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A REMARK ON INVARIANT PSEUDO-DIFFERENTIAL OPERATORS

ANDERS MELIN

In [4] Stetkær gave a simple characterization of all Lie groups G with the property that every bi-invariant (classical) pseudo-differential operator P on G is the sum of a differential operator and an operator with a smooth kernel [4, Theorem 3.5]. For some Lie groups this result was strengthened by proving the stronger property that the same conclusion could be drawn by just assuming the pseudo-local property of P [4, Corollary 4.6]. In this paper we shall prove that these two properties are in fact equivalent and therefore (as in [4, Theorem 3.5]) can be characterized by the non-compactness of the corresponding Lie algebra.

Now P, by the left invariance, can be written as a convolution with a distribution f on G. By the pseudo-local property f has to be smooth outside the identity element e of G and the bi-invariance of P forces f to be invariant (i.e. invariant under all inner automorphisms of G). Hence if f were the restriction of a function F in $C^\infty(G)$ to $G \setminus \{e\}$, then $f-F$ would be a finite sum of derivatives of the Dirac measure at e implying the decomposition of P as the sum of a differential operator and an operator with a smooth kernel. We are therefore led to introduce the following definition:

Definition 1. We shall say that the Lie group G (Lie algebra $\mathfrak{g}$) has the property (*) if every invariant function in $C^\infty(G \setminus \{e\})$ $(C^\infty(\mathfrak{g} \setminus \{0\}))$ is the restriction of an element in $C^\infty(G)$ $(C^\infty(\mathfrak{g}))$.

Invariance is of course defined with respect to the group of inner automorphisms of G respectively the adjoint group $\mathrm{Int}(\mathfrak{g})$ of $\mathfrak{g}$. If $\mathrm{Int}(\mathfrak{g})$ is compact we shall say that $\mathfrak{g}$ is a compact Lie algebra.

It will also be convenient for us to introduce a representation of all left invariant pseudo-differential operators on G by symbols defined on the dual of the Lie algebra of G. The bi-invariance of the operators will then be reflected by the invariance of the corresponding symbol. When doing this we shall use the characterization of a pseudo-differential operator

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as a Fourier integral operator with a special kind of phase function as introduced by Hörmander in [1, Section 2.3].

We can now state our Theorems.

THEOREM 2. a) *A connected Lie group G has the property (*) if and only if its Lie algebra is not compact.*

b) *A Lie algebra $\mathfrak{g}$ has the property (*) if and only if $\mathfrak{g}$ is not compact.*

THEOREM 3. *Let G be a connected Lie group.*

a) *If the Lie algebra of G is not compact then every bi-invariant pseudo-local operator on G is the sum of a differential operator and an operator with a smooth kernel.*

b) *If the Lie algebra of G is compact then one can find a bi-invariant (classical) pseudo-differential operator on G which does not have this decomposition.*

PROOF OF THEOREM 2b. We first remark that if $\mathrm{Int}(\mathfrak{g})$ is compact then we can find an Euclidean norm on $\mathfrak{g}$ which is invariant under this group. Since this norm has a singularity at the origin only, we see, that the non-compactness of $\mathfrak{g}$ is a necessary condition. On the other hand as was proved by Stetkær [4, p. 112] if $\mathfrak{g}$ is not compact it will contain a non-trivial nilpotent element w, that is an element w with $\mathrm{ad}w \neq 0$ but $(\mathrm{ad}w)^m = 0$ for some power m. Theorem 2b will therefore follow from the following result:

LEMMA 4. *Let $N \neq 0$ be a nilpotent linear transformation on R^n and $f \in C^\infty(\mathsf{R}^n \setminus \{0\})$ be invariant under the corresponding one-parameter group $\{\exp tN\}$ generated by N. Then f is the restriction of a function in $C^\infty(\mathsf{R}^n)$.*

PROOF. When $Nx \neq 0$ then $\exp tN\, x$ is a non-constant polynomial in t and therefore tends to infinity when t tends to infinity. If x belongs to the unit ball B then the orbit of x must intersect the boundary of B. Let P be a differential operator invariant under $\exp tN$. Then by a continuity argument $P^m f$ being constant along the orbits must be bounded in the punctured unit ball. If M is the set of elements in $C^\infty(\mathsf{R}^n \setminus \{0\})$ which represent functions in $L^2{}_{\mathrm{loc}}(\mathsf{R}^n)$, the space of all locally square integrable functions, then we let Ext denote the extension map from M to the set of distributions on R^n.

