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GALOISIAN APPROACH TO COMPLEX OSCILLATION THEORY OF SOME HILL EQUATIONS

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Abstract

We apply Kovacic's algorithm from differential Galois theory to show that all complex non-oscillatory solutions (finite exponent of convergence of zeros) of certain Hill equations considered by Bank and Laine using Nevanlinna theory must be Liouvillian solutions. These are solutions obtainable by suitable differential field extension constructions. In particular, we establish a full correspondence between solutions of non-oscillatory type and Liouvillian solutions for a particular Hill equation. Explicit closed-form solutions are obtained via both methods for this Hill equation whose potential is a combination of four exponential functions in the Bank-Laine theory. The differential equation is a periodic form of a biconfluent Heun equation. We further show that these Liouvillian solutions exhibit novel single and double orthogonality, and satisfy Fredholm integral equations over suitable integration regions in $\mathbb{C}$ that mimic single/double orthogonality for the corresponding Liouvillian solutions of the Lamé and Whittaker-Hill equations, discovered by Whittaker and Ince almost a century ago.

1. Introduction

Differential Galois theory ([1], [2], [34], [35], [33]) has been demonstrated to be a powerful tool to study, amongst various research areas, monodromy of linear differential equations. In particular, it can be used to identify *Liouvillian* solutions of the differential equations. These solutions correspond to the associated differential Galois group of the differential equations being solvable and are closely related to finding closed-form solutions of the differential equations. A well-known theorem of Kimura [25], [34] states that Gauss hypergeometric equation either admits algebraic solutions or else *Jacobi polynomials* are amongst the Liouvillian solutions of the equation. The classification given by Kimura contains the celebrated *Schwarz list* for when the monodromy group of the hypergeometric equation is *finite*. The study of Liouvillian solutions has been extended to the classical Lamé equation

$$w''(z) + \big[h - n(n+1)k^2\mathrm{sn}^2 z\big]w(z) = 0, \tag{1.1}$$

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where the sn z is a *Jacobi elliptic function*. The classical *Lamé polynomials* are amongst the Liouvillian solutions. One can find a detailed account of Lamé theory in [34, Chap. 2]. Let $n(r, f)$ be the number of zeros of an entire function f in $D(0, r) := \{z : |z| < r\}$. In this paper, we study the interplay between the Liouvillian solutions to the Hill equation

$$f''(z) + \big(-e^{4z} + K_3e^{3z} + K_2e^{2z} + K_1e^z + K_0\big)f(z) = 0, \tag{1.2}$$

where $K_j \in \mathbb{C}$, $j = 0, 1, 2, 3$, and its *non-oscillatory solutions*, i.e., those solutions with $\lambda(f) = \limsup_{r\to\infty} \log n(r, f)/\log r < \infty$, or *finite exponent of convergence of zeros* in Bank-Laine's *complex oscillation theory* that studies solutions of linear second order differential equations with transcendental entire coefficients under the framework of Nevanlinna's value distribution theory [20], [28], [43].

Bank & Laine [6], [7] showed that if the Hill equation

$$f''(z) + A(z)f(z) = 0, \tag{1.3}$$

where $A(z) = B(e^z)$,

$$B(\zeta) = K_s/\zeta^s + \cdots + K_0 + \cdots + K_\ell\zeta^\ell,$$

and K_j $(j = s, \ldots, \ell)$ are complex constants, admits a non-trivial entire solution f with $\lambda(f) < +\infty$ (i.e., zero as a *Borel exceptional* value), then

$$f(z) = \psi(e^{z/h}) \exp\big(d\,z + P(e^{z/h})\big), \tag{1.4}$$

where $h = 2$ if ℓ is odd and $h = 1$ otherwise, $\psi(\zeta)$ is a polynomial, $P(\zeta)$ is a Laurent polynomial and d is a constant. Obviously, both the polynomial ψ and P depend on the coefficient $B(\zeta)$. See also [39] for higher-order differential equations.

Let $x = e^{z/h}$. Then equation (1.3) can be transformed to the following equation in x:

$$x^2\Psi''(x) + x\Psi'(x) + h^2\left(\sum_{j=s}^{\ell} K_jx^{jh}\right)\Psi(x) = 0. \tag{1.5}$$

The main purpose of this paper is to show the solutions $\Psi(x) = f(z)$ to (1.5) must be *Liouvillian solutions* in the sense of the *Picard-Vessiot theory* of Kolchin [26] when the solutions $f(z)$ of (1.3) have zero as the Borel exceptional value, that is, f can be represented in the form (1.4). In the special case of equation (1.2), which is labelled PBHE, we further show that the converse of the above statement also holds, namely, that each Liouvillian solution $\Psi(x)$

to the differential equation (1.5) must correspond to a solution $f(z) = \Psi(x)$ of the (1.2) that assume the form (1.4), that is, $\lambda(f) < +\infty$. Indeed, the equivalence problem has been solved for the equation $f''(z) + (e^z - K_0)f(z) = 0$ and the *Morse equation* in [8], [11] and [7], and their differential Galois theory counterparts are dealt with, for example, in [1, p. 275], [2, p. 351] and [2, p. 355] respectively. We next show that if K_3, K_2 are real and $K_0 < 0$, then these Liouvillian solutions possess new single and double orthogonality for the non-oscillatory solutions, as well as novel Fredholm integral equations that resemble the single and double orthogonality for the Lamé polynomials (already recorded in [21, §95, §98]) and Fredholm integral equation for the Lamé equation discovered by Whittaker [42]. We also mention that the orthogonal eigenfunctions over $(-\infty, \infty)$ satisfy the Sturm-Liouville type boundary condition

$$\lim_{z\to\pm\infty} f(z) = 0. \tag{1.6}$$

In [9] the authors applied differential Galois theory to study Sturm-Liouville problems of Schrödinger equations in connection with the boundary condition (1.6).

Our argument is heavily based on the celebrated algorithm due to Kovacic [27] specifically for differential equations written in the normalised form

$$y'' = ry, \tag{1.7}$$

where $r \in \mathbb{C}(x)$, where $\mathbb{C}(x)$ denotes the field of rational functions. The algorithm is based on a key theorem of Kovacic which is tailored from Kolchin's theory [26].

THEOREM 1.1 (Kovacic, 1986). *There are four cases to distinguish.*

(1) *The differential equation* (1.7) *has a solution of the form* $\exp(\int \omega)$, *where* $\omega(x) \in \mathbb{C}(x)$.

(2) *The differential equation* (1.7) *has a solution of the form* $\exp(\int \omega)$, *where* $\omega(x)$ *is algebraic over* $\mathbb{C}(x)$ *and case* (1) *does not hold.*

(3) *All solutions of the differential equation* (1.7) *are algebraic over* $\mathbb{C}(x)$ *and cases* (1) *and* (2) *do not hold.*

(4) *The differential equation* (1.7) *has no Liouvillian solution.*

We state our first result for the general Hill equation (1.3).

THEOREM 1.2. *Let* f *be an entire function solution to the Hill equation* (1.3) *with finite exponent of convergence of zeros, i.e.,* $\lambda(f) < +\infty$. *Then the equation* (1.5), *where* $h = 2$, *if both* $\ell \geq s \geq 0$ *are odd, and* $h = 1$, *if both*

$\ell \geq s$ are even (see e.g., [10]), admits a Liouvillian solution $\Psi(x) = f(z)$ to (1.5), where $x = e^{z/h}$, that belongs to the case (1) of Theorem 1.1.

