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BOUNDARIES FOR NATURAL SYSTEMS II

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1. Introduction.

In this paper, we consider the following problem. Let (X,A) be a natural system with X a locally compact, Hausdorff space. Suppose B_0 is a unitary subalgebra of A such that

(i) B_0 is k-dense in A,
(ii) B_0 consists of bounded functions only, and
(iii) the natural injection of X into M_B is topological, where B is the uniform closure of B in $C_b(X)$.

What is the relation between the Shilov boundaries of A and B? Specifically, we want to compare $\partial_B \cap X$ with $\partial(X,A)$ and ∂_B with the limit in M_B of the family $\{\partial(K,A):\ K \text{ compact},\ K \subseteqq X\}$ ($\partial(X,A)$ is the limit of this family in X). This is the problem one encounters when a uniform F-algebra is generated by a finite family of elements, each with bounded spectrum. In this case, B_0 can be any algebra between the algebra of polynomials in the generating family and the (Banach) algebra of all elements which have bounded spectra.

Our main results are (with (X,A), B_0, and B as above):

(1) $\partial(X,A)\cap X^0 \subseteqq \partial_B \cap X \subseteqq \partial(X,A)$, where each containment can be proper,
(2) if X is dense in M_B, then $\partial_B \cap X = \partial(X,A)$, although it is often the case that $\partial(X,A)$ is not dense in ∂_B, and
(3) $\partial_B \subseteqq \operatorname{top\,lim\,inf}\{\partial(K,A):\ K \text{ compact},\ K\subseteqq X\}$, and in general, this is the most one can say.

2. Preliminaries.

We include here the statements of the relevant results from [2], [4], and [8].

Definition 2.1. A *system* is a pair (X,A) where X is a Hausdorff space and A is a subalgebra of $C(X)$ which satisfies (i) A contains the

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constant functions, and (ii) the weak topology on X determined by A is the given topology. If, in addition, (iii) every compact-open (k-) continuous homomorphism of A onto C is a point evaluation mapping determined by a point of X, we call the system *natural.*

DEFINITION 2.2. Let (X,A) be a natural system, and let K be a compact subset of X. The *A-convex hull* of K is the set

$$K^{\wedge} = \{x \in X \ : \ |a(x)| \leqq \|a\|_K,\ a \in A\}\,,$$

where $\|\cdot\|_K$ denotes the supremum-on-K seminorm on $C(X)$.

DEFINITION 2.3. Let (X,A) be a natural system, $x \in X$.

$$\mathscr{K}(x) = \{K \ : \ K \text{ compact}, K \subseteqq X, x \in K^{\wedge}\}\,.$$

Minimal elements of the set are called *supports for* x. The point x is called *independent* if $\{x\}$ is the only support for x.

LEMMA 2.4. *Let (X,A) be a natural system, $x \in X$. The following statements are equivalent.*

(1) *x is independent.*
(2) *If K is a compact subset of X and if $x \in K^{\wedge}$, then $x \in K$.*
(3) *If K is a compact subset of X and if $x \in K$, then $x \in \partial(K,A)$, the Shilov boundary of $A|K$ in K; equivalently, the Shilov boundary of the Banach algebra $(A|K)^-$.*

DEFINITION 2.5. The closure of the set of independent points (of a natural system) will be denoted $\partial_r(X,A)$ and called the Rickart boundary of (X,A).

DEFINITION 2.6. Let (X,A) be a natural system. The type II boundary $\partial_2(X,A)$ is top lim sup $\{\partial(K,A) : \ K \subseteqq X,\ K \text{ compact}\}$.

This is not the definition of $\partial_2(X,A)$ originally given in [2], but is shown in [4] to be equivalent. Definitions 2.1, 2.3, and 2.5 are given by Rickart (our name for the boundary) in [8]. The following results are proved in [4].

