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PRIMITIVE IDEALS IN TENSOR PRODUCTS OF BANACH ALGEBRAS

JUN TOMIYAMA

Let A and B be Banach algebras. Recently Laursen [1] has established a bijection

$$\mathscr{M}(A \otimes_\gamma B) \leftrightarrow \mathscr{M}(A) \times \mathscr{M}(B)$$

when one of A or B is commutative. Here we mean by $\mathscr{M}(A \otimes_\gamma B)$ the set of maximal modular ideals of $A \otimes_\gamma B$, the greatest cross norm tensor product of A and B and by $\mathscr{M}(A)$ and $\mathscr{M}(B)$ the sets of those ideals in A and B. The result extends the fact which is previously known in case where both A and B are commutative. However, in the latter case it is also known that there exists a bijection

$$\mathscr{M}(A \otimes_\beta B) \leftrightarrow \mathscr{M}(A) \times \mathscr{M}(B)$$

for an arbitrary compatible cross norm β (cf. [6]). Thus we may expect the same result for $A \otimes_\beta B$ in the former case. In the present article we shall show that this is actually the case. The proof is a direct modification of the method employed in [6] and does not depend on the presence of a unit, which will show an advantage of our proof. The result for strong semisimplicity of product algebras will be also naturally extended to this situation, so that it may cover the cases such as $C(X,B)$ where X is a compact Hausdorff space, etc.

A cross norm β in $A \odot B$, algebraic tensor product of A and B, which is not less than the λ-norm (cf. [4]) is called a compatible norm if it is compatible with multiplications in $A \odot B$, that is,

$$\|xy\|_\beta \leqq \|x\|_\beta \|y\|_\beta$$

for any $x,y \in A \odot B$. In this case $A \otimes_\beta B$ becomes a Banach algebra. Take a bounded linear functional φ of A and define the mapping R_φ on $A \odot B$ by

$$R_\varphi(\textstyle\sum_{i=1}^n a_i \otimes b_i) = \sum_{i=1}^n \langle a_i, \varphi\rangle b_i \, .$$

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Since $\beta \geqq \lambda$ it will be easily seen that this mapping is continuous and can be extended to $A \otimes_\beta B$. We can also define the mapping

$$L_\psi\colon\ A \otimes_\beta B \to A$$

for a bounded linear functional ψ on B by

$$L_\psi(\textstyle\sum_{i=1}^n a_i \otimes b_i) = \sum_{i=1}^n \langle b_i, \psi\rangle a_i\ .$$

These mappings concern each other in the following way:

$$\langle x, \varphi\otimes\psi\rangle = \langle R_\varphi(x), \psi\rangle = \langle L_\psi(x), \varphi\rangle$$

for any $x \in A \otimes_\beta B$ where $\varphi\otimes\psi$ means a product functional. We call this equality Fubini type principle and those mappings right and left Fubini mappings.

Fubini mappings have been defined in [6].

Let $\Pi_0(A)$ be the set of modular primitive ideals in A. We shall state our result in the following way.

THEOREM 1. *Suppose that A is a commutative Banach algebra and β a compatible cross norm, then there is a bijection*

$$\Pi_0(A \otimes_\beta B) \leftrightarrow \Pi_0(A) \times \Pi_0(B)$$

given by $P = R_\varphi^{-1}(P_B)$ where $P \in \Pi_0(A \otimes_\beta B)$, $P_B \in \Pi_0(B)$, and R_φ is a right Fubini mapping induced by a homomorphism φ on A. The restriction of the above mapping to the sets of maximal modular ideals yields a bijection

$$\mathcal{M}(A \otimes_\beta B) \leftrightarrow \mathcal{M}(A) \times \mathcal{M}(B)\ .$$

PROOF. We notice first that R_φ is a homomorphism from $A \otimes_\beta B$ onto B. Take an ideal $P_B \in \Pi_0(B)$ and $\varphi \in \Pi_0(A) = \mathcal{M}(A)$ (identifying homomorphisms and modular maximal ideals) and put

$$P = R_\varphi^{-1}(P_B)\ .$$

Then the quotient algebra $A \otimes_\beta B/P$ is isomorphic to B/P_B by the isomorphism induced by R_φ, hence P is also a modular primitive ideal. Suppose we have

$$P = R_{\varphi_1}^{-1}(P'_B)$$

for a homomorphism φ_1 and an ideal $P'_B \in \Pi_0(B)$. If $\varphi \neq \varphi_1$ there exists an element $a \in A$ such that $\langle a, \varphi\rangle = 0$ and $\langle a, \varphi_1\rangle \neq 0$. Take $b \in B$ so that $b \notin P'_B$. Then applying R_φ and R_{φ_1} to $a\otimes b$ we see that

$$a\otimes b \in P \quad \text{and} \quad a\otimes b \notin P\ ,$$

a contradiction. Therefore $\varphi=\varphi_1$, and $P_B=P'_B$, that is, the above expression is unique.

