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ON HÖRMANDER'S THEOREM ABOUT SURJECTIONS OF $\mathscr{D}'$

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Summary.

Consider an operator $P\colon \mathscr{D} \to \mathscr{D}$. It is proved here that for $P'\mathscr{D}' = \mathscr{D}'$ it is necessary and sufficient that P^{-1} is sequentially continuous in a certain sense and that it "holds singularities". Roughly speaking, the latter means that for a certain extension $\bar{P}$ of P, acting no more over infinitely smooth functions, to every compact K_1 it is possible to assign a compact K_2 such that $\operatorname{sing\,supp} \bar{P}u \subset K_1$ implies $\operatorname{sing\,supp} u \subset K_2$. Similar results are actually established not only for $\mathscr{D}$ but for a certain wide class of $(\mathscr{L}\mathscr{F})$-spaces. The paper is made selfcontained and includes some results announced in [5] and [6]. It provides an answer to some questions raised by Trèves in the introduction to [8].

For topological spaces (W,μ) and (V,ν) we write $(W,\mu) \leqq (V,\nu)$ if $V \subset W$ and if the identical injection of V into W is continuous.

Denote by K the compact subsets of the N-dimensional Euclidean space equal to the closure of their interior. We write

$$\mathscr{D}(K) = \{f \in \mathscr{D}\colon \operatorname{supp} f \subset K\},$$

and by $\tau_{\mathscr{D}}$ we denote the usual topology of $\mathscr{D}$. In what follows, $(\mathscr{E}', \tau_{\mathscr{E}'})$ shall denote the space of distributions with compact supports with the usual topology of $\mathscr{E}'$.

We shall say that a Banach space $(L, \|\cdot\|)$, briefly (L), *carries singularities over K* if

$$(*) \qquad (\mathscr{D}(K), \tau_{\mathscr{D}}) \geqq (L) \geqq (\mathscr{E}', \tau_{\mathscr{E}'})$$

and if $\mathscr{D}(K)$ is dense in (L).

Consider an open set Ω in the N-dimensional Euclidean space. A family ξ of Banach spaces is said to be a *projective component of $\mathscr{D}(\Omega)$* if every space from ξ carries singularities over some compact $K \subset \Omega$ and if the following conditions hold.

1) To every compact $K \subset \Omega$ there corresponds an $(L) \in \xi$ which carries singularity over K.

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2) The family ξ contains a sequence which is decreasing and cofinal with respect to the relation $\geqq$.

With every projective component ξ we associate the space L_ξ which is the union of all L for $(L)\in\xi$. A sequence is said to be convergent in L_ξ if it converges in an $(L)\in\xi$.

Consider open subsets Ω_1 and Ω_2 of the N_1- and N_2-dimensional Euclidean spaces respectively and a linear mapping P from $\mathscr{D}(\Omega_1)$ to $\mathscr{D}(\Omega_2)$ continuous with respect to the usual topologies. Take components ξ_1 and ξ_2 of $\mathscr{D}(\Omega_1)$ and $\mathscr{D}(\Omega_2)$ respectively. Given $(L_2)\in\xi_2$, call $\{(f_n, g_n)\}\in L_{\xi_1}\times(L_2+\mathscr{D}(\Omega_2))$ convergent if $\{f_n\}$ converges in L_{ξ_1}, $\{g_n\}$ converges in L_{ξ_2} and in addition $\{g_n\}$ converges uniformly with all derivatives off a compact for which (L_2) carries singularities. Let $\overline{P}_{L_2}$ denote the sequential closure of P in $L_{\xi_1}\times(L_2+\mathscr{D}(\Omega_2))$. We say that P^{-1} *holds singularities from* ξ_2 *to* ξ_1 if the following condition holds

$(H^*)_{\xi_2,\xi_1}$: To every $(L_2)\in\xi_2$ there corresponds an $(L_1)\in\xi_1$ such that the domain of $\overline{P}_{L_2}$ is contained in $L_1+\mathscr{D}(\Omega_1)$.

THEOREM 1. *The adjoint mapping P' is a surjection, that is, $P'\mathscr{D}'(\Omega_2)=\mathscr{D}'(\Omega_1)$ iff P^{-1} is sequentially continuous from each L_{ζ_2} to some L_{ζ_1} and there exist projective components ξ_1 and ξ_2 such that P^{-1} holds singularities from ξ_2 to ξ_1.*

Theorem 1 can easily be expressed also for Ω_1 and Ω_2 being differentiable manifolds. However, the most important fact is that in the Theorem the condition for P is invariant with respect to automorphisms of $\mathscr{D}$. This makes it possible to formulate an analogue to Theorem 1 for P acting within a certain pretty large class of $(\mathscr{L}\mathscr{F})$-spaces and then provide a proof using purely functional analytic tools.

The case where P is the convolution operator was investigated by Hörmander in [2]. One of the results of Section 4 of [2] can be expressed as follows.

THEOREM. *If P is a convolution operator transforming $\mathscr{D}(\Omega_1)$ into $\mathscr{D}(\Omega_2)$, then for P' to map $\mathscr{D}'(\Omega_2)$ onto $\mathscr{D}'(\Omega_1)$ it is necessary and sufficient that P^{-1} is sequentially continuous from $\mathscr{D}(\Omega_2)$ to $\mathscr{D}(\Omega_1)$ and that to every compact $K_2\subset\Omega_2$ there corresponds a compact $K_1\subset\Omega_1$ such that for $u\in\mathscr{E}'(\Omega_1)$*

$$(*)\qquad \operatorname{sing\,supp}\overline{P}u\subset K_2 \quad \textit{implies} \quad \operatorname{sing\,supp}u\subset K_1,$$

where $\overline{P}$ is the natural extension of P.

It is easy to see that $(*)$ implies $(H^*)_{\xi_2,\xi_1}$ for $L_{\xi_2}=\mathscr{E}'(\Omega_2)$ and $L_{\xi_1}=\mathscr{E}'(\Omega_1)$.

Because of the particular choice of the projective components, this version is not invariant under automorphisms of $\mathscr{D}$. It would be interesting to find out if for a P which admits an extension to a continuous mapping from $\mathscr{E}'(\Omega_1)$ to $\mathscr{E}'(\Omega_2)$ and has P^{-1} sequentially continuous from $\mathscr{D}(\Omega_2)$ to $\mathscr{D}(\Omega_1)$, fulfilment of the condition $(H^*)_{\xi_2,\xi_1}$ for any pair of components implies its fulfilment for the special components which decompose $\mathscr{E}'(\Omega_1)$ and $\mathscr{E}'(\Omega_2)$ respectively.