LEMMA 2.7. *Let (X,A) be a natural system with X locally compact. Let K be a compact subset of X. Then*

$$\partial_2(X,A) \cap K^0 \subseteqq \partial(K,A) \subseteqq (\partial_2(X,A) \cap K^0) \cup \operatorname{bdy} K\,.$$

LEMMA 2.8. *Let (X,A) be a natural system with X locally compact and let $x \in X$. The following statements are equivalent.*

(1) $x \in X \setminus \partial_2(X,A)$.

(2) *There exists a base for the topology at x consisting of relatively compact neighborhoods U such that $\partial(\overline{U},A) \subseteq \mathrm{bdy}\,\overline{U}$.*

THEOREM 2.9. *Let (X,A) be a natural system with X locally compact. Then $\partial_r(X,A) = \partial_2(X,A)$.*

We denote this common boundary $\partial(X,A)$.

3. Main results.

In this section, we shall always assume that the space underlying our natural system is locally compact. We shall use extensively the following easily verified fact. Let Y be a locally compact Hausdorff space, and let $S \subseteq Y$. Then S is locally compact if and only if S is open in $\overline{S}$. Finally, when we have $X \subseteq Y$, we shall write "$K \Subset X$" for "K is compact and $K \subseteq X$", always reserving the letter "K" for compact sets.

THEOREM 3.1. *Let (X,A) be a natural system with X locally compact and Hausdorff. Let B_0 be a subalgebra of A such that*

(a) $1 \in B_0$,

(b) *B_0 is k-dense in A ("k" = compact-open topology),*

(c) *$i\colon X \to M_B$ is topological, where B is the uniform closure of B_0 in $C_b(X)$, the bounded continuous functions on X to $\mathbf{C}$, and M_B is the spectrum of the uniform Banach algebra B, and i is the natural injection.*

Then (on identifying X and $i(X) \subseteq M_B$):

(1) $\partial(X,A) \cap X^0 \subseteq \partial_B \cap X \subseteq \partial(X,A)$,

(2) $\partial(X,A) \cap X^0 = \partial_B \cap X^0$,

(3) *if X is dense in M_B, then $\partial(X,A) = \partial_B \cap X$, and*

(4) *each containment in* (1) *can be proper.*

PROOF. We consider X to be a locally compact subset of M_B and will denote by $b^\wedge$ the extension of $b \in B$ to M_B.

(1a) Fix $x \in X^0$. Assume $x \notin \partial_B$. There exists an open subset U of X^0 such that $x \in U$, $\overline{U} = \overline{U}^X$ is compact and contained in X and such that $\overline{U} \cap \partial_B = \emptyset$. Let V be any X-neighborhood of x contained in U. (We show that $\partial(\overline{V}^X, A) \subseteq \mathrm{bdy}_X\,\overline{V}^X$.) We know that $\partial(\overline{V},B) \subseteq \mathrm{bdy}_{M_B}(\overline{V})$ (Rossi's local maximum principle, [9]). But $\mathrm{bdy}_{M_B}(V) = \partial_{B\overline{V}}$. By definition, $B_{\overline{V}}$ is the uniform closure of $B|\overline{V}$. This is just $\overline{A|\overline{V}}=A_{\overline{V}}$, since $B_0\subseteqq A$ is k-dense, and $\overline{V}$ is a compact subset of X. Hence, we have $\partial(\overline{V},A)\subseteqq \mathrm{bdy}_X\overline{V}$ for all V containing x such that $V\subseteqq U$. By Lemma 2.8, we have that $x\notin\partial(X,A)$.

(1b) Let $x\in X$ be a strong boundary point for B. Fix a compact set K_0 in X such that $x\in K_0$ and let $K_0\subset K\Subset X$. (We show that $x\in\partial(K,A)$.) Let W be an X-neighborhood of x. There exists an M_B-neighborhood W' of x such that $W'\cap\overline{X}=W$ (since W is open in X which is open in $\overline{X}$). Choose $b\in B$ such that

$$\|b^\wedge\|_{M_B} = |b^\wedge(x)| > \|b^\wedge\|_{M_B\setminus W'}\,.$$

We can assume that $|b^\wedge(x)|=1$ and $\|b^\wedge\|_{M_B\setminus W'}<\frac{1}{4}$. Choose $a\in A$ such that $\|a-b\|_K<\frac{1}{4}$. Then

$$\|a\|_{K\setminus W} < |a(x)| \leqq \|a\|_{W\cap K}\,.$$

Since $\{W\cap K:\ W$ is an X-neighborhood of $x\}$ is a neighborhood base at x in K, we have $x\in\partial(K,A)$. Hence, if $x\in X$ is a strong boundary point for B, then $x\in\partial(X,A)$.