Now take an element $P \in \Pi_0(A \otimes_\beta B)$ and let π be the quotient mapping to $A \otimes_\beta B/P$, which is a Banach algebra containing a unit. We choose an element $k=\sum_{i=1}^n a_i \otimes b_i$ such that $\pi(k)$ is invertible and write as

$$\varrho(x) = \pi\left(\textstyle\sum_{i=1}^n xa_i \otimes b_i\right) \pi(k)^{-1}$$

for $x \in A$. We note first that $\varrho(x)$ commutes with $\pi(k)$, hence with $\pi(k)^{-1}$ too. Clearly ϱ is a linear mapping from A into $A \otimes_\beta B/P$. For $x,y \in A$ we have

$$\begin{aligned}\varrho(x)\,\varrho(y) &= \pi(\textstyle\sum_{i=1}^n xa_i \otimes b_i)\,\pi(k)^{-1}\,\pi(\sum_{i=1}^n ya_i \otimes b_i)\,\pi(k)^{-1} \\ &= \pi(\textstyle\sum_{i,j} xya_ia_j \otimes b_ib_j)\,\pi(k)^{-2} \\ &= \pi(\textstyle\sum_{i=1}^n xya_i \otimes b_i)\,\pi(k)^{-1} = \varrho(xy)\end{aligned}$$

and moreover

$$\pi(y \otimes bc)\varrho(x) = \pi(\textstyle\sum_{i=1}^n xya_i \otimes bcb_i)\,\pi(k)^{-1} = \pi(xy \otimes bc)$$

for arbitrary elements $x,y \in A$ and $b,c \in B$. Therefore ϱ is a non-zero homomorphism from A to $A \otimes_\beta B/P$. We shall show that $\varrho(A)$ is contained in the center. In fact,

$$\begin{aligned}\pi(y \otimes b)\,\varrho(x) = \pi(xy \otimes b) &= \pi(k)^{-1}\,\pi(k)\,\pi(xy \otimes b) \\ &= \pi(k)^{-1}\,\pi(\textstyle\sum_{i=1}^n xya_i \otimes b_ib) = \varrho(x)\,\pi(y \otimes b)\end{aligned}$$

for any $y \in A$ and $b \in B$. Since $A \otimes_\beta B/P$ is a primitive algebra, $\varrho(A)$ is isomorphic to the complex number field (cf. [3, p. 61]) and there is a (complex valued) homomorphism φ such that

$$\varrho(x) = \langle x, \varphi\rangle 1$$

where 1 is the identity in the algebra. Take an element $e \in A$ such as $\langle e, \varphi\rangle = 1$ and put

$$\pi_B(x) = \pi(e \otimes x) \quad \text{for } x \in B\,.$$

We have

$$\begin{aligned}\pi_B(xy) = \pi(e \otimes xy) &= \pi(e^2 \otimes xy) \\ &= \pi(e \otimes x)\,\pi(e \otimes y) = \pi_B(x)\,\pi_B(y)\,.\end{aligned}$$

Hence π_B is a continuous homomorphism from B to $A \otimes_\beta B/P$, and moreover we get the following relation,

$$\begin{aligned}\pi(a \otimes b) = \pi(ae \otimes b) &= \pi(e \otimes b)\varrho(a) \\ &= \langle a, \varphi\rangle \pi_B(b) \\ &= \pi_B(\langle a, \varphi\rangle b) = \pi_B \circ R_\varphi(a \otimes b)\,.\end{aligned}$$

Hence,

$$\pi = \pi_B \circ R_\varphi .$$

Putting $P_B = \pi_B^{-1}(0)$, we have $P_B \in \Pi_0(B)$ and the expression

$$P = R_\varphi^{-1}(P_B) .$$

Finally, the canonical isomorphism between $A \otimes_\beta B/P$ and B/P_B shows that the restriction of the above bijection induces a bijection.

$$\mathcal{M}(A \otimes_\beta B) \leftrightarrow \mathcal{M}(A) \times \mathcal{M}(B) .$$

This completes the proof.