Consider the following property concerning families of sets. Given any two families $\mathscr{F}_1$ and $\mathscr{F}_2$ of subsets of a fixed set, we say that $\mathscr{F}_1$ *does not overrun* $\mathscr{F}_2$ if to every $U_1 \in \mathscr{F}_1$ there corresponds an $U_2 \in \mathscr{F}_2$ such that $U_1 \cap \bigcup \mathscr{F}_2 \subset U_2$, where $\bigcup \mathscr{F}_2$ denotes the union of the sets from $\mathscr{F}_2$.

In this paper we shall consider the $(\mathscr{L}\mathscr{F})$-spaces (X, τ) as in [1] fulfilling the following additional requirement.

There exists an $(\mathscr{L}\mathscr{F})$-space $(L, \jmath) \leqq (X, \tau)$ which is the strict inductive limit of Banach spaces such that the family of Banach subspaces of $(L, \jmath)$ does not overrun the family of Fréchet subspaces of (X, τ), where by a Banach (Fréchet) subspace we understand any subspace which is Banach (Fréchet) in the induced topology. We shall denote by $\mathscr{F}(X, \tau)$ the family of all Fréchet subspaces of an $(\mathscr{L}\mathscr{F})$-space (X, τ).

A locally convex space $(L_\xi, \jmath_\xi)$ is said to be a strict p-component of (X, τ) if $(X, \tau) \geqq (L_\xi, \jmath_\xi)$, $(L_\xi, \jmath_\xi)$ is the strict inductive limit of a sequence of Banach spaces, and if to every $Z \in \mathscr{F}(L_\xi, \jmath_\xi)$ there corresponds a $U \in \mathscr{F}(X, \tau)$ such that Z is contained in the closure of U in $(L_\xi, \jmath_\xi)$. Clearly, all elements of $\mathscr{F}(L_\xi, \jmath_\xi)$ are Banach spaces.

A strict p-component is fully described by the family $\mathscr{F}(L_\xi, \jmath_\xi)$. We shall consider the family of all unit balls of spaces from $\mathscr{F}(L_\xi, \jmath_\xi)$. This family consists of all closed, absolutely convex bounded subsets C of $(L_\xi, \jmath_\xi)$. Then

$$L_C = \bigcup_1^\infty nC \in \mathscr{F}(L_\xi, \jmath_\xi) .$$

We shall denote by ξ the family of all C for which additionally $X \cap L_C$ is dense in $(L_C, \jmath_\xi)$. Notice that the Minkowski functional $\|\cdot\|_C$ of C induces on L_C the topology $\jmath_\xi$. We shall often write briefly ξ for $(L_\xi, \jmath_\xi)$.

The definition of ξ in [7] does not coincide with the one given here. It is, however, easily verified that every family, as defined here, can be uniquely extended to fulfil requirements of [7] with preservation of the space $(L_\xi, \jmath_\xi)$. Conversely, to every ξ, as defined in [7], there corresponds a strict p-component $(L, \jmath) \geqq (L_\xi, \jmath_\xi)$. Notice that in [7], p-components are not assumed Hausdorff while here we deal only with Hausdorff components.

The family of all strict p-components ξ of (X,τ) such that $\mathscr{F}(L_\xi, \downharpoonleft_\xi)$ does not overrun $\mathscr{F}(X,\tau)$ we denote by $\mathscr{P}(X,\tau)$. For $\xi_1,\xi_2 \in \mathscr{P}(X,\tau)$ we write $\xi_1 \leqq \xi_2$ if $(L_{\xi_1}, \downharpoonleft_{\xi_1}) \leqq (L_{\xi_2}, \downharpoonleft_{\xi_2})$.

For locally convex spaces (V_i,ν_i), $i=1,2$, we write

$$(V_1,\nu_1) \wedge (V_2,\nu_2) = (V_1+V_2,\nu_1\wedge\nu_2)$$

for the inductive limit of those spaces, that is for $\nu_1\wedge\nu_2$ is set the finest locally convex topology such that $(V_1+V_2, \nu_1\wedge\nu_2) \leqq (V_i,\nu_i)$ for $i=1,2$.

Consider a pair (Y,σ) and (X,τ) of $(\mathscr{L}\mathscr{F})$-spaces and a continuous linear mapping P from (Y,σ) to (X,τ). Take $\xi \in P(Y,\sigma)$ and $\xi \in P(X,\tau)$. We say that P^{-1} is *$\wedge$-continuous from ξ to ζ* or we write briefly $P^{-1} \in (M)_{\xi,\zeta}$ if the following condition holds.

$(M)_{\xi,\zeta}$: To every $B \in \xi$ there corresponds a $C \in \zeta$ such that to every $U \in \mathscr{F}(X,\tau)$ there corresponds a $V \in \mathscr{F}(Y,\sigma)$ in such a way that P^{-1} maps $PY \cap (L_B+U)$ into L_C+V and that it is continuous from $(L_B, \downharpoonleft_\xi) \wedge (U,\tau)$ to $(L_C, \downharpoonleft_\xi) \wedge (V,\sigma)$.

We say that P^{-1} is *$\wedge$-continuous from (X,τ) to (Y,σ)* or we write briefly $P^{-1} \in (M)$ if the following condition holds.

(M): To every $\zeta \in \mathscr{P}(Y,\sigma)$ there corresponds a $\xi \in \mathscr{P}(X,\tau)$ such that $P^{-1} \in (M)_{\xi,\zeta}$.

We say that P^{-1} *holds singularities from ξ to ζ* or we write briefly $P^{-1} \in (H)_{\xi,\zeta}$ if the following condition holds.

$(H)_{\xi,\zeta}$: To every $B \in \xi$ there corresponds a $C \in \zeta$ such that for every $\{y_n\} \subset Y$ tending to y in $(L_\zeta, \downharpoonleft_\zeta)$ with $\{Py_n\}$ converging in $(L_B, \downharpoonleft_\xi) \wedge (X,\tau)$ we have y belonging to L_C+Y.