Acosta-Humánez and Blázquez-Sanz applied differential Galois theory in [1] to conclude that both the *Mathieu* equation

$$f''(z) + (A + B\cos 2z)f(z) = 0$$

and the *extended Mathieu equation*

$$f''(z) + (A + B\cos 2z + C\sin 2z)f(z) = 0$$

have no Liouvillian solutions.

In addition to the Lamé equation (see §9) and the Mathieu equation, Whittaker appears to be the first one to derive a Fredholm type integral equation of the second type for the Whittaker-Hill equation [41] (see also [30]), namely that periodic solutions of the equation

$$f''(z) + \left[A - (n+1)\ell\cos 2z + \frac{1}{8}\ell^2\cos^2 4z\right]f(z) = 0, \tag{1.8}$$

are eigensolutions to the

$$f(z) = \lambda\int_0^{2\pi}\cos^n(z-t)\, e^{\frac{1}{2}\ell(\sin^2 z + \sin^2 t)}f(t)\,dt,$$

with symmetric kernel. We shall extend Ince's method from [23] to derive an analogous Fredholm type integral equation of the second type for the Liouvillian solutions for the PBHE (1.2) with symmetric kernel written in terms of a confluent hypergeometric function. These integral equations possess rich monodromy information of the eigensolutions concerned that are awaiting serious study, despite their long history.

This paper is organised as follows. We shall briefly review the Picard-Vessiot theory, the meaning of Liouvillian solutions and Kovacic's algorithm in §2 before we can state the main results about the equivalence between non-oscillatory solutions of (1.2) and Liouvillian solutions of (1.5) in §3. The proofs of Theorems 3.1 and 1.2 are given in §3 and §4 respectively. We introduce main results about orthogonality in §6 and their proofs are given in §7 and §8. Finally we discuss a new Fredholm type integral equation for the Liouvillian solutions from Theorem 3.1 in §9 and its derivation is given in §9. We include the part of the Kovacic algorithm that we need to use in the Appendix of this paper.

2. The Kovacic algorithm

2.1. Differential Galois theory

We give a brief description on basic terminology and fundamentals of differential Galois theory that lead to Kovacic's Theorem 1.1.

A *differential field* $\mathbb{F}(\supseteq \mathbb{C}(x))$ is *Liouvillian* if there is a tower of differential extension fields $\mathbb{C}(x) = \mathbb{F}_0 \subset \mathbb{F}_2 \subset \cdots \subset \mathbb{F}_n = \mathbb{F}$ such that for each $i = 1, \ldots, n$:

(1) $\mathbb{F}_k = \mathbb{F}_{k-1}[\alpha]$, where $\alpha'/\alpha \in \mathbb{F}_{k-1}$; or

(2) $\mathbb{F}_k = \mathbb{F}_{k-1}[\alpha]$, where $\alpha' \in \mathbb{F}_{k-1}$; or

(3) $\mathbb{F}_k$ is finite algebraic over $\mathbb{F}_{k-1}$.

A function is said to be *Liouvillian* if it is contained in some Liouvillian field defined above.

Suppose we are given a linear differential equation $aY'' + bY' + cY = 0$. Then we can always rewrite it in the so-called normal form: $y'' = ry$, where r belongs to the same differential field as that of a, b, c, that is in $\mathbb{C}(x)$. Let y_1, y_2 be a fundamental set of solutions to the DE $y'' = ry$. Let

$$\mathbb{G} = \mathbb{C}(x)\langle y_1, y_2\rangle = \mathbb{C}(x)(y_1, y_2, y_1', y_2')$$

be the extension field of $\mathbb{C}(x)$ generated by y_1 and y_2. Then we define the (*differential*) *Galois group* of **G** over $\mathbb{C}(x)$ to be the set of all *differential automorphisms* σ of $\mathbb{G}$ that leaves $\mathbb{C}(x)$ invariant. Here, an automorphism σ of $\mathbb{G}$ is *differential* if $\sigma(y') = (\sigma y)'$ for all $\sigma \in \mathbb{G}$.

Without going into the details, the fundamental theorem of differential Galois theory of a linear differential equation is about the isomorphisms between the intermediate fields of Picard-Vessiot extension and the closed subgroups of its differential Galois group. It is fundamental, and demonstrated by Kovacic [27], that the representations of these subgroups are *linear algebraic groups*, namely that they are subgroups of $GL(2, \mathbb{C})$, in that each matrix element satisfies a polynomial equation. A fundamental theorem of Lie-Kolchin [34, p. 8], is that a connected linear algebraic group is *solvable* if and only if it is conjugate to a triangular group. See also Laine [28, Chap. 14]. Then a classification of algebraic subgroups of $SL(2, \mathbb{C})$ is that:

(1) **G** is triangularisable,

(2) **G** is conjugate to a subgroup of the infinite dihedral group and case (1) does not hold,

(3) **G** is finite and cases (1) and (2) do not hold, or

(4) $\mathbf{G} = SL(2, \mathbb{C})$.

2.2. Kovacic's algorithm

Kovacic tailored the above theorem specifically to differential equations written in the normal form $y'' = ry$ already given in Theorem 1.1.

After transforming a given linear differential equation to the normal form (1.7), Kovacic's algorithm gives a *necessary condition* that tests the plausibility of the equation (1.7) having a Liouvillian solution. We first write $r = s/t$ with $s, t \in \mathbb{C}[x]$, relatively prime. Then it is clear that the finite poles of r are located at the zeros of t. We understand the order of r at infinity in the usual sense to be the order of pole of $r(1/x)$ at $x = 0$. That is, the integer $\deg t - \deg s$.

THEOREM 2.1 (Kovacic [27]). *The following conditions are necessary for the respective cases mentioned in Theorem 1.1 to hold.*

(1) *Every pole of r must have even order or else have order* 1. *The order of r at ∞ must be even or else be greater than* 2.

(2) *r must have at least one pole that either has odd order greater than* 2 *or has order* 2.

(3) *The order of a pole of r cannot exceed* 2 *and the order of r at ∞ must be at least* 2. *If the partial fraction expansion of r is*

$$r = \sum_i \frac{\alpha_i}{(x-c_i)^2} + \sum_j \frac{\beta_j}{x-d_j},$$

then $\sqrt{1+4\alpha_i} \in \mathbb{Q}$, for each i, $\sum_j \beta_j = 0$, and if

$$\gamma = \sum_i \alpha_i + \sum_j \beta_j d_j,$$

then $\sqrt{1+4\gamma} \in \mathbb{Q}$.

One observes from Theorem 1.1 that the most general solution has the form $y = P \exp\big(\int \omega\big) = \exp\big(\int P'/P + \omega\big)$, where $P \in \mathbb{C}[x]$ and $\omega \in \mathbb{C}(x)$, so that the logarithmic derivative of y satisfies the Riccati equation

$$\frac{P''}{P} + 2\omega\frac{P'}{P} + (\omega' + \omega^2) = r.$$

The task would then be to determine the *main part* of ω using the above Theorem 2.1 by constructing a rational function denoted by $\big[\sqrt{r}\,\big]_c$ for various poles c of r. The remaining task would be to determine the polynomial component P in the differential equation

$$P'' + 2\omega P' + (\omega' + \omega^2 - r)P = 0, \tag{2.1}$$

if any. An affirmative answer would imply a Liouvillian solution. Since the full Kovacic algorithm is lengthy, we refer readers to the appendix of this paper.