Since $\mathrm{Ext} f$ is smooth outside the origin, Lemma 4 will be established as soon as we can construct a (constant coefficient) differential operator Q on R^n which is stronger than all first order differential operators and such that $Q^m \mathrm{Ext} f$ belongs to $L^2_{\mathrm{loc}}(\mathsf{R}^n)$ for all non-negative integers m. (See [3, Section 3.2].)

If $P(\xi)$ is a complete polynomial, that is if

$$\Lambda(P) = \{\eta \in \mathsf{R}^n : P(\xi+t\eta)=P(\xi) \text{ for all } \xi \in \mathsf{R}^n,\ t \in \mathsf{R}\} = \{0\}$$

then as is proved in [2, Theorem 2.17] some positive power of $P(D)$ (where D with $D_j = -i\partial/\partial x_j$ is the gradient vector) will be stronger than all first order differential operators. Therefore Lemma 4 will follow from the next two lemmas. Notice that when $n=2$ or more generally when $\operatorname{rank} N=1$ then f is independent of one of the coordinates in a suitable basis and is therefore smooth, so we do not have to bother about these cases.

LEMMA 5. *Let f satisfy the hypothesis of Lemma 4 and let $P(\xi)$ of degree $<n$ be a polynomial invariant under $\{\exp tN^*\}$ where N^* denotes the adjoint of N. Then $(P(D))^m \mathrm{Ext} f \in L^2_{\mathrm{loc}}(\mathsf{R}^n)$ for all non-negative integers m. This is also true if P is replaced by a product of polynomials satisfying the same hypotheses.*

LEMMA 6. *Let N be a nilpotent linear transformation on R^n with rank $N \geqq 2$ (so that $n \geqq 3$). Then there exists a polynomial P on R^n with the following properties:*

(i) *P is a product of homogeneous polynomials of degree less than n.*

(ii) *Each factor of P is invariant under $\{\exp tN^*\}$.*

(iii) *P is complete.*

PROOF OF LEMMA 6. We shall first prove Lemma 6 when $\operatorname{rank} N = n-1$. Choose a vector $v \in \mathsf{R}^n$ such that $N^{n-1}v \neq 0$ and set $\xi_k = \langle N^{k-1}v, \xi\rangle$ when $\xi \in \mathsf{R}^n$ and $1 \leqq k \leqq n$. Then there is a unique polynomial P on R^n satisfying the conditions:

$$P \text{ is homogeneous of degree } n-1 . \tag{1}$$

$$P(\xi) = \langle v, \exp(-\xi_{n-1}N^*)\, \xi\rangle \quad \text{when } \xi_n = 1 . \tag{2}$$

Notice that the degree of the right-hand side of (2) is at most $n-1$ when $\xi_n=1$ and that the hyperplanes $\xi_n = \text{constant}$ are invariant under $\{\exp tN^*\}$. Hence when $\xi_n=1$

$$\begin{aligned} P(\exp tN^*\xi) &= \langle v, \exp(-\xi_{n-1}N^* - t\xi_n N^*) \exp tN^*\xi\rangle \\ &= \langle v, \exp(-\xi_{n-1}N^*)\xi\rangle , \end{aligned}$$

so $P_t(\xi) = P(\exp tN^*\xi)$ satisfies (1) and (2) and therefore equals $P(\xi)$.

It remains to check the completeness of P. We may choose a vector ξ in R^n with $\xi_n=1$ and $\xi_i=0$ for $i<n$. Since $\Lambda(P)$ being a linear subspace is invariant under $\{\exp tN^*\}$ and therefore under N^* the coefficient of t in the polynomial $P(\xi+tN^*\eta)$ must vanish if $\eta \in \Lambda(P)$. But this coefficient is just $\langle Nv,\eta\rangle$, for by (2)

$$P(\xi+tN^*\eta) = \langle v, \exp(-t\eta_n N^*)(\xi+tN^*\eta)\rangle$$

and $\langle v,N^*\xi\rangle=\xi_2=0$ since $n>2$. Replacing η by $(N^*)^i\eta$ $(i\geqq 0)$ we conclude that $\eta_k=0$ for $k\geqq 2$, and then repeating the same arguments with η instead of $N^*\eta$ we get $\eta=0$.