We say that P^{-1} *totally holds singularities* or we write briefly $P^{-1} \in (H)$ if the following condition holds.

(H): To every $\zeta \in \mathscr{P}(Y,\sigma)$ there corresponds a $\xi \in \mathscr{P}(X,\tau)$ such that $P^{-1} \in (H)_{\xi,\zeta}$.

We say that P^{-1} is sequentially continuous on components if to every $\zeta \in \mathscr{P}(Y,\sigma)$ there corresponds $\xi \in \mathscr{P}(X,\tau)$ such that P^{-1} is sequentially continuous from $(L_\xi. \downharpoonleft_\xi)$ to $(L_\xi, \downharpoonleft_\xi)$.

A p-component is said to be reflexive if $\mathscr{F}(L_\xi, \downharpoonleft_\xi)$ consists of reflexive Banach spaces. From now on we shall assume that to every $\xi_1 \in \mathscr{P}(X,\tau)$ there corresponds a reflexive $\xi_2 \in \mathscr{P}(X,\tau)$ with $\xi_1 \leqq \xi_2$.

THEOREM 2. *The adjoint P' of P maps the dual X' of (X,τ) onto the dual Y' of (Y,σ) iff P^{-1} is $\wedge$-continuous from (X,τ) to (Y,σ).*

THEOREM 3. *The mapping P^{-1} is $\wedge$-continuous from (X,τ) to (Y,σ) iff it is sequentially continuous on components and totally holds singularities.*

THEOREM 4. *Take* $\zeta,\lambda \in \mathscr{P}(Y,\sigma)$. *If* $\zeta \geqq \lambda$ *and* λ *does not overrun* ζ, *then if* P^{-1} *holds singularities from some* $\xi \in \mathscr{P}(X,\tau)$ *to* λ, *it holds singularities from* η *to* ζ *for any* $\xi \leqq \eta \in \mathscr{P}(X,\sigma)$.

REMARK. Denoting by $\bar{P}_B$ the sequential closure of P in $(L_\zeta, \downarrow_\zeta) \times [(L_B, \downarrow_\xi) \wedge (X,\tau)]$ the condition $(H)_{\xi,\zeta}$ is equivalent to the following condition.

$(H^*)_{\xi,\zeta}$: To every $B \in \xi$ there corresponds a $C \in \zeta$ such that the domain of $\bar{P}_B$ is contained in $L_C + Y$.

The rest of this paper shall be devoted to verifying Theorems 2, 3 and 4, and here we notice that Theorem 1 is an easy consequence of these theorems. Indeed, the not overrunning conditions are certainly fulfilled for projective components of $\mathscr{D}(\Omega)$ as they were defined here. This is because neither the family of Fréchet subspaces of $\mathscr{D}(\Omega)$ nor any component of $\mathscr{D}(\Omega)$ can be overrun by the "ultimate" component which is the decomposition of $\mathscr{E}'(\Omega)$ into Banach spaces (of course not a strict decomposition). Joining Theorems 2, 3 and 4 with the Remark, we obtain Theorem 1 as a trivial corollary.

PROPOSITION 1. *Let* $\xi \in \mathscr{P}(X,\tau)$ *and* $C \in \xi$. *Then* $(L_C, \downarrow_\xi) \wedge (X,\tau)$ *is again an* $(\mathscr{L}\mathscr{F})$*-space and it is the inductive limit of* $(L_C, \downarrow_\xi) \wedge (X_n,\tau)$ *for every decomposition* $\{X_n\} \subset \mathscr{F}(X,\tau)$ *of* X.

PROOF. Since $\{(X_n,\tau)\}$ is strict, $\{(L_C, \downarrow_\xi) \wedge (X_n,\tau)\}$ is strict as well and thus $\operatorname{lim\,ind}(L_C, \downarrow_\xi) \wedge (X_n,\tau)$ is an $(\mathscr{L}\mathscr{F})$-space. A seminorm on $L_C + X$ is continuous in $(L_C, \downarrow_\xi) \wedge (X,\tau)$ iff it is continuous in every $(L_C, \downarrow_\xi) \wedge (X_n,\tau)$ and the Proposition follows.

We shall write $P\zeta \geqq \eta$ if for every $B \in \zeta$ there is a $C \in \eta$ such that $P(Y \cap B) \subset C$. This simply means that P is continuous from $(L_\zeta, \downarrow_\zeta)$ to $(L_\eta, \downarrow_\eta)$. We also put

$$L_{(U\cap C)^\circ} =_{df} \bigcup_1^\infty n(U \cap C)^\circ ,$$

where $^\circ$ denotes the polar in U'.

LEMMA 1. *If* P^{-1} *is* $\wedge$*-continuous from* ξ *to* ζ, *and if* $\eta \in \mathscr{P}(X,\tau)$ *is such that* $\eta \leqq \xi$ *and* $P\zeta \geqq \xi$, *then setting* $U = PY$ *we have* $U \in (ACC)_{\eta,\xi}$, *that is,* U *fulfils the following condition.*

$(ACC)_{\eta,\xi}$: *To every* $B \in \xi$ *there corresponds a* $D \in \eta$ *such that for every* $Z \in \mathscr{F}(X,\tau)$ *we have*

$$L_B \cap (U \cap Z)^- \subset (U \cap L_D)^-,$$

where the closures $^-$ *are taken subsequently in* $(L_B, \downharpoonleft_\xi) \wedge (X, \tau)$ *and in* $(L_D, \downharpoonleft_\eta)$.

Proof. Take $B \in \xi$ and adjust $C \in \zeta$ according to $(M)_{\xi,\zeta}$. Subsequently, take $D \in \eta$, $D \supset B$, in such a way that it is $P(Y \cap C) \subset D$. Fix Z and take a sequence $\{Py_n\} \subset U \cap Z$ convergent to some x in $(L_B, \downharpoonleft_\xi) \wedge (Z, \tau)$. From $(M)_{\xi,\zeta}$ it follows that $\{y_n\}$ converges to some y in $(L_C, \downharpoonleft_\zeta) \wedge (V, \sigma)$ for some $(\mathscr{F})$-subspace V of (Y, σ). Hence, $y_n = c_n + v_n$, where $\{c_n\}$ tends to some c in $(L_C, \downharpoonleft_\zeta)$ and $\{v_n\}$ tends to some v in (V, σ). Since $P(Y \cap C) \subset D$, the sequence $\{Pc_n\}$ has a limit d in $(L_D, \downharpoonleft_\eta)$ and we have $Py_n = Pc_n + Pv_n$ converging to $x = d + Pv \in L_B \subset L_D$. Thus $Pv = x - d \in U \cap L_D$ and $P(c_n + v)$ tends to x in $(L_D, \downharpoonleft_\eta)$ which concludes the proof.