3. Complex non-oscillatory and Liouvillian solutions

It is easy to check that the transformation $\Psi(x) = f(z)$, $x = e^z$ (i.e., $h = 1$ in (1.4)), transforms the equation (1.2) to the equation

$$x^2\Psi''(x) + x\Psi'(x) + \big(-x^4 + K_3x^3 + K_2x^2 + K_1x + K_0\big)\Psi(x) = 0, \quad (3.1)$$

which we call a *generalised Bessel equation*. It is easy to see from the format of the equation that (3.1) has a regular singularity at the origin and an irregular singularity at ∞ which is of higher rank than that of the Bessel equation. The well-known formula

$$\Psi = y \cdot \exp\left(-\frac{1}{2}\int \frac{1}{x}\,dx\right) = x^{-1/2} \cdot y \quad (3.2)$$

transforms equation (3.1) to a standard normal form

$$y'' = \left(x^2 - K_3x - K_2 - \frac{K_1}{x} - \frac{1/4 + K_0}{x^2}\right)y, \quad (3.3)$$

to which we may apply Kovacic's algorithm to identify Liouvillian solutions to the normal form of generalised Bessel equation (3.3), if any. This would lead to a criterion for entire solutions to the Hill equation (1.2) with finite exponent of convergence of zeros $\lambda(f) < +\infty$. The origin is a regular singularity of equation (3.3), but ∞ is an irregular singularity. We mention that the authors of [17] applied differential Galois theory to study the Stoke phenomenon of the *prolate spheroidal wave equation*. It would be interesting to see if their approach could also be applied to this equation we consider in this paper.

THEOREM 3.1. *Let K_3, K_2, K_1 and K_0 be arbitrary constants in $\mathbb{C}$. Then the Hill equation* (1.2)*, when written in the operator form,*

$$-Hf(z) := -\left(e^{-z}\frac{d^2}{dz^2} - e^{3z} + K_3e^{2z} + K_2e^z + K_0e^{-z}\right)f(z) = K_1 f(z), \quad (3.4)$$

admits an entire solution with $\lambda(f) < +\infty$ if and only if the normalised generalised Bessel equation (3.3) *admits a Liouvillian solution in the case* (1) *of Kovacic's Theorem 1.1 above. Moreover, when this happens, there exists a non-negative integer n such that the following equation*

$$\frac{K_3^2}{4} + K_2 + 2\epsilon_0\epsilon_\infty\sqrt{-K_0} = -2\epsilon_\infty(n+1) \quad (3.5)$$

holds amongst the K_3, K_2 *and* K_0, *where* $\epsilon_\infty^2 = \epsilon_0^2 = 1$. *Furthermore, there are either*

(1) $n+1$, *possibly repeated, choices of* K_1, *or*

(2) *precisely* $n+1$ *distinct real values of* K_1, *labelled by* $(K_1)_\nu$, $\nu = 0, \dots, n$, *provided, in addition, that* K_3, K_2, K_0 *are real,* $K_0 < 0$, $\epsilon_\infty = -1$ *and*

$$1 + 2\epsilon_0\sqrt{-K_0} > 0 \tag{3.6}$$

holds.

The $(K_1)_\nu$, $\nu = 0, \dots, n$, *consist of the roots of the* $(n+1)\times(n+1)$ *determinant* $D_{n+1}(K_1) = 0$, *where* $D_{n+1}(K_1)$ *is tridiagonal and it is equals to*

$$\begin{vmatrix} b_0 & 1 & & & & \\ c_1 & b_1 & 1 & & & \\ & c_2 & b_2 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & c_{n-1} & b_{n-1} & 1 \\ & & & & c_n & b_n \end{vmatrix}, \tag{3.7}$$

with

$$\begin{aligned} b_k &= k_1 - \epsilon_\infty k K_3, && k = 0, 1, 2, \dots, n, \\ c_k &= k\big(k + 2\epsilon_0\sqrt{-K_0}\,\big)(k_2 + 2\epsilon_\infty(k-1)), && k = 1, 2, \dots, n, \end{aligned}$$

and k_1, k_2 *are given by*

$$\begin{aligned} k_1 &= K_1 - \frac{\epsilon_\infty}{2}\big(1 + 2\epsilon_0\sqrt{-K_0}\,\big)K_3, \\ k_2 &= 2\epsilon_\infty\big(1 + \epsilon_0\sqrt{-K_0}\,\big) + K_2 + \frac{1}{4}K_3^2. \end{aligned} \tag{3.8}$$

Moreover, we can write the solution f *in the form*

$$\mathrm{BH}_{n,\nu}(z) = Y_{n,\nu}(e^{-z})\exp\left[\frac{\epsilon_\infty}{2}\big(e^{2z} - K_3e^z\big) + \big(n + \epsilon_0\sqrt{-K_0}\,\big)z\right], \tag{3.9}$$

where $Y_{n,\nu}(x)$ *is a polynomial of degree* n, *and* $\nu = 0, 1, \dots, n$ *respectively for the different roots of* $D_{n+1}(K_1) = 0$. *Clearly, each* $y(x) = \mathrm{BH}_{n,\nu}(z)$, $\nu = 0, \dots, n$, *is a Liouvillian solution to the differential equation* (3.3).

REMARK 3.2. We remark that if $\epsilon_0 = 1$, so that $1 + 2\epsilon_0\sqrt{-K_0} > 1 > 0$ holds trivially. If, however, $\epsilon_0 = -1$, then $1 - 2\sqrt{-K_0} > 0$, so that $0 < \sqrt{-K_0} < 1/2$.

4. Proof of Theorem 1.2

We suppose that f is a solution of (1.2) with $\lambda(f) < +\infty$. Then Bank & Laine's result [7] asserts that f is given by (1.4), where $f(z) = \Psi(x)$, $x = e^{z/h}$. Hence

$$\Psi(x) = x^{hd}\psi(x)\exp\bigl(P(x)\bigr) = \exp\biggl(\int \omega\biggr),$$

where

$$\omega = P' + \frac{\psi'}{\psi} + \frac{hd}{x},$$

which belongs to $\mathbb{C}(x)$. According to the transformation (3.2), we have

$$y = \exp\biggl(\int \tilde{\omega}\biggr),$$

where

$$\tilde{\omega} = P' + \frac{\psi'}{\psi} + \frac{2hd+1}{2x} \in \mathbb{C}(x)$$

matches precisely the case (1) of Theorem 1.1. Hence the equation

$$x^2 y'' = h^2\biggl(-\sum_{j=s}^{\ell} K_j x^{jh} - \frac{1}{4}\biggr)y$$

and (1.5) possess Liouvillian solutions.

5. Proof of Theorem 3.1

Suppose that a solution f of (1.2) has $\lambda(f) < +\infty$. Bank & Laine's result asserts that there exist complex constants d, d_j and a polynomial $P_n(\zeta)$ of degree n such that

$$f(z) = P_n(e^z)\exp\bigl(d_1 e^z + d_2 e^{2z} + d\,z\bigr), \tag{5.1}$$

with

$$P_n(\zeta) = \sum_{k=0}^{n} q_k \zeta^k, \qquad q_n \neq 0.$$

Substitution of (5.1) into (1.2) results in

$$4d_2^2 - 1 = 0, \quad 4d_1 d_2 + K_3 = 0, \quad d^2 + K_0 = 0.$$

Then we introduce notation $\epsilon_\infty = \pm 1$, $\epsilon_0 = \pm 1$ and write $d_2 = \epsilon_\infty/2$, $d_1 = -\epsilon_\infty K_3/2$, $d = \epsilon_0\sqrt{-K_0}$. Thus we obtain the solution (3.9).

