MATH. SCAND. 30 (1972), 324–330

CERTAIN q-TRANSFORMS

SNEH D. PRASAD

1. Introduction.

In this note we have obtained certain inversion theorems for some series transforms, which can be expressed as basic integrals. Use has been made of the q-Laplace transform, its inversion and convolution theorems.

2. Notations.

Let

$$[a]_r = [q^a]_r = (1-q^a)(1-q^{a+1})\ldots(1-q^{a+r-1}),\ [a]_0=1,\ |q|<1\ .$$

$$[a]_{-r} = \frac{(-1)^r q^{\frac{1}{2}r(r+1)}}{q^{ar}[1-a]_r}.$$

$$(a+b)_\alpha = a^\alpha \prod_{j=0}^{\infty} \frac{(1+ba^{-1}q^j)}{(1+ba^{-1}q^{j+\alpha})} \text{ for any arbitrary index } \alpha\ .$$

The basic hypergeometric ${}_1\Phi_1$-series is defined as

$${}_1\Phi_1[a;b;x] = \sum_{r=0}^{\infty} \frac{[a]_r x^r}{[b]_r[1]_r}, \quad |x|<1\ .$$

The basic integral is defined as

$$\int_0^t f(x)\,d(x;q) = t(1-q)\sum_{j=0}^{\infty} q^j f(tq^j)\ .$$

The rth basic differential of $f(x)$ is given by

$$f^{(r)}(x) = (q-1)^{-r}x^{-r}q^{-\frac{1}{2}r(r-1)}\sum_{j=0}^{r} \frac{[r-j+1]_j(-1)^j q^{\frac{1}{2}j(j-1)}}{[1]_j} f(q^{r-j}x)\ .$$

Received January 12, 1971.

The functions $e_q(x)$ and $E_q(x)$ are defined as follows:

$$e_q(x) = \frac{1}{(1-x)_\infty} = \sum_{r=0}^{\infty} \frac{x^r}{[1]_r}, \quad |x|<1 .$$

$$E_q(x) = (1-x)_\infty = \sum_{r=0}^{\infty} \frac{(-1)^r q^{\frac{1}{2}r(r-1)} x^r}{[1]_r}.$$

As a q-analogue of the Gamma function we define

$$\Gamma_q(\alpha) = \frac{e_q(q^\alpha)}{e_q(q)\,(1-q)^{\alpha-1}}, \quad \alpha \neq 0,-1,-2,-3,\ldots .$$

The basic analogues of $\sin x$ and $\cos x$ are given as follows:

$$\sin_q(x) = \sum_{r=0}^{\infty} \frac{(-1)^r x^{2r+1}}{[1]_{2r+1}}.$$

$$\cos_q(x) = \sum_{r=0}^{\infty} \frac{(-1)^r x^{2r}}{[1]_{2r}}.$$

${}_ql_s f(t)$, the q-Laplace transform of $f(t)$, as defined by Hahn [1] is given by

$${}_ql_s f(t) = \frac{1}{1-q} \int_0^{s^{-1}} E_q(qsx) f(x)\, d(x;q) = \frac{[1]_\infty}{s} \sum_{i=0}^{\infty} \frac{q^i f(s^{-1}q^i)}{[1]_i}.$$

The convolution theorem for q-Laplace transforms of the functions $F(t)$ and $G(t)$ as stated by Hahn [1] can be expressed as

$${}_ql_s F(t)\, {}_ql_s G(t) = {}_ql_s \left\{ \frac{t}{1-q} \int_0^1 F(tx)\, G[t-txq]\, d(x;q) \right\},$$

where

$$G[t-txq] = \sum_{r=0}^{\infty} A_r (t-txq)_r \quad \text{if} \quad G(t) = \sum_{r=0}^{\infty} A_r t^r .$$

3. Results.

THEOREM 1. *If*

$$g(t) = \frac{At}{1-q} \int_0^1 f^{(m)}[t-txq]\, (xt)^{\alpha-1}\, {}_1\Phi_1[a;\alpha;cxt]\, d(x;q)\,, \tag{3.1}$$

then

$$(3.2)\quad f(t) = \frac{Bt}{1-q}\int_0^1 g^{(n)}[t-txq]\,(xt)^{n+m-\alpha-1}\,{}_1\Phi_1[-a;n+m-\alpha;xctq^{a}]\,d(x;q)$$

provided that

(i) $|tc|<1$ *and* $|tcq^{a}|<1$,

(ii) n *and* m *are positive integers such that* $n+m>R(\alpha)>0$,

(iii) $f(0)=0=f'(0)=\ldots=f^{(m-1)}(0)$,
$g(0)=0=g'(0)=\ldots=g^{(n-1)}(0)$,

(iv) $AB=\dfrac{(1-q)^{m+n}}{(1-q)_{\alpha-1}(1-q)_{n+m-\alpha-1}}$,

where

$$f[t-txq] = \textstyle\sum_{r=0}^{\infty} A_r(t-txq)_r \quad \text{if} \quad f(t) = \sum_{r=0}^{\infty} A_r t^r ,$$

and

$$g[t-txq] = \textstyle\sum_{r=0}^{\infty} B_r(t-txq)_r \quad \text{if} \quad g(t) = \sum_{r=0}^{\infty} B_r t^r .$$

PROOF. Let ${}_q l_s f(t)=F(s)$ and ${}_q l_s g(t)=G(s)$. Now under the stated conditions, we have

$${}_q l_s f^{(m)}(t) = \frac{s^m F(s)}{(1-q)^m}.$$

This can be obtained by a repeated application of a known result due to Hahn [1, 9.7].