Put $(L_A', \|\cdot\|_A') =$ the adjoint of $(L_A, \|\cdot\|_A)$. Write $U \in (A_0)_{\eta,\xi}$ if the following condition holds.

$(A_0)_{\eta,\xi}$: To every $B \in \xi$ there corresponds a $D \in \eta$, $D \supset B$, such that to every $\varepsilon > 0$, every $Z \in \mathscr{F}(X, \tau)$ and every $z' \in L_D'$ vanishing on $U \cap D$ there corresponds an $x' \in X'$ bounded on $X \cap B$ and vanishing on $U \cap Z$ such that $\|\bar{x}' - \bar{z}'\|_B' < \varepsilon$, where $\bar{x}' \in L_B'$ denotes the extension of the restriction of x' to $X \cap L_B$ and $\bar{z}'$, denotes the restriction of z' to L_B.

We then prove the following lemma, cf. [5, Proposition 2]. (The property (A_0) seems to be related to the notion of orthogonality introduced in [3] by Pták.)

Lemma 2. *If* ξ *is reflexive, then* $(ACC)_{\eta,\xi} \subset (A_0)_{\eta,\xi}$.

Proof. Notice that $(A_0)_{\eta,\xi}$ amounts to the following statement. Given $B \in \xi$, we can find $D \in \eta$, $D \supset B$, such that for every Fréchet subspace Z of (X, τ) the closure of the subspace

$$V_1 = \{\bar{x}' \in L_B' : x' \in L_{(X \cap B)^\circ}, x'(U \cap Z) = \{0\}\}$$

with respect to the norm $\|\cdot\|_B'$ contains the subspace

$$V_2 = \{\bar{z}' \in L_B' : z' \in L_D', z'(U \cap L_D) = \{0\}\} .$$

Due to reflexivity of ξ, it is sufficient to show weak* density of V_1, that is, if for $z \in L_B$ all functionals from V_1 vanish on z, then all functionals from V_2 vanish on z as well.

The space V_1 consists of the restrictions to L_B of functionals from

$$V_1 = \{x' \in M' :\ x'(U \cap Z) = \{0\}\} ,$$

where $(M,\mu) = (L_B, \downarrow_\xi) \wedge (X,\tau)$. Hence, to prove the Lemma, we have to show that from $(ACC)_{\eta,\xi}$ it follows that if for $z \in L_B$ all functionals from V_1 vanish on z, then all the functionals from V_2 vanish on z as well. The first part of this means that $z \in (U \cap Z)^-$, where the closure $^-$ is taken in (M,μ), and the second part amounts to $z \in (U \cap L_D)^-$, where the closure $^-$ is taken in $(L_D, \downarrow_\eta)$, so that the above-stated implications amounts to the inclusion from $(ACC)_{\eta,\xi}$ and the Lemma follows.

Lemma 3. (Cf. [7, Theorem 5.1].) *Consider* $\xi,\eta \in \mathscr{P}(X,\tau)$, $\xi \geqq \eta$, *and* $\lambda \in \mathscr{P}(Y,\sigma)$. *If* $P^{-1} \in (M)_{\eta,\lambda}$ *and* $PY \in (A_0)_{\eta,\xi}$, *then* $P' \in (NO^*)_{\xi,\lambda}$, *that is,* P' *satisfies the following condition.*

$(NO^*)_{\xi,\lambda}$: To every $B \in \xi$ there corresponds a $C \in \lambda$ such that to every $y' \in (Y \cap C)^\circ$ and every $Z \in \mathscr{F}(Y,\sigma)$ there corresponds an $x' \in (X \cap B)^\circ$ such that $y'y = (P'x')y$ for $y \in Z$.

Proof. Take $B \in \xi$ and adjust $D \in \eta$, $D \supset B$, to fulfil $(A_0)_{\eta,\xi}$. Subsequently, adjust to D a $C \in \lambda$ to fulfil $(M)_{\eta,\lambda}$. The condition $(M)_{\eta,\lambda}$ allows us to make it so that

$$(*) \qquad Px \in D \quad \text{implies} \quad x \in \tfrac{1}{2}C .$$

Fix a $Z \in \mathscr{F}(Y,\sigma)$, $Z \supset Y \cap L_C$. Since P is continuous, we can find a $V \in \mathscr{F}(X,\tau)$ such that $PZ \subset V$. Additionally, we shall require that $V \supset X \cap L_D$. Take $y' \in (Y \cap C)^\circ$. Since $y'P^{-1}$ is continuous in

$$(L_D, \downarrow_\eta) \wedge (V,\tau) ,$$

we can extend it over V to a $\downarrow_\eta \wedge \tau$-continuous functional and then we can still extend the obtained functional over to $u' \in X'$. Denoting by $\|u'\|$ the sup norm of u' in $((PY) \cap L_D, \|\cdot\|_D)$, we get from $(*)$

$$\|u'\| \leqq \tfrac{1}{2}\|P'u'\|_{(Y \cap C)^\circ} .$$

Thus, for the norm-preserving extension $z' \in L_D'$ of the restriction of u' to $(PY) \cap L_D$ we obtain

$$\|z'\|_D' \leqq \tfrac{1}{2}\|P'u'\|_{(Y \cap C)^\circ} .$$

Denoting by $\bar{z}'$ the restriction of z' to L_B, we obtain

$$\begin{aligned} \|\bar{z}'\|_B' \leqq \|z'\|_D' &\leqq \tfrac{1}{2}\|P'u'\|_{(Y \cap C)^\circ} \\ &\leqq \tfrac{1}{2}(\|y' - P'u'\|'_{(Y \cap C)^\circ} + \|y'\|'_{(Y \cap C)^\circ}) \leqq \tfrac{1}{2} . \end{aligned}$$

Writing $u^* \in L_D'$ for the extension of the restriction of u' to $X \cap L_D$, we notice that $z' - u^* \in L_D'$ vanishes on $(PY) \cap L_D$. Using $(A_0)_{\eta,\xi}$ we find $v' \in X'$ bounded on $X \cap B$ and vanishing on $(PY) \cap V$ such that

$$\|(\bar{z}' - \bar{u}') - \bar{v}'\|_B' < \tfrac{1}{2},$$

where $\bar{z}', \bar{u}', \bar{v}' \in L_B'$ denote the extensions of the restrictions to $X \cap L_B$ of z', u', v' respectively. Since $PZ \subset V$, we have $(P'v')y = 0$ for $y \in Z$. Since $(y' - P'u')y = 0$ for $y \in Z$, setting $x' = u' + v'$, we obtain $y'y = (P'x')y$ for $y \in Z$. Moreover,

$$\|x'\|_{(X \cap B)^\circ} \leqq \|\bar{v}' - (\bar{z}' + \bar{u}')\|_B' + \|\bar{z}'\|_B' \leqq 1,$$

and the Lemma follows.