It follows from Theorem 1.2 that if a solution f of (1.2) has $\lambda(f) < +\infty$, then the function $y = x^{1/2}\Psi(x)$, where $\Psi(x) = f(z)$, $x = e^z$, is a Liouvillian solution to the differential equation (3.3). Thus, we apply Kovacic's algorithm, case (1), to the equation (5.2). It turns out that we can identify (3.3) with the *biconfluent Heun equation* (BHE) [36], written in the following normalised form,

$$y'' = \left(x^2 + \beta x + \frac{\beta^2}{4} - \gamma + \frac{\delta}{2x} + \frac{\alpha^2 - 1}{4x^2}\right)y, \tag{5.2}$$

where α, β, γ, δ are some constants. Equation

$$xy'' + (1+\alpha-\beta x-2x^2)y' + \left((\gamma-\alpha-2)x - \frac{1}{2}(\delta+(1+\alpha)\beta)\right)y = 0, \tag{5.3}$$

is called the canonical form of BHE. Duval and Loday-Richard [14] have already applied Kovacic's algorithm to equation (5.2) to obtain criteria for Liouvillian solutions. In addition to the fact that they did not give details of their computation in [14, Prop. 13], see also [35], it would be difficult for us to elaborate on the dependence of the different set of coefficients, namely $K_3, \ldots, K_0$ from the (1.2). In particular, we shall derive four different sets of $n+1$ solutions of the from (3.9) as emphasised in the remark after the theorem, so we judge it appropriate to working out details of the algorithm here for the sake of completeness, especially given our very different motivation, context and notation.

Kovacic's algorithm consists of identifying the set of poles $\{c\}$ (both finite and ∞) of r, constructing a suitably defined rational function denoted by $\left[\sqrt{r}\,\right]_c$ based on the Laurent expansion of r at the pole c, and the corresponding tuple (α_c^+, α_c^-) of complex numbers. In the case of (3.3) under consideration, our r can only have poles at 0 and ∞.

According to Kovacic's algorithm (see the Appendix), there are three cases to consider:

(1) $K_0 = -1/4$, $K_1 = 0$,

(2) $K_0 = -1/4$, $K_1 \neq 0$,

(3) $K_0 \neq -1/4$.

We use the notation $\mathbb{Z}$ and $\mathbb{N}$ to denote all integers and all positive integers, respectively. The generalised Bessel equation (3.3) has Liouvillian solutions if and only if it falls into the case (1) of Kovacic's algorithm and we claim that one of the following conditions is fulfilled.

(1) $K_0 = -1/4$, $K_1 = 0$ and $-\epsilon_\infty\big(K_2 + \frac{1}{4}K_3^2\big) \in 2\mathbb{Z} + 1$. The monic

polynomial $P_n(x)$ of degree n in the Liouvillian solution satisfies

$$P_n'' + 2\epsilon_\infty\left(x - \frac{1}{2}K_3\right)P_n' + \left(\epsilon_\infty + K_2 + \frac{1}{4}K_3^2\right)P_n = 0. \tag{5.4}$$

(2) $K_0 = -1/4$, $K_1 \neq 0$ and $-\epsilon_\infty\big(K_2 + \frac{1}{4}K_3^2\big) = 2n + 3 \in 2\mathbb{N} + 1$. Then $P_n(x)$ is a Liouvillian solution that satisfies

$$\begin{aligned} &xP_n''(x) + (2 - \epsilon_\infty K_3 x + 2\epsilon_\infty x^2)P_n' \\ &\qquad + \left[K_1 - \epsilon_\infty K_3 + \left(3\epsilon_\infty + K_2 + \frac{1}{4}K_3^2\right)x\right]P_n = 0. \end{aligned} \tag{5.5}$$

(3) $K_0 \neq -1/4$ and $\epsilon_0, \epsilon_\infty \in \{\pm 1\}$ such that $-\epsilon_\infty(K_2 + \frac{1}{4}K_3^2) - \epsilon_0\sqrt{-4K_0} = 2(n+1) \in 2\mathbb{N}$. The $P_n(x)$ is a Liouvillian solution that satisfies

$$\begin{aligned} &xP_n''(x) + \big(1 + 2\epsilon_0\sqrt{-K_0} - \epsilon_\infty K_3 x + 2\epsilon_\infty x^2\big)P_n' \\ &\qquad + \left[K_1 - \frac{1}{2}\epsilon_\infty\big(1 + 2\epsilon_0\sqrt{-K_0}\big)K_3 \right. \\ &\qquad \left. + \left(2\epsilon_\infty + 2\epsilon_0\epsilon_\infty\sqrt{-K_0} + K_2 + \frac{1}{4}K_3^2\right)x\right]P_n = 0. \end{aligned} \tag{5.6}$$

The case (5.4) reduces essentially to a Hermite equation whose structure is so degenerate that it falls outside the scope of being a genuine BHE case. The case (5.5) is a special case of (5.6) and so we omit the details. In the following we consider the third case when $K_0 \neq -1/4$ to illustrate the Kovacic algorithm. In this situation, we have

$$r(x) = \frac{-\frac{1}{4} - K_0}{x^2} + \frac{-K_1}{x} - K_2 - K_3 x + x^2.$$

We first need to identify the singularities of r, and they are $\{0, \infty\}$. So the set $\Gamma = \Gamma' \cup \{\infty\} = \{0\} \cup \{\infty\}$ and the only finite pole is $c = 0$. Then it is obvious that the order of $r(x)$ at $x = 0$ is two, i.e., $o(r_0) = 2$. According to case (1), Step 1 in the Appendix, we deduce that the rational function $\big[\sqrt{r}\,\big]_0 \equiv 0$, $r = \cdots + bx^{-2}$ with $b = -1/4 - K_0$, so that the tuple (α_0^+, α_0^-) of numbers are given by the expression

$$\alpha_0^{\epsilon_0} = \frac{1 + \epsilon_0\sqrt{1 + 4b}}{2} = \frac{1 + \epsilon_0\sqrt{-4K_0}}{2} = \frac{1}{2} + \epsilon_0\sqrt{-K_0}, \qquad \epsilon_0 = \pm 1.$$

Similarly, it is easy to spot that the only other pole being at ∞ also has order -2, i.e., $o(r_\infty) = 2 - 4 = -2$. Then the case (1), Step 1 of Kovacic's algorithm in the Appendix, again asserts that we have $\left[\sqrt{r}\,\right]_\infty \equiv x - \frac{K_3}{2}$, and that the tuple $(\alpha_\infty^+, \alpha_\infty^-)$ of numbers are given by

$$\alpha_\infty^{\epsilon_\infty} = \frac{1}{2}\left[-\epsilon_\infty\left(K_2 + \frac{1}{4}K_3^2\right) - 1\right], \qquad \epsilon_\infty = \pm 1.$$