We now fix $x\in X\cap\partial_B$, and fix an X-neighborhood U of x. Let U' be an M_B-neighborhood of x such that $U'\cap\overline{X}=U$. By Theorem 3.3.15 of [7], there is a strong boundary point φ for B in U'. But then $\varphi\in\partial_B$ and $\partial_B\subseteqq\overline{X}$ since B is a uniform algebra on X, so $\varphi\in\overline{X}\cap U'=U$ and $U\cap\partial(X,A)\neq\emptyset$. Since U was an arbitrary X-neighborhood of x, we have $x\in\partial(X,A)^{-X}=\partial(X,A)$.

(2) By (1), $\partial(X,A)\cap X^0\subseteqq\partial_B\cap X^0\subseteqq\partial(X,A)\cap X^0$.

(3) If X is dense in M_B, it is open and (2) yields the conclusion of (3).

(4) This will be shown in Examples 3.6 and 3.7 below for situations even more specialized than natural systems.

Let A be a uniform F-algebra with identity. Then the Gelfand transform is a topological isomorphism from A onto $(A^\wedge,k)$. We identify A and $A^\wedge$, and let A_0 be the set of all $a\in A$ such that $\|a\|_{M_A}<\infty$. It is easily verified that $(A_0,\|\cdot\|_{M_A})$ is a uniform Banach algebra.

COROLLARY 3.2. *Let A be a uniform F-algebra with identity and assume that the spectrum M_A of A is locally compact. Suppose B is a (uniformly) closed subalgebra with identity of $(A_0,\|\cdot\|_{M_A})$ such that*

(a) *B is k-dense in A, and*

(b) *$i^*\colon M_A\to M_B$ (induced by $i\colon B\subseteqq A$) is topological.*

Then (on identifying M_A and $i^*(M_A)$), we have the same conclusions as in Theorem 3.1.

DEFINITION 3.3. Let A be a uniform F-algebra with identity. The subset $S \subseteqq A$ generates A if the smallest k-closed subalgebra of A which contains S is A itself. If S generates A, then S induces a continuous injection $s\colon M_A \to \mathbf{C}^S$ by $s(\varphi) = \{\varphi(a)\}_{a \in S}$. We shall call S a *faithful generating family* for A if this map s is topological.

We note that there are uniform F-algebras for which (1) A is singly-generated, but cannot be faithfully generated by any countable set, and (2) A is singly-generated and there exist $a, b \in A$ such that $\{a\}$ is faithful while $\{b\}$ is not. For examples of these, see [3]. In the first example, M_A is not locally compact. We conjecture that if M_A is locally compact and A is singly-generated, then one can find a faithful generator. This is made at least plausible by the fact that if M_A is locally compact and A is separable, then M_A is metrizible. Since the requirement that a generating family for A be faithful is important in the next theorem, we give some equivalent statements of this fact.

LEMMA 3.4. *Let A be a uniform F-algebra with identity. Suppose that S generates A and S is at most countable. The following statements are equivalent.*

(1) *S is faithful.*

(2) *$\{a \circ s^{-1} : a \in A\} \subseteqq C(\Delta)$, where $\Delta = s(M_A)$, with the product topology from $\mathbf{C}^S$.*

(3) *The Gelfand topology is the weak topology on M_A determined by S.*

(4) *s maps discrete subsets of M_A to discrete subsets of Δ.*

PROOF. (1) and (3) are clearly equivalent. (1) $\Leftrightarrow$ (4) is easily verified since Δ is a k-space (it is metrizible) and s is topological if and only if it is proper (Lemma 1.3 of [3]). Finally, (1) $\Leftrightarrow$ (2) can be proved by a simple neighborhood chase.