LEMMA 4. *We have* $(NO^*)_{\xi,\lambda} \subset (O)_{\xi,\lambda}$, *where* $P' \in (O)_{\xi,\lambda}$ *if the following condition holds.*

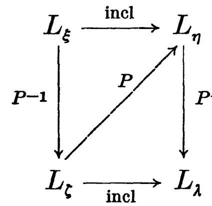
$(O)_{\xi,\lambda}$: To every $B \in \xi$ there corresponds a $C \in \lambda$ such that to every $y' \in (Y \cap C)^\circ$ there corresponds an $x' \in (X \cap B)^\circ$ such that $y' = P'x'$.

PROOF. Take an ascending sequence $\{B_n\} \subset \xi$ cofinal with ξ such that $B_1 = B$. Subsequently, to every B_n assign $C_n \in \lambda$ according to the requirements of $(NO^*)_{\xi,\lambda}$. Finally, let $\{Z_n\}$, $Y = \bigcup_1^\infty Z_n$, be an ascending sequence of $(\mathscr{F})$-subspaces of (Y, σ) such that $Z_n \supset Y \cap C_n$ for $n = 1, 2, \ldots$.

Take $y' \in (Y \cap C)^\circ$, $C = C_1$, and assign to it $x_1' \in 2^{-1}(X \cap B_1)^\circ$ such that $y'y = (P'x_1')y$ for $y \in Z_2$. Hence $y_2' = y' - P'x_1' = (Y \cap C_2)^\circ$ and we can find $x_2' \in 2^{-2}(X \cap B_2)^\circ$ such that $y_2'y = (P'x_2')y$ for $y \in Z_3$. Continuing this way, we produce a sequence $x_n' \in 2^{-n}(X \cap B_n)^\circ$ such that $y'y = (P'(\sum_1^n x_i'))y$ for $y \in Z_{n+1}$. It is easy to see that $x' = \sum_1^\infty x_n'$ is a well defined functional belonging to $(X \cap B)^\circ$ such that $y' = P'x'$, and this concludes the proof of Lemma 4.

PROPOSITION 2. *If P is such that to every $\zeta \in \mathscr{P}(Y, \sigma)$ there corresponds $\xi \in \mathscr{P}(X, \tau)$ such that P^{-1} is $\wedge$-continuous from ξ to ζ, then to every $\lambda \in \mathscr{P}(Y, \sigma)$ there corresponds a $\xi \in \mathscr{P}(X, \tau)$ such that P' fulfils $(O)_{\xi,\lambda}$.*

PROOF. To a given $\lambda \in \mathscr{P}(Y, \sigma)$ we first assign $\eta \in \mathscr{P}(X, \tau)$ such that $P^{-1} \in (M)_{\eta,\lambda}$. Then we find $\zeta \in \mathscr{P}(Y, \sigma)$, $\zeta \geqq \lambda$, such that $P\zeta \geqq \eta$ and finally to ζ we assign a reflexive $\xi \in \mathscr{P}(X, \tau)$, $\xi \geqq \eta$, such that $P^{-1} \in (M)_{\xi,\zeta}$. The following commutative diagram describes the situation,



From Lemma 1 we obtain $PY \in (ACC)_{\eta,\xi} \subset (A_0)_{\eta,\xi}$ and applying subsequently Lemmas 3 and 4, we obtain $P' \in (NO^*)_{\xi,\lambda} \subset (O)_{\xi,\lambda}$ which concludes the proof.

COROLLARY 1. *If P fulfils the requirements of Proposition 2, then $P'X' = Y'$.*

PROOF. It is sufficient to notice that to every $y' \in Y'$ there corresponds a $\lambda \in \mathscr{P}(Y,\sigma)$ such that y' is bounded on $Y \cap B$ for every $B \in \lambda$ and then apply Proposition 2.

Take an $(\mathscr{L}\mathscr{F})$-space (Z,δ) and $\lambda \in \mathscr{P}(Z,\delta)$. Let Z' be the adjoint of (Z,δ). We shall define a metric topological group $(Z',\varrho_\lambda{}^\circ)$ as follows. First we choose an ascending cofinal sequence $\{C_n\} \subset \lambda$. Then we put for $x' \in Z'$

$$\begin{aligned} \varrho_n(x') &= t_n/(1+t_n) \quad \text{for} \quad t_n = \|x'\|_{(Z\cap C)^\circ} < \infty \\ &= 1 \qquad\qquad\quad \text{otherwise} \end{aligned}$$

and then

$$\varrho_\lambda{}^\circ(x') = \textstyle\sum_{n=1}^\infty 2^{-n} \varrho_n(x') \ .$$

It is easy to see that the topology induced by the metric $\varrho_\lambda{}^\circ(x'-y')$ does not depend on the choice of the sequence $\{C_n\}$. The convergence of $\{x_n{}'\} \subset Z'$ to zero in $(Z',\varrho_\lambda{}^\circ)$ means that given $C \in \lambda$, there exists an n_C such that $x_n{}' \in C^\circ$ for $n > n_C$. Therefore, the set of polars C° of C from λ constitutes a basis of neighbourhoods of zero in $(Z',\varrho_\lambda{}^\circ)$.

It is left to the reader to verify that $(Z',\varrho_\lambda{}^\circ)$ is always complete. The object $(Z',\varrho_\lambda{}^\circ)$ was introduced already in [7] and was called the $(\mathscr{F})$-class polar to λ.