The above theorem shows that when both A and B have units there is a bijection

$$\Pi(A \otimes_\beta B) \leftrightarrow \Pi(A) \times \Pi(B) ,$$

where $\Pi(A \otimes_\beta B)$ is the structure space of $A \otimes_\beta B$, and $\Pi(A)$ and $\Pi(B)$ show the structure spaces of A and B. If we consider the correspondance between primitive ideals and maximal left ideals we can deduce Lebow's result [2] from the above bijection.

COROLLARY. *In the above notations, suppose both A and B have units, then a maximal left ideal L in $A \otimes_\beta B$ is expressed uniquely as*

$$L = R_\varphi^{-1}(L_B)$$

where φ is a homomorphism of A and L_B is a maximal left ideal of B.

PROOF. We may proceed the arguments about the uniqueness of expressions and the maximality of the left ideal $R_\varphi^{-1}(L_B)$ along the same line as in the beginning of the proof of Theorem 1, so that it is enough to show how to determine φ and L_B. For a maximal left ideal L, let ϱ be the canonical irreducible representation of $A \otimes_\beta B$ on $E = A \otimes_\beta B/L$. We put $P = \varrho^{-1}(0)$, then P is a primitive ideal and there exist a homomorphism φ and a primitive ideal P_B such that

$$P = R_\varphi^{-1}(P_B) .$$

This shows that we can find a (continuous) irreducible representation ϱ_B of B such that $\varrho_B^{-1}(0) = P_B$ and $\varrho = \varrho_B \circ R_\varphi$. Let ξ_0 be the canonical image of the identity of $A \otimes_\beta B$ in E. We have

$$L = \{x \in A \otimes_\beta B \mid \varrho(x)\xi_0 = 0\} .$$

Put

$$L_B = \{b \in B \mid \varrho_B(b)\xi_0 = 0\} .$$

Then $L = R_\varphi^{-1}(L_B)$ and L_B is clearly a maximal left ideal of B.

Now Theorem 1 will lead us to a natural generalization of [1, Theorem 2] to the case $A \otimes_\beta B$. For the sake of completeness we shall give the proof of our arguments.

THEOREM 2. *In the above notations, suppose A and B are strongly semisimple, then $A \otimes_\beta B$ is strongly semisimple if and only if the canonical mapping*

$$\tau: \; A \otimes_\beta B \to A \otimes_\lambda B$$

is one-to-one.

PROOF. We shall show that the strong radical R_0 of $A \otimes_\beta B$ coincides with $\tau^{-1}(0)$. In fact, we have the following equivalence:

$$\begin{aligned} x \in R_0 &\Leftrightarrow R_\varphi(x) \in M_B \quad \text{for any } \varphi \in \mathcal{M}(A) \quad \text{and} \quad M_B \in \mathcal{M}(B) \\ &\Leftrightarrow R_\varphi(x) = 0 \quad \text{for any } \varphi \in \mathcal{M}(A)\,. \end{aligned}$$

Since for any $\psi \in B^*$, by Fubini type principle

$$\langle R_\varphi(x), \psi\rangle = \langle L_\psi(x), \varphi\rangle = 0\,,$$

the above equivalence can be transferred to the identity,

$$\langle L_\psi(x), \varphi\rangle = \langle x, \varphi\otimes\psi\rangle = \langle \tau(x), \varphi\otimes\psi\rangle = 0$$

for any $\varphi \in A^*$ and $\psi \in B^*$. This is equivalent to $\tau(x) = 0$.

Theorem 2 covers, for example, the case $C_0(X, B)$, the space of all B-valued continuous functions on a locally compact space X vanishing at infinity as well as the case $L^1(G, B)$ where G is a locally compact abelian group and B is a Banach algebra, to the extent that they are strongly semi-simple if and only if B is strongly semisimple.

For an another special compatible norm such as C^*-cross norm more general results are known. For example, if we take A a separable type I C^*-algebra then for an arbitrary separable C^*-algebra B and the least C^*-cross norm α we get a bijection

$$\Pi(A \otimes_\alpha B) \leftrightarrow \Pi(A) \times \Pi(B)$$

and this correspondance is actually a homeomorphism with respect to hull-kernel topology (cf. [7, p. 225]).

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UNIVERSITY OF COPENHAGEN, COPENHAGEN, DENMARK

AND

YAMAGATA UNIVERSITY, YAMAGATA, JAPAN