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ON DISCONTINUOUS HOMOMORPHISMS OF $L^1(G)$

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It is still not known whether or not every algebra homomorphism of $C_0(X)$ where X is locally compact and Hausdorff is continuous. The same ignorance prevails with respect to the analogous question concerning the convolution algebra $L^1(G)$ where G is a locally compact abelian group. In this note we establish a connection between these questions: If G is a locally compact abelian group with dual group $\hat{G}$ and if $C_0(\hat{G})$ admits a discontinuous homomorphism then so does $L^1(G)$. This extends recent work of T. Slobko.

In the sequel X will be a locally compact Hausdorff space, $C_0(X)$ the algebra of continuous functions on X vanishing at infinity, G will be a locally compact abelian group with dual group $\hat{G}$. βX will be the Stone–Čech compactification of X and if $F \subset \beta X$ then $\mathscr{K}(F)$ will be the algebra of functions from $C(\beta X)$ which are constant on some neighborhood of each of the points of F, the neighborhoods depending on the function. Finally, $\mathscr{L}(F)$ is the ideal in $\mathscr{K}(F)$ of functions vanishing in a neighborhood of F.

The starting point of our discussion is the following result [5, Theorem 1]

Proposition. *If ν is a homomorphism of $C_0(X)$ then there is a finite set of points, F in βX and a constant M such that*

$$\|\nu(g)\| \leqq M\|g\| \quad \textit{for all } g \in \mathscr{L}(F) .$$

Applying the method of proof from [1, Theorem 4.1] to this result we easily obtain

Lemma 1. *Let ν be a homomorphism of $C_0(X)$ (into a Banach algebra). Let F be the singularity set in the sense of the above. Then ν is continuous on the dense subalgebra $\mathscr{K}(F) \cap C_0(X)$.*

Another extension to the locally compact case is contained in

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LEMMA 2. *Let ν be a homomorphism of $C_0(X)$ into a commutative Banach algebra B and let R be the radical of $\overline{\nu(C_0(X))}$. Let μ and λ be the continuous and singular parts of ν. Then*

(a) *The range of μ is closed in B and*

$$\overline{\nu(C_0(X))} = \mu(C_0(X)) \oplus R$$

where $\oplus$ denotes topological direct sum.

(b) $R = \overline{\lambda(C_0(X))}$.

PROOF. The argument is similar to that employed in [1, Theorem 4.3]: Let $K = \{f \in C_0(X)\,;\ \mu(f) = 0\}$. Since μ is continuous K is a closed ideal in $C_0(X)$ and since $C_0(X)$ is a closed ideal in $C(\beta X)$, K is a closed ideal in $C(\beta X)$. (Let $g \in K$, let $f \in C(\beta X)$ and let e_α be an approximate identity for $C_0(X)$. Then $\lim f g e_\alpha = fg$ since $fg \in C_0(X)$; moreover $fe_\alpha \in C_0(X)$, $g \in K$ implies $fge_\alpha \in K$, that is, $fg \in K$). This means there is a closed set $L \subseteqq \beta X$ such that $K = \{f \in C(\beta X)\,;\ f(L) = 0\}$. It is known that $C(\beta X)/K$, provided with the quotient norm $\|\cdot\|$ is isometrically isomorphic with $C(L)$. Consequently, $C_0(X)/K$ is isometrically isomorphic with $C_0(X \cap L)$. Since $\mu(f)$ is constant on the coset $f + K$ we may also norm $C_0(X)/K$ by

$$|f + K| = \|\mu(f)\| .$$

By [3]

$$\|f + K\| \leqq |f + K| \quad f \in C_0(X) ,$$

and since $|f + K| = \|\mu(f)\| \leqq C\|f\|$ for every f in $C_0(X)$ it follows that the quotient norm and $|\cdot|$ are equivalent. The rest of the proof proceeds exactly as in [1].

Let x_0 be a singularity point for ν in the sense of Slobko (that is, $x_0 \in \beta X$) and let $Z(x_0)$ be the set of functions in $C_0(X)$ that vanish on some open neighborhood of x_0. Then corresponding to [5, Lemma 3] we have

LEMMA 3. *Let ν be a discontinuous homomorphism of $C_0(X)$, X locally compact and let x_0 be a singularity point. Then ν induces a non-zero algebra semi-norm on $C_0(X)/Z(x_0)$.*

PROOF. Choose a function h in $C(\beta X)$ such that $h = 1$ in some neighborhood of x_0 and $= 0$ in a neighborhood of every other singularity point of ν. Define

$$\nu'(f) = \nu(fh)$$

for all $f \in C_0(X)$. Then ν' has only one singularity point, x_0. Let μ and λ be the continuous and singular parts of ν', respectively. Let $\tilde{}$ denote the canonical mapping $C_0(X) \to C_0(X)/Z(x_0)$ and define

$$p(x) = \inf\{\|\nu'(s)\| \; ; \; \tilde{s}=x\}$$

for every $x \in C_0(X)/Z(x_0)$. It is elementary to show that p is an algebra semi-norm. It remains to see that p is non-zero. If $p \equiv 0$ then $\nu'(Z(x_0))$ is dense in $\nu'(C_0(X))$, and since $\mu(C_0(X)) \supseteq \mu(Z(x_0)) = \nu'(Z(x_0))$ by Lemma 1 we conclude that $\mu(C_0(X))$ is dense in $\nu'(C_0(X))$. By Lemma 2 this implies that $R=\{0\}$ and hence that $\lambda=0$, contradicting the discontinuity of ν' and of ν.