Now, let us return to our original setup with two $(\mathscr{L}\mathscr{F})$-spaces (Y,σ), (X,τ) and a continuous mapping P from Y to X. We have the following

LEMMA 5. *Given $\lambda \in \mathscr{P}(Y,\sigma)$ and $\xi \in \mathscr{P}(X,\tau)$, $P' \in (O)_{\xi,\lambda}$ from Lemma 4 means that P' is an open mapping of $(X',\varrho_\xi{}^\circ)$ onto $(Y',\varrho_\lambda{}^\circ)$ and if $P' \in (O)_{\xi,\lambda}$, then P^{-1} is $\wedge$-continuous from ξ to λ.*

PROOF. Suppose that P' admits $(O)_{\xi,\lambda}$ and P^{-1} is not $\wedge$-continuous from ξ to λ. Then there exists a $B \in \xi$ such that for no $C \in \lambda$ the condition $(M)_{\xi,\lambda}$ is fulfilled. Choose $C \in \lambda$ which corresponds to B according to $(O)_{\xi,\lambda}$. Since, in particular, $(M)_{\xi,\lambda}$ does not hold for this choice of C, either there exists a $U \in \mathscr{F}(X,\tau)$ such that for no $V \in \mathscr{F}(Y,\sigma)$ the image by P^{-1} of $PY \cap (L_B + U)$ is contained in $V + L_C$, or, if for some V this image is contained in $V + L_C$, the mapping P^{-1} is not continuous from $(L_B, \downarrow_\xi) \wedge (U,\tau)$ to $(L_C, \downarrow_\lambda) \wedge (V,\sigma)$. In the first case we take a sequence $\{y_n\} \subset Y$ such that $\{Py_n\} \subset (L_B + U)$ and that for a cofinal ascending $\{V_n\} \subset \mathscr{F}(Y,\sigma)$ we have $y_n \in (L_C + V_{n+1}) - (L_C + V_n)$. Multiplying if necessary by $t_n > 0$, we can always make $\{Py_n\}$ bounded in the space $(L_B, \downarrow_\xi) \wedge (U,\tau)$, while $\{y_n\}$ cannot be bounded in the space $(L_C, \downarrow_\lambda) \wedge (Y,\sigma)$. Hence there exists a y' in the adjoint of $(L_C, \downarrow_\lambda) \wedge (Y,\sigma)$ on which $\{y_n\}$ is not bounded, and from $(O)_{\xi,\lambda}$ it follows that there must exist an x' in the adjoint of $(L_B, \downarrow_\xi) \wedge (X,\tau)$ such that $y'y = x'Py$ for $y \in Y$ and this contradicts boundedness of $\{Py_n\}$. In the alternative case, if P^{-1} maps $PY \cap (L_B + U)$ into $L_C + V$ for some $V \in \mathscr{F}(Y,\sigma)$ but P^{-1} is not continuous from $(L_B, \downarrow_\xi) \wedge (U,\tau)$ to $(L_C, \downarrow_\lambda) \wedge (V,\sigma)$, we take a bounded $\{Py_n\} \subset L_B + U$ such that $\{y_n\}$ is not bounded in $(L_C, \downarrow_\lambda) \wedge (V,\sigma)$ and choosing again y' in the adjoint of $(L_C, \downarrow_\lambda) \wedge (Y,\sigma)$ on which $\{y_n\}$ is not bounded, we arrive at a contradiction to the existence of x' in the adjoint of $(L_B, \downarrow_\xi) \wedge (X,\tau)$ such that $y'y = x'Py$ for $y \in Y$. This concludes the proof of Lemma 5.

LEMMA 4'. *For given $\lambda \in \mathscr{P}(Y,\sigma)$ and $\xi \in \mathscr{P}(X,\tau)$ we have the mapping P' open from $(X', \varrho_\xi{}^\circ)$ to $(Y', \varrho_\lambda{}^\circ)$ if $P' \in (NO)_{\xi,\lambda}$, that is, if P' fulfils the following condition.*

$(NO)_{\xi,\lambda}$: To every $B \in \xi$ there corresponds a $C \in \lambda$ such that for every $y' \in (Y \cap C)^\circ$, every $D \in \lambda$ and every $\varepsilon > 0$ there corresponds an $x' \in (X \cap B)^\circ$ with $\|y' - P'x'\|_{(Y \cap D)^\circ} < \varepsilon$.

PROOF. Though it is a consequence of Proposition 12 of [4], we shall prove it independently. (This is actually repetition of Banach's proof given for $(\mathscr{F})$-spaces.) Notice at first that $(NO)_{\xi,\lambda}$ amounts to the following statement. To every $\varepsilon > 0$ there corresponds a $\delta > 0$ such that the closure in $(Y', \varrho_\lambda{}^\circ)$ of the image by P' of the ball $\{x' \in X' : \varrho_\xi{}^\circ(x') < \varepsilon\}$ contains the ball $\{y' \in Y' : \varrho_\lambda{}^\circ(y') < \delta)$. Fix any $\varepsilon > 0$ and adjust a sequence $0 < t_n \to 0$ in such a way that the closure of $P'\{x' \in X' : \varrho_\xi{}^\circ(x') < 2^{-n}\varepsilon\}$ contains $\{y' \in Y' : \varrho_\lambda{}^\circ(y') < t_n\}$.

Then for $y' \in Y'$ with $\varrho^\circ(y') < t_1$ we can find $x_1' \in X'$ with $\varrho^\circ(x_1') < 2^{-1}\varepsilon$ such that $\varrho_\lambda{}^\circ(y' - P'x_1') < t_2$. Continuing this procedure, we define a sequence $\{x_n'\} \subset X'$ such that

$$\varrho_\xi^\circ(x_n') < 2^{-n}\varepsilon \quad \text{and} \quad \varrho_\lambda^\circ(y' - P'(x_1' + \ldots + x_n')) < t_{n+1}$$

for every n. Setting $x' = \sum_1^\infty x_n'$, we obtain $\varrho_\xi^\circ(x') < \varepsilon$ and $y' = P'x'$. Hence to every $\varepsilon > 0$ we assigned $\delta = t_1 > 0$ such that

$$P'\{x' \in X' : \varrho_\xi^\circ(x') < \varepsilon\} \supset \{y' \in Y' : \varrho_\lambda^\circ(y') < \delta\}$$

and this amounts to $(O)_{\xi,\lambda}$, that is, to the openness of P' from (X', ϱ_ξ°) to $(Y', \varrho_\lambda^\circ)$.