We continue the algorithm to the next step, the Step 2, where we define the set of positive integers n:

$$D = \left\{n \in \mathbb{N} : n = \alpha_\infty^{\epsilon_\infty} - \sum_{c\in\Gamma'} \alpha_c^{\epsilon_c},\ \forall(\epsilon_p)_{p\in\Gamma}\right\}.$$

and we can derive

$$n = \alpha_\infty^{\epsilon_\infty} - \alpha_0^{\epsilon_0} = -1 + \frac{1}{2}\left[-\epsilon_\infty\left(K_2 + \frac{1}{4}K_3^2\right) - \epsilon_0\sqrt{-4K_0}\right] \tag{5.7}$$

holds for some non-negative integer n satisfying the assumption (3.5). Hence $1 = \#(D) > 0$. We now construct the rational function ω according to Step 2:

$$\omega = \epsilon_\infty \cdot \left[\sqrt{r}\,\right]_\infty + \epsilon_0 \cdot \left[\sqrt{r}\,\right]_0 + \frac{\alpha_0^{\epsilon_0}}{x} = \epsilon_\infty\left(x - \frac{1}{2}K_3\right) + \frac{\alpha_0^{\epsilon_0}}{x},$$

so that

$$\omega' + \omega^2 - r = 2\epsilon_\infty + K_2 + \frac{1}{4}K_3^2 + \epsilon_\infty\epsilon_0\sqrt{-4K_0} + \frac{K_1 - \epsilon_\infty\alpha_0^{\epsilon_0}K_3}{x}, \tag{5.8}$$

with $\epsilon_\infty = \pm 1$. Substituting equation (5.7), or equivalently equation (3.5), into (5.8), it simplifies to

$$\omega' + \omega^2 - r = -2\epsilon_\infty n + \frac{K_1 - \epsilon_\infty\alpha_0^{\epsilon_0}K_3}{x}.$$

It remains to solve the differential equation (2.1) for a polynomial P_n as stated in the Step 3 in case (1) of Kovacic's algorithm in the Appendix. That is, we need to solve for the differential equation

$$\begin{aligned} xP_n''(x) &+ \left(1 + 2\epsilon_0\sqrt{-K_0} - \epsilon_\infty K_3 x + 2\epsilon_\infty x^2\right)P_n' \\ &+ (-2\epsilon_\infty n x + K_1 - \epsilon_\infty\alpha_0^{\epsilon_0}K_3)P_n = 0, \end{aligned} \tag{5.9}$$

which is a simplified form of equation (5.6). Substituting

$$P_n = \sum_{m=0}^{n} \frac{A_m}{\left(1 + 2\epsilon_0\sqrt{-K_0}\,\right)_m}\frac{x^m}{m!}, \qquad (\mu)_n = \frac{\Gamma(\mu+n)}{\Gamma(\mu)}, \qquad n \geq 0,$$

into (5.9), we obtain the three-term recursion relation

$$\begin{aligned}
&A_0 = 1,\\
&A_1 = -k_1\\
&A_{m+2} + (k_1 - \epsilon_\infty(m+1)K_3)A_{m+1}\\
&\qquad\qquad + (m+1)\big(m+1+2\epsilon_0\sqrt{-K_0}\big)(k_2 + 2\epsilon_\infty m)A_m = 0,
\end{aligned}\tag{5.10}$$

with k_1 and k_2 given by (3.8), which results in the determinant (3.7) being zero. That is, there are $n+1$, possibly repeated, roots for K_1. Let us first suppose the inequality (3.6) holds when $\epsilon_\infty = -1$. Hence $1 + 2\epsilon_0\sqrt{-K_0} > 0$. However, the resulting expression from equation (5.9) corresponds to the parameter $1+\alpha > 0$ of equation (5.3). It follows from a theorem of Rovder [37, Theorem 1] that the determinant associated with (5.10) admits $n+1$ distinct real values for $K_1 - \epsilon_\infty\alpha_0^{\epsilon_0}K_3$ and hence for K_1. It follows that equation (5.9) has $n+1$ distinct polynomial solutions.

Thus the Liouvillian solutions to (3.3) are expressed by

$$y(x) = P_n(x)\exp\bigg(\int \omega(x)\,dx\bigg) = P_n(x)x^{\alpha_0^{\epsilon_0}}\exp\bigg[\frac{\epsilon_\infty}{2}(x^2 - K_3x)\bigg].$$

From the transformation (3.2), we obtain the solution to the Hill equation (1.2)

$$f(z) = P_n(e^z)\exp\bigg(\frac{\epsilon_\infty}{2}e^{2z} - \frac{\epsilon_\infty}{2}K_3e^z + \epsilon_0\sqrt{-K_0}\,z\bigg).$$

The cases (1) and (2) can be considered similarly.

6. Orthogonal solutions of a Hill equation

We next show for the first time that there are novel orthogonality relations amongst the non-oscillatory/Liouvillian solutions $\mathrm{BH}_{n,\nu}(z)$ (3.9) for the periodic BHE (1.2) from the Bank-Laine theory. These are analogous results for the Lamé equation which we now review.

It is known that the classical Lamé polynomials [3, §9.2–9.3] are eigensolutions to the Lamé equation

$$w''(z) + \big[h - n(n+1)k^2\mathrm{sn}^2 z\big]w(z) = 0,$$

where n is chosen to be a non-negative integer, and that there are $n/2$ or $(n+1)/2$ distinct choices of h, depending on whether n is even or odd respectively, thus leading to $2n+1$ distinct *Lamé polynomials*, and each of which assumes the form

$$\mathrm{sn}^r z\,\mathrm{cn}^s z\,\mathrm{dn}^t z\,F_p(\mathrm{sn}^2 z),\tag{6.1}$$

where r, s, t take integer values 0 or 1 subject to the constraint $r+s+t+2p = n$. This leads to eight different combinations of (r, s, t), called *species* [3, p. 201], and a total of $2n + 1$ Lamé polynomials. The coefficients of the polynomial F_p in (6.1) satisfy a three-term recursion. It is known from 19th century texts that Lamé polynomials of the same type satisfy orthogonality relations given by, for example,

$$\int_{-2K}^{2K} E_n^{m_1}(z) E_n^{m_2}(z)\, dz = 0 \tag{6.2}$$

whenever $m_1 \neq m_2$ $(0 \leq m_1, m_2 \leq n)$, where the *real period* and *imaginary period* of $\operatorname{sn} z$ are denoted by $4K$ and $2iK'$ respectively (e.g. [21, p. 370], [22, p. 466]; see also [3, §9.4]). That is, two Lamé polynomials of the same species must have the same degree n, and their orthogonality is over the other parameter m $(0 \leq m \leq n)$, which differs from conventional orthogonality we encounter from other well-known examples. On the other hand, we have

$$\int_{-2K}^{2K} \big(E_n^m(z)\big)^2\, dz \neq 0$$

whenever $m_1 = m = m_2$ [3, p. 206].

We next show that for the parameters $(\epsilon_0, \epsilon_\infty) = (1, -1)$, the $n + 1$ eigensolutions $\mathrm{BH}_{n,\nu}(z)$ defined in (3.9) exhibit a similar orthogonality to the (6.2). That is, any two functions $\mathrm{BH}_{n,\nu}(z)$ of the same degree n out of the $n + 1$ solutions defined by (3.9) are orthogonal with respect to the weight e^z over $(-\infty, +\infty)$, and *different parameters* ν, $0 \leq \nu \leq n$, (instead of different degrees n).

Moreover, these eigensolutions $\mathrm{BH}_{n,\nu}(z)$ satisfy the (1.6), i.e.,

$$\lim_{z\to\pm\infty} \mathrm{BH}_{n,\nu}(z) = 0, \tag{6.3}$$

which serves as a boundary condition of a Sturm-Liouville problem for a time-independent linear Schrödinger equation, including our equation (6.5) below, studied by Blázquez-Sanz and Yagasaki [9]. There the authors showed that the corresponding eigensolutions correspond to discrete eigenvalues are Liouvillian. We note that we require $\mathrm{BH}_{n,\nu}(z)$ to satisfy the boundary condition (6.3) for the orthogonality result below. The results in [9] provide another connection between differential Galois theory and non-oscillatory solutions of (6.5) below.