We are now ready to state and prove the main result

THEOREM. *Let G be a locally compact abelian group with dual group $\hat{G}$. If $C_0(\hat{G})$ admits a discontinuous homorphism then so does $L^1(G)$.*

PROOF. Let x_0 be a singularity point of the discontinuous homomorphism ν, let p be the algebra-semi-norm on $C_0(\hat{G})/Z(x_0)$ (Lemma 3) and $\gamma: L^1(G) \to C_0(\hat{G})$ be the Gelfand transform. We shall show that if $\tilde{}$ is the canonical mapping $C_0(\hat{G}) \to (C_0(\hat{G})/Z(x_0), p)$ then $\tilde{} \circ \gamma$ is discontinuous. Choose $f \in C_0(\hat{G})$ such that $p(\tilde{f}) > 0$. By [2] we can find $h \in L^1(G)$ and $g \in C_0(\hat{G})$ such that $f = \gamma(h)g$. Consequently, since $p(\tilde{f}) \leqq p(\gamma(h)^{\sim})p(\tilde{g})$, $p(\gamma(h)^{\sim}) > 0$.

At this point we shall consider first the possibility that $x_0 \in \beta\hat{G} \setminus \hat{G}$. In this case [4, 2.6.6] applies, that is, given $\varepsilon > 0$ we can find $k \in L^1(G)$ such that $\gamma(k) \in \mathscr{L}(\{x_0\})$ and such that $\|h - k*k\|_1 < \varepsilon$. Since $\gamma(h)^{\sim} = \gamma(h)^{\sim} - \gamma(h)^{\sim}\gamma(k)^{\sim}$, this proves the discontinuity of $\tilde{} \circ \gamma$ in case $x_0 \in \beta\hat{G} \setminus \hat{G}$.

If $x_0 \in \hat{G}$ we shall again invoke the factorization theorem of [2], but this time we apply it to

$$W = \{f \in L^1(G) \; ; \; \gamma(f)(x_0) = 0\} .$$

To see that the theorem applies, let

$$X = \{f \in C_0(\hat{G}) \; ; \; f(x_0) = 0\}$$

and let $\sigma: W \to B(X)$ be defined in the canonical way

$$\sigma(f)(g) = \gamma(f)g \quad f \in W, \; g \in X .$$

By [4, 2.6.4] W is the closed linear span of functions vanishing in a neighborhood of x_0. Consequently, by [2] we get that $W \cdot X$ is closed in X. It is no problem to check the density of $W \cdot X$ in X; consequently, we have the needed result: given $f \in C_0(\hat{G})$ such that $f(x_0)=0$, there is $h \in L^1(G)$ with $\gamma(h)(x_0)=0$ and $g \in C_0(\hat{G})$ with $g(x_0)=0$ such that $f=\gamma(h)g$.

The argument is now quite similar to the first case: if $f \in C_0(\hat{G})$ and $p(\tilde{f})>0$ then there is no loss of generality in assuming that $f(x_0)=0$. If this is not the case, consider an h with compact support satisfying $h\equiv 1$ in a neighborhood of x_0. Then

$$f = f-f(x_0)h+f(x_0)h$$

and if $p\big((f-f(x_0)h)^{\sim}\big)=0$ then $\nu'(f-f(x_0)h) \in \mu(C_0(\hat{G}))$. But since $h\equiv 1$ in a neighborhood of x_0, we have $\nu'(f(x_0)h) \in \mu(C_0(\hat{G}))$, yielding the continuity of ν'. So we can assume the existence of $f \in C_0(\hat{G})$ with $f(x_0)=0$ and $p(\tilde{f})>0$. By the above remarks this means there is an $h \in L^1(G)$ with $\gamma(h)(x_0)=0$ and $p(\gamma(h)^{\sim})>0$. Again by [4, 2.6.4], given $\varepsilon>0$ we can find $k \in L^1(G)$ such that

$$\|h-k\|_1 < \varepsilon$$

and

$$\gamma(k) \in Z(x_0)\,,$$

that is,

$$p(\gamma(h)^{\sim}-\gamma(k)^{\sim}) = p(\gamma(h)^{\sim}) > 0$$

and this proves the discontinuity of $^{\sim}\circ\gamma$ in the second case.

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