Further, the convolution theorem when applied to functions $f^{(m)}(t)$ and $t^{\alpha-1}{}_1\Phi_1[a;\alpha;ct]$, gives

$$\frac{(1-q)_{\alpha-1}(1-q^{a}cs^{-1})_\infty s^{m-\alpha}F(s)}{(1-q)^m(1-cs^{-1})_\infty} = {}_q l_s\left\{\frac{t}{1-q}\int_0^1 f^{(m)}[t-txq]\,(xt)^{\alpha-1}{}_1\Phi_1[a;\alpha;cxt]\,d(x;q)\right\}.$$

Taking q-Laplace transform on both sides of (3.1), we get

$$G(s) = \frac{A(1-q)_{\alpha-1}(1-q^{a}cs^{-1})_\infty s^{m-\alpha}}{(1-q)^m(1-c\,s^{-1})_\infty}\,F(s).$$

This gives

$$F(s) = \frac{(1-q)^{m+n}}{A(1-q)_{\alpha-1}(1-q)_{n+m-\alpha-1}}\,\frac{(1-cs^{-1})_\infty(1-q)_{n+m-\alpha-1}}{(1-q^{a}cs^{-1})_\infty s^{n+m-\alpha}}\,\frac{s^n G(s)}{(1-q)^n},$$

that is,

$$
\begin{aligned}
&{}_q l_s f(t)\\
&= \frac{(1-q)^{m+n}}{A(1-q)_{\alpha-1}(1-q)_{n+m-\alpha-1}}\, {}_q l_s\{t^{n+m-\alpha-1}\,{}_1\Phi_1[-a;\, n+m-\alpha;\, xcq^a]\}\, {}_q l_s\{g^{(n)}(t)\}\\
&= \frac{(1-q)^{m+n}}{A(1-q)_{\alpha-1}(1-q)_{n+m-\alpha-1}}\cdot\\
&\qquad \cdot {}_q l_s \left\{\frac{t}{1-q}\int_0^1 g^{(n)}[t-txq]\,(xt)^{n+m-\alpha-1}\,{}_1\Phi_1[-a;\, n+m-\alpha;\, xtcq^a]\, d(x;q)\right\},
\end{aligned}
$$

which in turn implies

$$
f(t) = \frac{Bt}{1-q}\int_0^1 g^{(n)}[t-txq](xt)^{n+m-\alpha-1}\,{}_1\Phi_1[-a;\, n+m-\alpha;\, xtcq^a]\, d(x;q)\,.
$$

To verify theorem 1, let us take $m=0$, $n=1$ and $f(t)=t^{\beta-1}{}_1\Phi_1[b;\beta;dx]$ with $q^b d=c$, $|dx|<1$ and $R(\alpha+\beta-2)>0$. We have then

$$
f[t-txq] = \sum_{r=0}^{\infty} \frac{[b]_r(1-qx)_{r+\beta-1}}{[\beta]_r[1]_r}\, d^r t^{r+\beta-1},
$$

and hence (3.1) gives

$$
g(t) = \frac{A(1-q)_{\alpha-1}\,[\alpha+\beta]_\infty t^{\alpha+\beta-1}}{(1-q^\beta)_\alpha}\,{}_1\Phi_1[a+b;\, \alpha+\beta;\, dt]\,.
$$

This gives

$$
g'(t) = \frac{A(1-q)_{\alpha-1}(1-q)_{\alpha+\beta-1}[\alpha+\beta]_\infty t^{\alpha+\beta-2}}{(1-q)(1-q)_{\alpha+\beta}(1-q^\beta)_\alpha}\,{}_1\Phi_1[a+b;\, \alpha+\beta-1;\, dt]\,.
$$

If we put this value in the right hand side of (3.2) we get $f(t)$.

Special cases of theorem 1.