THEOREM 6.1. *Let n be a non-negative integer. Suppose that the coefficients K_3, K_2 and $K_0 < 0$ of the Hill operator* (3.4) *are all real and satisfy the relation*

$$\frac{1}{4}K_3^2 + K_2 - 2\sqrt{-K_0} = 2(n+1) \tag{6.4}$$

for each n. Then the equations

$$-Hf(z) := -\left(e^{-z}\frac{d^2}{dz^2} - e^{3z} + K_3e^{2z} + K_2e^z + K_0e^{-z}\right)f(z)$$
$$= (K_1)_{n,\nu}f, \qquad 0 \le \nu \le n, \tag{6.5}$$

admit $n+1$ linearly independent solutions $\mathrm{BH}_{n,\nu}(z)$, $0 \le \nu \le n$, of the form

$$\mathrm{BH}_{n,\nu}(z) = e^{nz}Y_{n,\nu}(e^{-z})\exp\left[-\frac{1}{2}\left(e^{2z} - K_3e^z\right) + \sqrt{-K_0}\,z\right], \tag{6.6}$$

given in (3.9)*, corresponding to the $n+1$ distinct real values of $(K_1)_{n,\nu}$, $0 \le \nu \le n$. These satisfy the orthogonality relations*

$$\int_{-\infty}^{\infty} \mathrm{BH}_{n,\mu}(x)\mathrm{BH}_{n,\nu}(x)e^x\,dx = 0 \tag{6.7}$$

whenever $\mu \neq \nu$, and

$$\int_{-\infty}^{\infty} \left(\mathrm{BH}_{n,\mu}(x)\right)^2 e^x\,dx \neq 0 \tag{6.8}$$

whenever $\mu = \nu$.

We further observe that one can define an inner product by integrating the Lamé polynomials over a *complex period* $[K - 2iK', K + 2iK']$ instead of the real period $[-2K, 2K]$ [3, §9.4] considered above. That is,

$$\int_{K-2iK'}^{K+2iK'} E_n^{m_1}(z)E_n^{m_2}(z)\,dz = 0$$

whenever $m_1 \neq m_2$.

We observe from (6.2) and (6.7) that although the corresponding Lamé polynomial solutions and periodic BHE polynomial solutions of same degree respectively are orthogonal with respect to the parameters μ, ν, it is not clear if polynomials of *different degrees* are orthogonal to each other. It turns out that an orthogonality exists for *polynomials* of different degrees when they are formed from products of two Lamé polynomials of the same degree but in different parameters. The following *double orthogonality* can be found in [21, pp. 379–381], [22, pp. 467–469], [3, §9.4]:

$$\int_{-2K}^{2K}\int_{K-2iK'}^{K+2iK'} E_m^{\mu}(z)E_m^{\mu}(s)E_n^{\nu}(z)E_n^{\nu}(s)(\mathrm{sn}^2z - \mathrm{sn}^2s)\,dz\,ds = 0$$

whenever $n \neq m$, while if $m = n$, then the orthogonality still holds when $\mu \neq \nu$, for $0 \leq \mu, \nu \leq n$. On the other hand,

$$\int_{-2K}^{2K} \int_{K-2iK'}^{K+2iK'} \big[E_m^\mu(z) E_m^\mu(s)\big]^2 (\mathrm{sn}^2 z - \mathrm{sn}^2 s)\, dz\, ds \neq 0.$$

We show below that suitable product of two non-oscillatory/Liouvillian solutions of equation (1.2) also possesses a double orthogonality, over $\mathbb{R} \times (\mathbb{R} + i\pi)$, that is analogous to that of the Lamé equation discussed above, and that appears to be new.

THEOREM 6.2. *Let K_3 and K_2 be given real numbers. Let m and n be non-negative integers such that $(K_0)_n < 0$ and $(K_0)_m < 0$ from Theorem 3.1 satisfy*

$$\frac{1}{4}K_3^2 + K_2 - 2\sqrt{-(K_0)_n} = 2(n+1) \tag{6.9}$$

and

$$\frac{1}{4}K_3^2 + K_2 - 2\sqrt{-(K_0)_m} = 2(m+1), \tag{6.10}$$

respectively. Suppose $\mathrm{BH}_{n,\mu}$, $0 \leq \mu \leq n$, *are solutions of the differential equation* (6.5) *as defined in Theorem 6.1 with finite exponent of convergence of zeros, and* $\mathrm{BH}_{m,\nu}$, $0 \leq \nu \leq m$, *are the corresponding solutions to equation* (6.5) *with n replaced by m. Then we have*

$$\int_{-\infty}^{\infty} \int_{-\infty+i\pi}^{\infty+i\pi} \mathrm{BH}_{n,\mu}(z)\mathrm{BH}_{n,\mu}(s)\mathrm{BH}_{m,\nu}(z)\mathrm{BH}_{m,\nu}(s)(e^z - e^s)\, dz\, ds = 0 \tag{6.11}$$

whenever $(n, \mu) \neq (m, \nu)$. *Moreover,*

$$\int_{-\infty}^{\infty} \int_{-\infty+i\pi}^{\infty+i\pi} \big[\mathrm{BH}_{n,\mu}(z)\mathrm{BH}_{n,\mu}(s)\big]^2 (e^z - e^s)\, dz\, ds \neq 0, \tag{6.12}$$

if $(n, \mu) = (m, \nu)$.

7. Proof of Theorem 6.1

We note that since we assume (6.4) and $\sqrt{-K_0} > 0$ (so that $1 + 2\sqrt{-K_0} > 0$), so the (3.5) and the inequality (3.6) are fulfilled respectively. Theorem 3.1 guarantees that there are $n+1$ distinct solutions to (1.2) of the form (3.9), where we recall that $Y_{n,\nu}$ is a polynomial. Hence we see that $\mathrm{BH}_{n,\nu}(z) \to 0$ as $z = x \to +\infty$. To see that $\mathrm{BH}_{n,\nu}(z) \to 0$ as $z = x \to -\infty$ we only need to note that $\sqrt{-K_0} > 0$ because from the assumption that K_0 is real and $K_0 < 0$. Moreover, the derivative assumes the form

$$\begin{aligned}\mathrm{BH}'_{n,\nu}(z) = \Big[-e^{-z}Y'_{n,\nu}(e^{-z}) + \Big(\Big(\frac{1}{2}K_3e^z - e^{2z}\Big) + \big(n+\sqrt{-K_0}\,\big)\Big)Y_{n,\nu}(e^{-z})\Big] \\ \times \exp\Big[\Big(\frac{1}{2}K_3e^z - \frac{1}{2}e^{2z}\Big) + \big(n+\sqrt{-K_0}\,\big)z\Big], \qquad (7.1)\end{aligned}$$

which is similar to that for $\mathrm{BH}_{n,\nu}(z)$. A similar analysis as that of $\mathrm{BH}_{n,\nu}(z)$ reveals that $\mathrm{BH}'_{n,\nu}(z) \to 0$ as $z = x \to \pm\infty$.

Let n, m be non-negative integers and let the coefficients K_3, K_2 and $K_0 < 0$ satisfy the relations (3.5) and (3.6). Then it follows from Theorem 3.1 that the equation (1.2) possesses $n+1$ distinct solutions $\mathrm{BH}_{n,\nu}(z)$, $0 \le \nu \le n$, with finite exponent of convergence of zeros, given by (3.9).