We now turn to the general case. Since N^* is nilpotent we can write $\mathsf{R}^n=\sum\oplus V_j$ where each V_j reduces N^* and the rank of the restriction of N^* to V_j equals $\dim V_j-1$. We may assume that $n_j=\dim V_j$ is greater than 1 for all j. Since there are no complete, invariant polynomials on the two-dimensional subspaces in this decomposition we can not build up P from polynomials defined on these subspaces directly. To each V_j we associate a vector v_j such that $N^{n_j-1}v_j\neq 0$ and such that v_j is orthogonal to V_k for $j\neq k$. If $\xi\in\mathsf{R}^n$ we set

$$\xi_j^- = \langle N^{n_j-2}v_j,\xi\rangle, \quad \xi_j^+ = \langle N^{n_j-1}v_j,\xi\rangle .$$

When $\dim V_j>2$ we construct a polynomial P_j as in the first part of the proof and set $P_j=1$ when $\dim V_j=2$. Then

$$P(\xi) = \textstyle\prod_j P_j(\xi)\prod_{r<s}(\xi_r^+\xi_s^- - \xi_s^+\xi_r^-) \tag{3}$$

satisfies (i)–(iii).

The invariance of P follows almost immediately from its definition if we note that

$$(\exp tN^*\xi)_j^- = \xi_j^- + t\xi_j^+, \quad (\exp tN^*\xi)_j^+ = \xi_j^+ .$$

Assume that η and $N^*\eta$ belong to $\Lambda(q_{rs})$ where $q_{rs}(\xi)=\xi_r^+\xi_s^- - \xi_s^+\xi_r^-$, $r<s$. Then the coefficient of t in $q_{rs}(\xi+tN^*\eta)$ equals $\xi_r^+\eta_s^+ - \xi_s^+\eta_r^+$ so $\eta_r^+=\eta_s^+=0$. Then inserting η instead of $N^*\eta$ we also conclude that $\eta_r^-=\eta_s^-=0$. Since

$$\Lambda(P) = \textstyle\bigcap_{j,r,s}(\Lambda(P_j)\cap\Lambda(q_{rs}))$$

we easily see that $\Lambda(P)=\{0\}$. This completes the proof of Lemma 6.

Proof of Lemma 5. We shall prove that

$$P^m \operatorname{Ext} f = \operatorname{Ext} P^m f \quad (P=P(D)) \tag{4}$$

for all $m\geqq 0$. (Notice that P has different meanings in the two sides of the equation.) Assuming that (4) holds for m we set $g=P^m f$. Then if $P\,\mathrm{Ext}g=\mathrm{Ext}Pg$ we get

$$P^{m+1}\,\mathrm{Ext}f = P\,\mathrm{Ext}g = \mathrm{Ext}Pg = \mathrm{Ext}P^{m+1}f\,.$$

Since g satisfies the same hypothesis as f it will therefore be sufficient to prove (4) with $m=1$. We have to show that if φ belongs to $C_0^\infty(\mathsf{R}^n)$ (i.e. belongs to $C^\infty(\mathsf{R}^n)$ and has compact support) then

$$\int f(x)P(-D)\varphi(x)\,dx = \int \varphi(x)P(D)f(x)\,dx\,. \tag{5}$$

After a change of variables the left-hand side of (5) can be written as a sum

$$\int f^\varepsilon(y)\varepsilon^n P(-D/\varepsilon)\big((1-\psi)\varphi^\varepsilon(y)\big)\,dy + \int f^\varepsilon(y)\varepsilon^n P(-D/\varepsilon)\big(\psi\varphi^\varepsilon(y)\big)\,dy \tag{6}$$

where $f^\varepsilon(y)=f(\varepsilon y)$ and $\psi\in C_0^\infty(\mathsf{R}^n)$ equals 1 in a neighbourhood of the origin. In the first term we can integrate by parts and get after another change of variables

$$\int \big(1-\psi(y/\varepsilon)\big)\varphi(y)P(D)f\,dy$$

which tends to the right-hand side of (5) when ε tends to 0 since $P(D)f$ is bounded. It remains to prove that the second term of (6) tends to zero. But this is obvious for the integrand is bounded by a constant times ε since $\deg P<n$ and it has its support in a fixed compact set independent of ε. The last statement of Lemma 5 follows by a slight modification of the proof.

The proof of Theorem 2b is now complete.

For the proof of Theorem 3 it will be convenient to have a simple description of the left-invariant pseudo-differential operators on G.

Definition 7. (Cf. [1, Def. 1.1.1 p. 83]) Let V be a real finite dimensional vector space, D be a gradient vector and $\|\cdot\|$ denote some norm on V. A symbol on V of order m and type ϱ is a function $a=a(\theta)\in C^\infty(V)$ such that

$$\sup_\theta\big(\|\theta\|^{\varrho\deg(Q)-m}\,|Q(D)a(\theta)|\big) < \infty \tag{7}$$

for all homogeneous polynomials Q. If (7) holds for all $m\in\mathsf{R}$ we shall say that a has order $-\infty$. Two symbols are called equivalent if they differ by a symbol of order $-\infty$.