(i) If $a=\alpha$, then under appropriate conditions

$$
g(t) = \frac{At}{1-q}\int_0^1 f^{(m)}[t-txq]\,(xt)^{\alpha-1}\, e_q(cxt)\, d(x;q)
$$

implies

$$f(t) = \frac{Bt}{1-q}\int_0^1 g^{(n)}[t-txq]\,(xt)^{n+m-\alpha-1}{}_1\Phi_1[-\alpha;\, n+m-\alpha;\, xtcq^{\alpha}]\, d(x;q)\,.$$

(ii) If $c=0$, $m=0$ and $A=(\Gamma_q(\alpha))^{-1}$ then, under appropriate conditions

$$g(t) = \frac{t}{(1-q)\Gamma_q(\alpha)}\int_0^1 f[t-txq]\,(xt)^{\alpha-1}\, d(x;q)$$

implies

$$f(t) = \frac{t}{(1-q)\Gamma_q(n-\alpha)}\int_0^1 g^{(n)}[t-txq]\,(xt)^{n-\alpha-1}\, d(x;q)\,,$$

which is the basic analogue of the following well-known result that, if,

$$g(t) = \frac{1}{\Gamma(\alpha)}\int_0^t f(t-x)\,x^{\alpha-1}\, dx\,,$$

then

$$f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t g^{(n)}(t-x)\,x^{n-\alpha-1}\, dx\,.$$

(iii) If $c=-\frac{1}{4}d$ and $a\to\infty$, we see that

$$g(t) = \frac{A(1-q)_{\alpha-1}2^{\alpha-1}t}{(1-q)d^{\frac{1}{2}(\alpha-1)}}\int_0^1 f^{(m)}[t-txq](xt)^{\frac{1}{2}(\alpha-1)}{}_q j_{\alpha-1}(xtd)^{\frac{1}{2}}\, d(x;q)$$

implies

$$f(t) = \frac{Bt}{1-q}\int_0^1 g^{(n)}[t-txq]\,(xt)^{n+m-\alpha-1}\sum_{r=0}^{\infty}\frac{q^{\frac{1}{2}r(r-1)}(-1)^r x^r t^r d^r}{[n+m-\alpha]_r[1]_r 4^r}\, d(x;q)\,,$$

where ${}_q j_{\alpha-1}(t)$ is the basic analogue of the Bessel function and is given by

$${}_q j_{\alpha-1}(t) = \frac{1}{(1-q)_{\alpha-1}}\,(\tfrac{1}{2}x)^{\alpha-1}\sum_{r=0}^{\infty}\frac{(-1)^r}{[\alpha]_r[1]_r}\,(\tfrac{1}{2}x)^{2r}\,.$$

(iv) If $a=\frac{1}{2}$, $\alpha=\frac{3}{2}$ and $c=-1$, then

$$g(t) = \frac{At\pi^{\frac{1}{2}}}{2(1-q)}\int_0^1 f^{(m)}[t-txq]\,\mathrm{Erf}_q(tx)^{\frac{1}{2}}\, d(x;q)$$

implies

$$f(t) = \frac{Bt}{1-q} \int_0^1 g^n[t-txq]\,(xt)^{n+m-\frac{5}{2}}\,{}_1\Phi_1[-\tfrac{1}{2};\, n+m-\tfrac{3}{2};\, -xtq^{\frac{1}{2}}]\, d(x;q)\ ,$$

where $\mathrm{Erf}_q(x)$ is the basic analogue of the error function and is defined as

$$\mathrm{Erf}_q(x) = 2x\pi^{-\frac{1}{2}}\,{}_1\Phi_1[\tfrac{1}{2};\, \tfrac{3}{2};\, -x^2], \qquad |x|<1\ .$$

THEOREM 2. *If*

$$g(t) = \frac{At}{1-q} \int_0^1 f^{(m)}[t-txq]\, \sin_q(axt)\, d(x;q)\ ,$$

then

$$f(t) = \frac{Bt}{1-q} \int_0^1 g^{(n)}[t-txq] \left[\frac{a^2(xt)^{m+n-1}}{[1]_{m+n-1}} + \frac{(xt)^{m+n-3}}{[1]_{m+n-3}}\right] d(x;q)$$

provided

(i) *m and n are positive integers,*
(ii) $AB = (1-q)^{m+n}a^{-1}$,
(iii) $f(0) = 0 = f'(0) = \ldots = f^{(m-1)}(0)$ *and*
$g(0) = 0 = g'(0) = \ldots = g^{(n-1)}(0)\ .$

THEOREM 3. *If*

$$g(t) = \frac{At}{1-q} \int_0^1 f^{(m)}[t-txq]\, \cos_q(axt)\, d(x;q)\ ,$$

then

$$f(t) = \frac{Bt}{1-q} \int_0^1 g^{(n)}[t-txq] \left[\frac{(tx)^{m+n-2}}{[1]_{m+n-2}} + \frac{a^2\,(tx)^{m+n}}{[1]_{m+n}}\right] d(x;q)$$

provided

(i) *n and m are positive integers,*
(ii) $AB = (1-q)^{m+n}$,
(iii) $f(0) = 0 = f'(0) = \ldots = f^{(m-1)}(0)$ *and*
$g(0) = 0 = g'(0) = \ldots = g^{(n-1)}(0)\ .$

Proofs of theorems 2 and 3 are similar to that of theorem 1.

I am grateful to Professor R. P. Agarwal for his kind guidance throughout the preparation of this note.

REFERENCE

1. W. Hahn, *Beiträge zur Theorie der Heineschen Reihen. Die 24 Integrale der hypergeometrischen q-Differenzengleichung. Das q-Analogon der Laplace Transformation,* Math. Nachr. 2 (1949), 340–79.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF LUCKNOW, LUCKNOW, INDIA