Let $f_{n,\nu} = \mathrm{BH}_{n,\nu}$ and $f_{n,\mu} = \mathrm{BH}_{n,\mu}$ $(0 \le \nu, \mu \le n)$ be two solutions to (1.2) with finite exponent of convergence. That is, $f_{n,\nu}$ and $f_{n,\mu}$ are solutions to

$$f''_{n,\mu}(z) + \big(-e^{4z} + K_3e^{3z} + K_2e^{2z} + (K_1)_{n,\mu}e^z + K_0\big)f_{n,\mu}(z) = 0, \qquad (7.2)$$

$$f''_{n,\nu}(z) + \big(-e^{4z} + K_3e^{3z} + K_2e^{2z} + (K_1)_{n,\nu}e^z + K_0\big)f_{n,\nu}(z) = 0, \qquad (7.3)$$

respectively.

We subtract the two equations that result from multiplying (7.2) by $f_{n,\nu}$ and (7.3) by $f_{n,\mu}$, respectively. Then we integrate the resulting equation over $(-\infty, \infty)$. This yields

$$\begin{aligned}\int_{-\infty}^{\infty}\big[f_{n,\nu}(z)f''_{n,\mu}(z) - f_{n,\nu}(z)f''_{n,\mu}(z)\big]\,dz \\ + \big[(K_1)_{n,\mu} - (K_1)_{n,\nu}\big]\int_{-\infty}^{\infty} f_{n,\nu}(z)f_{n,\mu}(z)e^z\,dz = 0.\end{aligned}$$

We deduce from the explicit representations (6.6) of $\mathrm{BH}_{n,\nu}(z)$ and (7.1) of $\mathrm{BH}'_{n,\nu}(z)$ that they both vanish at $\pm\infty$. So integration-by-parts of the above equation yields

$$\begin{aligned}&\int_{-\infty}^{\infty} f_{n,\nu}(z)f_{n,\mu}(z)e^z\,dz \\ &\quad= \big[(K_1)_{n,\mu} - (K_1)_{n,\nu}\big]^{-1}\int_{-\infty}^{\infty}\big[f_{n,\nu}(z)f''_{n,\mu}(z) - f_{n,\nu}(z)f''_{n,\mu}(z)\big]\,dz \\ &\quad= \big[(K_1)_{n,\mu} - (K_1)_{n,\nu}\big]^{-1}\big[f_{n,\nu}(z)f'_{n,\mu}(z) - f_{n,\nu}(z)f'_{n,\mu}(z)\big]_{-\infty}^{\infty} \\ &\quad= 0,\end{aligned}$$

thus proving that the two functions $f_{n,\nu}(z)$, $f_{n,\mu}(z)$ are orthogonal, since the corresponding characteristic values $(K_1)_{n,\mu}$, $(K_1)_{n,\nu}$ are distinct. That is, whenever $\mu \neq \nu$ for $0 \leq \mu, \nu \leq n$. This proves (6.7).

Let us now consider the case when $\mu = \nu$. Notice that the product between $\big(\mathrm{BH}_{n,\mu}(z)\big)^2$ and the weight function e^z yields, under the assumption $2\sqrt{-K_0}+1 > 0$, that

$$\begin{aligned}\big(\mathrm{BH}_{n,\mu}(z)\big)^2 e^z &= \big[Y_{n,\nu}(e^{-z})\big]^2 \exp\big[(K_3 e^z - e^{2z}) + 2\big(n + \sqrt{-K_0}\,\big)z\big]e^z \\ &= \big[Y_{n,\nu}(e^{-z})\big]^2 \exp\big[(K_3 e^z - e^{2z}) + \big(2n + 1 + 2\sqrt{-K_0}\,\big)z\big] \\ &> 0\end{aligned}$$

holds throughout the real-axis $\mathbb{R}$. But the representation (6.6) clearly shows that the product

$$\big(\mathrm{BH}_{n,\mu}(z)\big)^2 e^z = \big[Y_{n,\nu}(e^{-z})\big]^2 \exp\big[(K_3 e^z - e^{2z}) + \big(2n + 1 + 2\sqrt{-K_0}\,\big)z\big]$$

vanishes at $\pm\infty$ sufficiently fast to guarantee that the integral

$$\int_{-\infty}^{\infty} \big(\mathrm{BH}_{n,\mu}(x)\big)^2 e^x \, dx$$

converges and is non-vanishing. This proves (6.8).

8. Proof of Theorem 6.2

The idea of the proof is classical: construct a suitable pair of partial differential equations and apply integration-by-parts [22, §276] (see also [21, pp. 379–381]). Let K_3, K_2 be real, and let $(K_0)_n$, $(K_0)_m$ satisfy equations (6.9) and (6.10), respectively. Let $f_{n,\mu}(z) = \mathrm{BH}_{n,\nu}(z)$ and $f_{m,\nu}(z) = \mathrm{BH}_{m,\mu}(z)$ be two corresponding eigensolutions of (1.2). We define

$$F_{n,\mu}(z,s) := f_{n,\mu}(z) f_{n,\mu}(s), \qquad F_{m,\nu}(z,s) := f_{m,\nu}(z) f_{m,\nu}(s)$$

to be complex functions of two variables (z, s). Clearly they satisfy the following partial differential equations:

$$\begin{aligned}\frac{\partial^2 F_{n,\mu}}{\partial z^2} - \frac{\partial^2 F_{n,\mu}}{\partial s^2} &= \big[-(e^{4z} - e^{4s}) + K_3\,(e^{3z} - e^{3s}) \\ &\quad + K_2\,(e^{2z} - e^{2s}) + (K_1)_{n,\mu}(e^z - e^s)\big]F_{n,\mu} = 0, \qquad (8.1)\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 F_{m,\nu}}{\partial z^2} - \frac{\partial^2 F_{m,\nu}}{\partial s^2} &= \big[-(e^{4z} - e^{4s}) + K_3\,(e^{3z} - e^{3s}) \\ &\quad + K_2\,(e^{2z} - e^{2s}) + (K_1)_{m,\nu}(e^z - e^s)\big]F_{m,\nu} = 0. \qquad (8.2)\end{aligned}$$

Subtract the equation (8.1) after multiplying by $F_{m,\nu}$ from the equation (8.2) after multiplying by $F_{n,\mu}$. We then integrate the resulting difference with respect to the two variables (z, s) from $-\infty$ to ∞ and $-\infty + i\pi$ to $\infty + i\pi$ respectively. This yields

$$\int_{-\infty}^{+\infty}\int_{-\infty+i\pi}^{+\infty+i\pi}\Big\{F_{m,\nu}\big[(F_{n,\mu})_{zz}-(F_{n,\mu})_{ss}\big]-F_{n,\mu}\big[(F_{m,\nu})_{zz}-(F_{m,\nu})_{ss}\big]\Big\}\,dz\,ds$$
$$+\big[(K_1)_{n,\mu}-(K_1)_{m,\nu}\big]\int_{-\infty}^{+\infty}\int_{-\infty+i\pi}^{+\infty+i\pi}F_{n,\mu}(z,s)F_{m,\nu}(z,s)(e^z-e^s)\,dz\,ds=0.$$

We note that $(F_{m,\nu})_z = \mathrm{BH}'_{m,\nu}(z)\cdot \mathrm{BH}_{n,\nu}(s)$ and $(F_{n,\nu})_s = \mathrm{BH}_{n,\nu}(z)\cdot \mathrm{BH}'_{n,\nu}(s)$. It again follows from (6.6) and (7.1) that $\mathrm{BH}_{n,\nu}(z)$ and $\mathrm{BH}'_{n,\nu}(z)$ vanish at $\pm\infty$, so that both $(F_{m,\nu})$ and $(F_{m,\nu})_z$ also vanish at $\pm\infty$, and the same holds for the partial derivatives with z replaced by s.

