omega(E_n) \to 0$, and a sequence $\{g_n\}$ in $G$ such that $\omega(g_n(E_n)) \geqq \varepsilon$. But since $\omega$ is faithful, $\varrho(E_n) \to 0$ by [2, p. 62], hence

$$\varrho(g_n(E_n)) = \varrho(E_n) \to 0 .$$

Since $\varrho$ is faithful, another application of [2, p. 62] shows $\omega(g_n(P_n)) \to 0$, a contradiction.

Conversely, suppose given $\varepsilon > 0$ we can find $\delta > 0$ such that if $E$ is a projection in $\mathscr{R}$ with $\omega(E) < \delta$, then $\omega(g(E)) < \varepsilon$ for all $g \in G$. Let $\{P_n\}$

be an infinite orthogonal sequence of projections in $\mathscr{R}$. Then $\omega(P_n) \to 0$. Let $\varepsilon > 0$ be given and choose $\delta > 0$ as above. Choose $n_0$ such that if $n \geqq n_0$ then $\omega(P_n) < \delta$. But then $\omega(g(P_n)) < \varepsilon$ for $n \geqq n_0$, so that $\omega(g(P_n)) \to 0$ uniformly in $G$. By the lemma there is a normal $G$-invariant state $\varrho$ of $\mathscr{R}$ such that $\varrho \mid \mathscr{R}^G = \omega \mid \mathscr{R}^G$. As pointed out in the proof of the lemma the support of $\varrho$ majorizes that of $\omega$. Since $\omega$ is faithful, the support of $\varrho$ is the identity, hence $\varrho$ is faithful.

THEOREM. *Let $\mathscr{R}$ be a von Neumann algebra and $G$ a group of *-automorphisms of $\mathscr{R}$. Then $\mathscr{R}$ is $G$-finite if and only if $G$ is relatively compact in the relative weak-operator topology on $\mathscr{L}_*(\mathscr{R})$.*

PROOF. Suppose $\mathscr{R}$ is $G$-finite. Let $\{g_\alpha\}$ be a net of automorphisms in $G$. We have to show that this net has a subnet which converges in the weak-operator topology to a map in $\mathscr{L}_*(\mathscr{R})$. Since $G$ is contained in the unit ball in $\mathscr{L}(\mathscr{R})$, which is weak-operator compact [7], there is a subnet $\{g_\beta\}$ which converges to a positive linear map $\varphi$ in $\mathscr{L}(\mathscr{R})$ such that $\varphi(I) = I$, see [7]. In order to show $\varphi \in \mathscr{L}_*(\mathscr{R})$ it suffices to show that if $\omega$ is a normal state of $\mathscr{R}$ then so is $\omega \circ \varphi$. Let

$$K = \{\omega \circ g \ : \ g \in G\} .$$

Then $\omega \circ \varphi$ is in the $w^*$-closure of $K$. Since $\mathscr{R}$ is $G$-finite, there is a faithful normal $G$-invariant projection map $\Phi$ of $\mathscr{R}$ onto $\mathscr{R}^G$, and every normal $G$-invariant state of $\mathscr{R}$ is of the form $\varrho = \varrho \mid \mathscr{R}^G \circ \Phi$ (see [8] or [3]). In particular, $\varrho = \omega \mid \mathscr{R}^G \circ \Phi$ satisfies the condition in the lemma, so by the lemma and the theorem of Akemann [1] referred to there the set $K$ is weakly relatively compact in $\mathscr{R}_*$. Therefore the $w^*$-closure of $K$ is contained in $\mathscr{R}_*$, hence $\omega \circ \varphi$ is normal. Therefore $\varphi \in \mathscr{L}_*(\mathscr{R})$, and $G$ is relatively compact.

Conversely, assume $G$ is relatively compact in the relative weak-operator topology on $\mathscr{L}_*(\mathscr{R})$. Let $\omega$ be a normal state of $\mathscr{R}$. Then $K = \{\omega \circ g : g \in G\}$ is weakly relatively compact in $\mathscr{R}_*$. Indeed, if $\{\omega \circ g_\alpha\}$ is a net in $K$ converging to a state $\varrho$ in the $w^*$-topology, we can choose a subnet $\{g_\beta\}$ of $\{g_\alpha\}$ converging in the weak-operator topology to $\varphi \in \mathscr{L}_*(\mathscr{R})$. Then $\omega \circ g_\beta \to \omega \circ \varphi$, hence $\varrho = \omega \circ \varphi$ is a normal state, and $K$ is weakly relatively compact. From the lemma and its proof $\mathscr{R}$ is $G$-finite.

COROLLARY. *Let $\mathscr{R}$ be a von Neumann algebra. Let $G$ be a group of *-automorhisms of $\mathscr{R}$ leaving the center of $\mathscr{R}$ element-wise fixed and con-*

*taining the group of inner automorphisms. Then $\mathscr{R}$ is finite if and only if $G$ is relatively compact in the relative weak-operator topology on $\mathscr{L}_*(\mathscr{R})$.*

PROOF. $\mathscr{R}$ is finite if and only if there exists a unique faithful normal center valued trace on $\mathscr{R}$ [2, pp. 267, 319], hence by uniqueness, if and only if there is a unique normal $G$-invariant projection of $\mathscr{R}$ onto its center, if and only if $\mathscr{R}$ is $G$-finite [8], hence by the theorem, if and only if $G$ is weakly relatively compact.

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