THEOREM 3.5. *Let A be a uniform F-algebra with identity and assume that M_A is locally compact. Suppose there exists a faithful generating family S for A such that $S \subseteqq A_0$. Let B be the closed subalgebra of A_0 generated by S. Then we have the conclusions of Theorem* 3.1.

PROOF. By Corollary 3.2, it is sufficient to prove that $i^*\colon M_A \to M_B$ is topological. We have the following commutative diagram

$$\begin{tikzcd} M_A \arrow[r, "i^*"] \arrow[d, "\alpha"] & M_B \arrow[d, "\beta"'] \\ \mathbf{C}^S \arrow[r, "\mathrm{id}"] & \mathbf{C}^S , \end{tikzcd}$$

where $\alpha(\varphi)=\{\varphi(a)\colon a \in S\}$, $\beta(\psi)=\{\psi(a)\colon a \in S\}$ for $\varphi \in M_A$, $\psi \in M_B$. By assumption α is topological, and β is always topological (M_B is compact). Hence, i^* is topological and Theorem 3.5 follows.

EXAMPLE 3.6. We give an example showing that the first containment in Theorem 3.1 (1) can be proper. Let

$$X = \{\xi \in \mathbf{C} : |\xi|=1, \arg\xi \neq 0\},$$

and let A be the k-closure on X of the polynomials in z. Then (X,A) is a natural system, X is locally compact and if we let B denote the uniform closure on X of the polynomials in z, then M_B is the closed unit disc, ∂_B the unit circle, and X^0 is empty (relative to M_B). Thus,

$$\emptyset = \partial(X,A) \cap X^0 \subsetneqq \partial_B \cap X = \partial(X,A) = X .$$

EXAMPLE 3.7. We now show that the second containment can be proper. Let $X=\{\xi \in \mathbf{C}\colon \frac{1}{2} \leqq |\xi| \leqq 1, \arg\xi \neq 0\}$, and let A be the k-closure on X of the polynomials in z. Then (X,A) is a natural system, X is locally compact,

$$\partial(X,A) = \{\xi \in X : |\xi|=\tfrac{1}{2} \text{ or } |\xi|=1\} .$$

We let B be the uniform closure on X of the polynomials in z. Then B is again the (closed) disc algebra, M_B the unit disc, ∂_B the unit circle. Further, $\partial(X,A) \cap X^0=\{\xi \in X\colon |\xi|=1\}$, $\partial_B \cap X=\partial(X,A) \cap X^0$, but $\partial(X,A)$ is properly larger.

We note that in both examples, we were in the very special situation of Theorem 3.5; hence, our examples work for all of Theorem 3.1, Corollary 3.2 and Theorem 3.5.

We now consider the hypotheses of Theorem 3.1; namely that B_0 be k-dense in A and that $i\colon X \to M_B$ be topological. Without the first of these assumptions, the problem would be meaningless. For example, we could have $A=\mathrm{Hol}(\mathbf{C})$ in which case $A_0 \cong \mathbf{C}$, and there is no relation between M_A and M_{A_0} from which any information about A can be obtained.

EXAMPLE 3.8. Let $X=\{(s,t) \in \mathbf{R}^2\colon 0 \leqq s < 2\pi,\ 0 \leqq t \leqq 1\}$, let $g\colon X \to \mathbf{C}$ be defined by $g(s,t)=(-\frac{1}{2}t+1)e^{is}$. Then X is locally compact, σ-compact, and g is continuous. Let A be the k-closure of $P(g)$ in $C(X)$. Then (X,A) is a natural system (in fact, A is a uniform F-algebra and X is M_A): Let $\varphi\colon A \to \mathbf{C}$ be k-continuous. Then φ is completely determined by the value of φ at g, call it λ. If $\lambda \in g(X)$, then φ is evaluation at some $x \in X$. Suppose $\lambda \notin g(X)$. We shall show that the continuous function $g_\lambda\colon X \to \mathbf{C}$ defined by $g_\lambda(x) = [g(x)-\lambda]^{-1}$ is in A. Fix a compact subset K of X and $\varepsilon > 0$. We may assume K is of the form $\{(s,t) \in X\colon\ 0 \leqq s \leqq c < 2\pi\}$ for some c. Then