We shall also say that two pseudo-differential operators are equivalent if their difference has a smooth kernel.

Theorem 8. *Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ and exponential map* exp. *Then P is a left invariant pseudo-differential operator* *on G of order m and type $\varrho, 1-\varrho$, $(\varrho > \frac{1}{2})$ if and only if there is a symbol $a(\theta)$ on $\mathfrak{g}^*$ of order m and type ϱ so that the difference $P-T$ where*

$$(8) \qquad Tu(h) = \iint_{\mathfrak{g}\times\mathfrak{g}^*} e^{-iY\cdot\theta}\, a(\theta)\psi(Y)u(h \exp Y)\, dY\, d\theta, \qquad u \in C_0^\infty(G)$$

has a smooth left invariant kernel whenever $\psi \in C_0^\infty(\mathfrak{g})$ equals 1 in a neighbourhood of the origin. The correspondence between P and a is one-to-one after reduction to equivalence classes and then differential operators correspond to polynomials. If both ψ and a are invariant under the action of $\mathrm{Int}(\mathfrak{g})$ *then T is a bi-invariant pseudo-differential operator.*

Proof. That a left invariant pseudo-differential operator can be represented by (8) follows from the pseudo-local property of P and the fact that P via composition with the map exp (which is locally a diffeomorphism near $0 \in \mathfrak{g}$) can be regarded as a pseudo-differential operator in a neighbourhood of $0 \in \mathfrak{g}$. The choice of ψ is of course inessential.

On the other hand the singular integral operator T in (8) is left invariant. It is therefore sufficient to prove that T is a pseudo-differential operator in a neighbourhood of the identity element e of G. We may therefore again regard T as an operator on $\mathfrak{g}$.

Introducing $v(X) = u \circ \exp(X)$, where $h = \exp X$ when the support of u is close to e, we have to prove that

$$(9) \qquad Lv(X) = \iint_{\mathfrak{g}\times\mathfrak{g}^*} e^{-iY\cdot\theta}\, a(\theta)u(\exp X \exp Y)\, dY\, d\theta$$

is a pseudo-differential operator in some neighbourhood of $0 \in \mathfrak{g}$. After a change of variables, $Y = \Phi(X,Z)$, defined by the equation

$$\exp X \exp Y = \exp Z\,,$$

L has the form of a Fourier integral operator with phase function $\varphi(X,Z,\theta) = \Phi(X,Z)\cdot\theta$. The restriction of φ to the θ-variable is critical at (X,Z,θ) if and only if $\exp X = \exp Z$ and therefore if and only if $X = Z$ when X and Z are close to the origin. Therefore L is a pseudo-differential operator by [1, Section 2.3].

The remaining statements follow from standard facts about pseudo-differential operators together with some computations involving changes of variables in (8). Notice that $a(\theta)$ is the restriction to $0 \times \mathfrak{g}^*$ of the symbol of P when P is regarded as a pseudo-differential operator on $\mathfrak{g}$.

With these preparations it is now easy to prove the remaining assertions, Theorem 2a and Theorem 3. We let $\mathfrak{g}$ denote the Lie algebra of G as before.

PROOF OF THEOREM 3b. Set $a(\theta)=(1+\|\theta\|^2)^{-1}$ where $\|\cdot\|$ denotes an Euclidean norm on $\mathfrak{g}^*$ invariant under the compact group $(\mathrm{Int}(\mathfrak{g}))^*$ and choose $\psi \in C_0^\infty(\mathfrak{g})$ invariant under $\mathrm{Int}(\mathfrak{g})$. The corresponding pseudo-differential operator T defined by (8) is then bi-invariant and not equivalent to a differential operator by Theorem 8.

PROOF OF THEOREM 2a AND THEOREM 3a. The sufficiency part of Theorem 2a follows from Theorem 2b since the composition of an invariant function on G with the exponential map gives an invariant function on $\mathfrak{g}$. Note that in the proof of Theorem 2b we only need the fact that f is smooth in some punctured neighbourhood of 0 to show that the singularity at 0 is removable. This part of Theorem 2a also implies Theorem 3a by the remarks in the introduction. The necessity part of Theorem 2a follows from Theorem 3b and the same remarks. In fact, if $\mathrm{Int}(\mathfrak{g})$ is compact the operator T defined in the proof of Theorem 3b gives an invariant function f in $C^\infty(G \setminus \{e\})$ which is not the restriction of an element in $C^\infty(G)$.

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UNIVERSITY OF LUND, SWEDEN