LEMMA 6. (Cf. [7, Theorem 7.1].) *Suppose that $P'X' = Y'$. Then to every $\lambda \in \mathscr{P}(Y,\sigma)$ there corresponds a $\xi \in \mathscr{P}(X,\tau)$ such that $P' \in (NO)_{\xi,\lambda}$.*

PROOF. Let $\{Z_n\} \subset \mathscr{F}(X,\tau)$ be an ascending sequence of (F)-subspaces cofinal with $\mathscr{F}(X,\tau)$. For every n we provide a point-wise non-decreasing sequence of norms $\{\| \ \|_{n,m} : m = 1,2,\ldots\}$ inducing the topology τ on Z_n. Denote by $Z^*_{n,m}$ the subspace of X' consisting of all functionals from X which on Z_n are continuous with respect to the norm $\| \ \|_{n,m}$. Clearly, for every n we have $X' = \bigcup_{m=1}^\infty Z^*_{n,m}$. Hence, there exists an m_1 such that $P'Z^*_{1,m_1}$ is a second category subset of $(Y', \varrho_\lambda^\circ)$. Suppose that we have selected $m_1, \ldots, m_n$ such that $P'(Z^*_{1,m_1} \cap \ldots \cap Z^*_{n,m_n})$ is a second category subset of $(Y', \varrho_\lambda^\circ)$. Then, since

$$P'(Z^*_{1,m_1} \cap \ldots \cap Z^*_{n,m_n}) = \bigcup_{m=1}^\infty P'(Z^*_{1,m_1} \cap \ldots \cap Z^*_{n,m_n} \cap Z^*_{n+1,m}),$$

we can still find m_{n+1} such that $P'(Z^*_{1,m_1} \cap \ldots \cap Z^*_{n+1,m_{n+1}})$ is a second category subset of $(Y', \varrho_\lambda^\circ)$. Continuing this procedure, we produce a sequence $\{Z^*_{n,m_n}\}$ of sets with the above property. Notice that we can then produce a norm $\|\cdot\|$ such that $\|x\|_{n,m_n} \leqq a_n \|x\|$ for $x \in Z_n$, $n = 1,2,\ldots$, and that $\|\cdot\|$ is continuous in every (Z_n, τ). Let $\xi \in \mathscr{P}(X,\tau)$ be such that $(Z_n, \downarrow_\xi) = (Z_n, \|\cdot\|)$. Then to every $B \in \xi$ there corresponds n such that

$$L_{(X \cap B)^\circ} \supset Z^*_{1,m_1} \cap \ldots \cap Z^*_{n,m_n},$$

that is, for every B, $P'L_{(X\cap B)^\circ}$ is a second category subset of $(Y', \varrho_\lambda^\circ)$. Here we could directly apply Proposition 10 of [4], but to make the paper self-contained, we repeat the argument essentially due to Banach. Fix $\varepsilon > 0$ and define $H_m = P'\{x' \in X' : \varrho_\xi^\circ(x'/m) < \frac{1}{2}\varepsilon\}$. For this ε we can find a $B \in \xi$ such that

$$P'L_{(X\cap B)^\circ} \subset \bigcup_1^\infty H_m,$$

and since the set on the left is a second category subset, there exists m_0 such that H_{m_0} is not nowhere dense in $(Y', \varrho_\lambda^\circ)$. Hence, there are $y_0' \in Y'$ and $r > 0$ such that

$$H_{m_0} \supset \{y' \in Y' : \varrho_\lambda^\circ(y' - y_0') < r\},$$

where $^-$ denotes the closure in $(Y', \varrho_\lambda^\circ)$. Since $\varrho_\lambda^\circ(y' - y_0'/m_0) < r/m_0$ implies

$$\varrho_\lambda^\circ(m_0 y' - y_0') = \varrho_\lambda^\circ(m_0(y' - y'/m_0)) \leqq m_0 \varrho_\lambda^\circ(y' - y_0'/m_0) < r ,$$

we have $m_0 y' \in H_{m_0}^-$ and thus $y' \in H_1^-$. Taking $y_1' \in H_1$ with

$$\varrho_\lambda^\circ(y_1' - y_0'/m_0) < \eta = r/2m_0 ,$$

we obtain $H_1^- \supset \{y' \in Y' : \varrho_\lambda^\circ(y_1' - y') < \eta\}$. For $\varrho_\lambda^\circ(y') < \eta$, we have $\varrho_\lambda^\circ((y_1' - y') - y_1')$. Hence $y_1' - y' \in H_1^-$, and thus also $y' - y_1' \in H_1^-$ so that finally

$$(*) \qquad \{y' \in Y' : \varrho_\lambda^\circ(y') < \eta\} \subset (y_1' - H_1)^- .$$

Moreover, if $y' = y_1' - u' \in y_1' - H_1$, then there are $x_1', v' \in X'$ with $\varrho_\xi^\circ(x_1')$, $\varrho_\xi^\circ(v') < \frac{1}{2}\varepsilon$ such that $y_1' = P'x_1'$ and $u' = P'v'$. Setting $x' = x_1' - v'$, we have $y' = P'x'$ and $\varrho_\xi^\circ(x') < \varepsilon$ so that

$$(**) \qquad y_1' - H_1 \subset P'\{x' \in x' : \varrho_\xi^\circ(x') < \varepsilon\} .$$

Joining (*) and (**), we obtain

$$\{y' \in Y' : \varrho_\lambda^\circ(y') < \eta\} \subset (P'\{x' \in X' : \varrho_\xi^\circ(x') < \varepsilon\})^-,$$

and due to the arbitrariness of the choice of $\varepsilon > 0$ this amounts exactly to $(NO)_{\xi,\lambda}$. Hence, the Lemma 6 is proved.