$$g(K) = \{\xi \in \mathbf{C}\ :\ \tfrac{1}{2} \leqq |\xi| \leqq 1,\ 0 \leqq \arg\xi \leqq c\}$$

is polynomially convex and compact. The function

$$f_\lambda = g_\lambda \circ g^{-1}\colon g(K) \to \mathbf{C}$$

is just $f_\lambda(\xi) = (\xi-\lambda)^{-1}$ which is holomorphic in a neighborhood of $g(K)$. By Runge's theorem (see, for example, p. 257 of [10]), there exists a polynomial p such that $\|f_\lambda - p\|_{g(K)} < \varepsilon$. Then for $x \in K$,

$$|g_\lambda(x) - p(g)(x)| = |f_\lambda(g(x)) - p(g(x))| < \varepsilon ,$$

and $g_\lambda \in A$. So $g-\lambda$ is invertible in A. But $\varphi(g-\lambda) = \varphi(g) - \lambda = 0$, a contradiction. Hence, $\varphi(g) \in g(X)$ for each $\varphi \in M_A$ and $M_A = X$.

We let B be the uniform closure on X, hence on $\bar{X}$, of polynomials in g. All such functions identify the lines $\{(0,t)\colon\ 0 \leqq t \leqq 1\}$ and $\{(2\pi,t)\colon\ 0 \leqq t \leqq 1\}$ in the obvious way so we can consider B on $R = \{\xi\colon\ \tfrac{1}{2} \leqq |\xi| \leqq 1\}$, and B is the closure there of polynomials in z. Thus, M_B is the closed disc, ∂_B the unit circle.

Finally, g injects X into M_B,

$$g(\partial(X,A)) = \{\xi \in \mathbf{C}\ :\ |\xi| = \tfrac{1}{2},\ |\xi| = 1, \text{ or } \arg\xi = 0\} ,$$

since $\partial(X,A)$ is just $\mathrm{bdy}_{\mathbf{C}} X \cap X$, and $g(\partial(X,A)) \cap g(X)^\circ$ contains

$$\{\xi\ :\ \tfrac{1}{2} < |\xi| \leqq 1,\ \arg\xi = 0\}$$

which is not contained in $\partial_B \cap g(X)$. Thus, the conclusions of 3.1 cannot in general be obtained without restricting the nature of the map of X to M_B.

We now consider the additional hypothesis which allows the sharper result $\partial_B \cap X = \partial(X,A)$: X is dense in M_B. We shall look at this hypothesis in the setting of Theorem 3.5, where S is a *finite* faithful generating family for A, $S \subseteqq A_0$, and B is the Banach algebra generated by S.

Since S is faithful, we consider $M_A \subseteqq M_B \subseteqq \mathbf{C}^n$ for some $n (= \operatorname{card} S)$. Let $S = \{a_1, \ldots, a_n\}$. The image of M_A in $\mathbf{C}^n$ under s is just the joint spectrum $\sigma_A(a_1, \ldots, a_n)$. Similarly, $\sigma_B(a_1, \ldots, a_n)$ is the image of M_B

under the "same" map. We know that $\sigma_A(a_1,\ldots,a_n)$ is polynomially convex. It is dense in $\sigma_B(a_1,\ldots,a_n)$ if and only if $\sigma_A(a_1,\ldots,a_n)^-$ is also polynomially convex. In case $n=1$, we can say a little more: $\sigma_A(a)^-$ is exactly $\sigma_{A_0}(a)$ (see [1]). Thus, M_A is dense in M_B exactly when the A_0-spectrum of the bounded generator for A is polynomially convex.

The question of the density of X in M_B for larger B is much more difficult. For example, if $A=\mathrm{Hol}(U)$ for U open in $\mathbf{C}$ or open and holomorphically convex in $\mathbf{C}^n$, then $A_0=H^\infty(U)$ and the density problem is just the corona problem.

We now turn to the second problem from the introduction: the relationship between ∂_B and $M_B-\lim\{\partial(K,A)\colon K \Subset X\}$.

THEOREM 3.10. *Let (X,A), B_0, and B be as in Theorem 3.1. If X is dense in M_B, then*

$$\partial_B \subseteqq \text{top lim inf}\{\partial(K,A)\colon K \Subset X\}\,,$$

where the limit is taken in M_B.

PROOF. We first show that if φ_0 is a strong boundary point for B, then φ_0 belongs to top lim inf$\{\partial(K,A)\}$. Fix an open set U containing φ_0, and choose $b \in B$ such that

$$1 = \|b^\wedge\|_{M_B} = |b^\wedge(\varphi_0)| > \tfrac{1}{4} > \|b^\wedge\|_{M_B\setminus U}\,.$$

Choose $K_0 \Subset X$ such that $K_0{}^0 \cap \{\varphi\colon |b^\wedge(\varphi)|>\tfrac{3}{4}\}\neq\emptyset$. (We can do this because X is locally compact and dense in M_B.) Fix $K \Subset X$ such that $K_0 \subseteqq K$ (and show $U\cap\partial(K,A)\neq\emptyset$). If $U\cap\partial(K,A)=\emptyset$, then $U\cap K$ is open in K and $\partial(\overline{U\cap K}^K,A)\subseteqq \mathrm{bdy}_K(\overline{U\cap K}^K)$ by the local maximum principle. But it is easily verified that $\mathrm{bdy}_K(\overline{U\cap K}^K)\subseteqq \mathrm{bdy}_{M_B} U \subseteqq M_B\setminus U$. We can choose $a\in A$ so that $\|a-b\|_K<\tfrac{1}{4}$. If $x \in \mathrm{bdy}_K(\overline{U\cap K}^K)$, then

$$|a(x)| \leqq |b^\wedge(x)|+\tfrac{1}{4} < \tfrac{1}{2}$$

($|b^\wedge|<\tfrac{1}{4}$ on $M_B\setminus U$). But there are points X in $K\cap U$ where $|a^\wedge(x)|>\tfrac{1}{2}$, a contradiction of the fact that we should have

$$\|a\|_{\overline{U\cap K}^K} \leqq \|a\|_{\mathrm{bdy}_K(\overline{U\cap K}^K)}\,.$$

Thus,

$$\varphi_0 \in \text{top lim inf}\{\partial(K,A)\}\,.$$

Since this set is closed, and ∂_B is the closure of the strong boundary points, the conclusion follows.

EXAMPLE 3.11. We show that in general one cannot obtain

$$\text{top}\lim\sup\{\partial(K,A)\} \subseteq \partial_B$$

(hence, it is not in general the case that $\partial_B = \text{top}\lim\{\partial(K,A)\}$). Consider $A = \text{Hol}(U)$, $B = H^\infty(U)$, where U is the open unit disc. Then $B = A_0$ is k-dense in A, and by Carleson's result (see [5]) $i: U \to M_B$ is topological and has dense range. It is shown on pages 174–175 of [6] that ∂_B is a proper subset of $M_B \setminus U$. We shall show that

$$\text{top}\lim\sup\{\partial(K,A)\} = M_B \setminus U .$$

Fix $\varphi \in M_B \setminus U$, $K_0 \Subset U$, and a neighborhood V of φ. (We must show that there exists $K \Subset X$ such that $K_0 \subseteq K$ and $V \cap \partial(K,A) \neq \emptyset$). Choose $\zeta \in V \cap U$ such that

$$|\zeta| > \sup\{|\xi| \ : \ \xi \in K_0\} .$$

Let $K = \{\xi \in U : |\xi| \leqq |\zeta|\}$. Then $K_0 \subseteq K \Subset U$, $\zeta \in \text{bdy}_U K \cap V = \partial(K,A) \cap V$.

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