PROPOSITION 3. *If $P'X' = Y'$, then to every $\lambda \in \mathscr{P}(Y,\sigma)$ there corresponds a $\xi \in \mathscr{P}(X,\tau)$ such that P^{-1} is $\wedge$-continuous from ξ to λ.*

PROOF. Given $\lambda \in \mathscr{P}(Y,\sigma)$, we apply first Lemma 6 and find $\xi \in \mathscr{P}(X,\tau)$ such that $(NO)_{\xi,\lambda}$ holds. Then from Lemma 4′ we conclude that also $(O)_{\xi,\lambda}$ holds for this choice of λ. Then it remains only to apply Lemma 5 to find that for the pair λ and ξ the condition $(M)_{\xi,\lambda}$ holds as well, and this concludes the proof of Proposition 3.

PROOF OF THEOREM 2. This proof amounts to combining Corollary 1 and Proposition 3.

PROPOSITION 4. *Fix arbitrary $\zeta \in \mathscr{P}(Y,\sigma)$ and $\xi \in \mathscr{P}(X,\tau)$. Then P^{-1} is $\wedge$-continuous from ξ to ζ if and only if P^{-1} is sequentially continuous from ξ to ζ and holds singularities from ξ to ζ.*

PROOF. Take $B \in \xi$ and assign to it $C \in \zeta$ according to $(H)_{\xi,\zeta}$. Subsequently, fix $U \in \mathscr{F}(X,\tau)$ and take $D \in \xi$ such that $D \supset B$ and $L_D \supset U$.

There exists an $A \in \zeta$, $A \supset C$, such that P^{-1} is sequentially continuous from $(L_D, \downharpoonleft_\xi)$ to $(L_A, \downharpoonleft_\zeta)$ so that we can produce an extension S of P^{-1} to an operator closed in $(L_D, \downharpoonleft_\xi) \times (L_A, \downharpoonleft_\zeta)$. We shall show that S which is defined on the closure of $PY \cap (L_B + U)$ transforms this closure into $L_C + Y$ and is closed in $[(L_B, \downharpoonleft_\xi) \wedge (U, \tau)] \times [(L_C, \downharpoonleft_\zeta) \wedge (Y, \sigma)]$. Indeed, if $\{Py_n\}$ tends to some x in $(L_B, \downharpoonleft_\xi) \wedge (U, \tau)$, then it tends to x in $(L_D, \downharpoonleft_\xi)$ as well. Hence $\{y_n\}$ tends to some y in $(L_A, \downharpoonleft_\zeta)$ and by $(H)_{\xi,\zeta}$ we have $y \in L_C + Y$. Clearly, $Sx = y$. Applying the closed graph theorem, we obtain continuity of S from $(L_B, \downharpoonleft_\xi) \wedge (U, \tau)$ to $(L_C, \downharpoonleft_\zeta) \wedge (Y, \sigma)$. Hence $P^{-1} \in (M)_{\xi,\zeta}$ which concludes the proof of sufficiency. To verify necessity, notice first that $\wedge$-continuity of P^{-1} from ξ to ζ trivially implies its sequential continuity from $(L_\xi, \downharpoonleft_\xi)$ to $(L_\zeta, \downharpoonleft_\zeta)$. Indeed, by $(M)_{\xi,\zeta}$ one can assign to $B \in \xi$ a $C \in \zeta$ in such a way that P^{-1} is continuous from $(L_B, \downharpoonleft_\xi)$ to $(L_C, \downharpoonleft_\zeta) \wedge (V, \sigma)$ for some $V \in \mathscr{F}(Y, \sigma)$, and thus from $(L_B, \downharpoonleft_\xi)$ to some $(L_D, \downharpoonleft_\zeta)$ for a suitable $D \in \zeta$. Since $(H)_{\xi,\zeta}$ is also an immediate consequence of $(M)_{\xi,\zeta}$, Proposition 4 holds.

PROOF OF THEOREM 3. Suppose P^{-1} is sequentially continuous on components and totally holds singularities from (X, τ) to (Y, σ). Take $\zeta \in \mathscr{P}(Y, \sigma)$. Since P^{-1} is continuous on components there exists a $\xi \in \mathscr{P}(X, \tau)$ such that P^{-1} is sequentially continuous from $(L_\xi, \downharpoonleft_\xi)$ to $(L_\zeta, \downharpoonleft_\zeta)$. Since $P^{-1} \in (H)$, ξ can be chosen such that $P^{-1} \in (H)_{\xi,\zeta}$, and then applying Proposition 4, we get $P^{-1} \in (M)_{\xi,\zeta}$, and by arbitrariness of $\zeta \in \mathscr{P}(Y, \sigma)$ we conclude that $P^{-1} \in (M)$. Conversely, from Proposition 4 it follows that $P^{-1} \in (M)$ implies $P^{-1} \in (H)$, and implies its sequential continuity as well, and this concludes the proof of Theorem 3.

LEMMA 7. *Given $\lambda, \zeta \in \mathscr{P}(Y, \sigma)$ such that $\lambda \leqq \zeta$ and that does not overrun ζ, to every $C \in \lambda$ there corresponds a D, $D \in \zeta$, such that*

$$L_\zeta \cap (L_C + Y) \subset L_D + Y .$$

PROOF. Take $D \in \zeta$ such that $L_\zeta \cap L_C \subset L_D$. If $u \in L_\zeta \cap (L_C + Y)$, then $u = c + y$, where $c \in L_C$ and $y \in Y$. Hence $c = u - y \in L_\zeta \cap L_C \subset L_D$, and consequently, $u \in L_D + Y$ which finishes the proof.

PROOF OF THEOREM 4. Let $P^{-1} \in (H)_{\xi,\lambda}$. Take $B \in \xi$ and assign to it a $C \in \lambda$ according to $(H)_{\xi,\lambda}$. By Lemma 7 we can find $D \in \zeta$ such that $L_\zeta \cap (L_C + Y) \subset L_D + Y$. To verify $(H)_{\xi,\zeta}$, take $\{y_n\} \subset Y$ tending to y in $(L_\zeta, \downharpoonleft_\zeta)$ with $\{Py_n\}$ convergent in $(L_B, \downharpoonleft_\xi) \wedge (X, \tau)$. Since $(L_\zeta, \downharpoonleft_\zeta) \geqq (L_\lambda, \downharpoonleft_\lambda)$, $\{y_n\}$ converges in $(L_\lambda, \downharpoonleft_\lambda)$ as well, and thus by $(H)_{\xi,\lambda}$ we have $y \in L_C + Y$. However, $y \in L_\zeta$ and thus $y \in L_\zeta \cap (L_C + Y) \subset L_D + Y$, and the Theorem follows.

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