Integration of the above equation by parts with respect to both variables (z, s) and utilisation of our observation in the last paragraph that $(F_{m,\nu})$ and $(F_{m,\nu})_z$ also vanish simultaneously at $\pm\infty$ yields

$$\begin{aligned}
0 &= \int_{-\infty}^{\infty}\int_{-\infty+i\pi}^{\infty+i\pi}\big[F_{m,\nu}(F_{n,\mu})_{zz}-F_{n,\mu}(F_{m,\nu})_{zz}\big]\\
&\quad+\big[F_{n,\mu}(F_{m,\nu})_{ss}-F_{m,\nu}(F_{n,\mu})_{ss}\big]\,dz\,ds\\
&\quad+\big[(K_1)_{n,\nu}-(K_1)_{m,\mu}\big]\int_{-\infty}^{\infty}\int_{-\infty+i\pi}^{\infty+i\pi}F_{n,\mu}(z,s)F_{m,\nu}(z,s)(e^z-e^s)\,dz\,ds\\
&= \int_{-\infty+i\pi}^{\infty+i\pi}\big[F_{m,\nu}(F_{n,\mu})_z-F_{n,\mu}(F_{m,\nu})_z\big]_{\infty}^{-\infty}\,ds\\
&\quad-\int_{-\infty}^{\infty}\int_{-\infty+i\pi}^{\infty+i\pi}(F_{m,\nu})_z(F_{n,\mu})_z-(F_{n,\mu})_z(F_{m,\nu})_z\,dz\,ds \qquad (8.3a)\\
&\quad+\int_{\infty}^{-\infty}\big[F_{n,\mu}(F_{m,\nu})_s-F_{m,\nu}(F_{n,\mu})_s\big]_{-\infty+i\pi}^{\infty+i\pi}\,dz\\
&\quad-\int_{-\infty}^{\infty}\int_{-\infty+i\pi}^{\infty+i\pi}(F_{m,\nu})_s(F_{n,\mu})_s-(F_{n,\mu})_s(F_{m,\nu})_s\,dz\,ds \qquad (8.3b)\\
&\quad+\big[(K_1)_{n,\nu}-(K_1)_{m,\mu}\big]\int_{-\infty+i\pi}^{\infty+i\pi}\int_{\infty}^{-\infty}F_{n,\mu}(z,s)F_{m,\nu}(z,s)(e^z-e^s)\,dz\,ds\\
&= \int_{-\infty+i\pi}^{\infty+i\pi}\big[F_{m,\nu}(F_{n,\mu})_z-F_{n,\mu}(F_{m,\nu})_z\big]_{-\infty}^{\infty}\,ds - 0
\end{aligned}$$

(*cont. next page*)

$$+\int_{-\infty}^{\infty}\big[F_{n,\mu}(F_{m,\nu})_s - F_{m,\nu}(F_{n,\mu})_s\big]_{-\infty+i\pi}^{\infty+i\pi}\,dz - 0$$

$$+\big[(K_1)_{n,\nu}-(K_1)_{m,\mu}\big]\int_{-\infty}^{\infty}\int_{-\infty+i\pi}^{\infty+i\pi} F_{n,\mu}(z,s)F_{m,\nu}(z,s)(e^z-e^s)\,dz\,ds,$$

where the integrands on lines (8.3a) and (8.3b) are identically zero. Moreover, both of the two single integrals after the last equal sign are also identically zero since the $F_{n,\mu}$, $F_{m,\nu}$ and their partial derivatives vanish simultaneously at $\pm\infty$. That is, we have shown that

$$\big[(K_1)_{n,\nu}-(K_1)_{m,\mu}\big]\int_{-\infty}^{\infty}\int_{-\infty+i\pi}^{\infty+i\pi} F_{n,\mu}(z,s)F_{m,\nu}(z,s)(e^z-e^s)\,dz\,ds = 0,$$

proving that (6.11) holds whenever $n \neq m$ and irrespective of the choices of ν and μ.

It remains to consider the case $m = n$ in (6.11). We can easily rewrite

$$\int_{-\infty}^{\infty}\int_{-\infty+i\pi}^{\infty+i\pi}\big(\mathrm{BH}_{n,\nu}(s)\mathrm{BH}_{n,\nu}(z)\big)^2(e^z-e^s)\,dz\,ds$$

into the following double integral

$$\int_{-\infty}^{\infty}\int_{-\infty+i\pi}^{\infty+i\pi}\mathrm{BH}^2_{n,\nu}(z)\cdot\mathrm{BH}^2_{n,\nu}(s)(e^z-e^s)\,dz\,ds$$
$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\big[\mathrm{BH}^2_{n,\nu}(z)\cdot\mathrm{BH}^2_{n,\nu}(\xi+i\pi)(e^z+e^\xi)\big]\,dz\,d\xi.$$

We note that it follows from (6.6) that $\mathrm{BH}^2_{n,\nu}(\xi+i\pi)$ converges to 0 rapidly when $\xi \to \pm\infty$, and that the weight (e^z+e^ξ) is positive throughout the integration region. Hence we deduce that the double integral above, in much the same way as we have done for the validity of (6.8), converges and is non-vanishing. This proves the (6.12).

9. Fredholm integral equations

Whittaker commented in [41, p. 15] that unlike the hypergeometric equation where its solutions have integral representations, certain solutions of Heun equations satisfy *homogeneous Fredholm-type integral equations of the second kind* of the form

$$u(x) = \lambda\int_a^b K(x,t)u(t)\,dt,$$

instead (see, e.g., [36, Part A, §6]), where λ is the eigenvalue of the solution $u(x)$, and the kernel $K(x, t)$ is symmetric in x and t, if any. Such integral equations are of fundamental importance for Heun equations (see, for example, [36, Part A]), which play the role of integral representations for the hypergeometric equation.

Whittaker [42] appears to be the first one who showed that the Lamé polynomials, being eigensolutions to the Lamé equation (1.1), satisfy the following *Fredholm integral equation of the second kind*:

$$y(z) = \lambda \int_{-2K}^{2K} P_n(k \operatorname{sn} z \operatorname{sn} t) y(t)\, dt,$$

where the $P_n(t)$ is the n-th Legendre polynomial and λ the corresponding eigenvalue. The reader may consult [36] for further improvements to Whittaker's result.

It has long been known that *Laplace* and analogous transformation methods can solve some linear differential equations via definite integrals. The kernels of these integral equations are constructed by solving specifically designed linear *partial differential equations*, which are based on the adjoint form of the original differential equation, by the method of *separation of variables*. See for example, Ince [24, XVIII]. Such *kernel function* approaches, see Garabedian [19, Chapter 10] for a modern treatment, in the words of Ince [23], differ from Green's function considerations where the kernel of the integrals involved usually has discontinuities along the "diagonal", while the Laplace-type integral transform method produces "continuous" kernels. Our principal concern here is to obtain such an explicit Fredholm-type integral equation and solutions for the PBHE (1.2), using the Laplace method.

THEOREM 9.1. *Let K_3, K_2 and K_0 be real such that*

$$K_0 = -m^2, \qquad m \in \mathbb{N}.$$

For each non-negative integer n, there are $n + 1$ distinct pairs of generalised eigenvalues $\big((K_1)_{n,\nu}, \lambda_{n,\nu}\big)$, $0 \le \nu \le n$, such that the Fredholm integral equation of the second kind

$$\begin{aligned} y(z) = \lambda \int_0^{4\pi i} & e^{\frac{1}{2}\epsilon_\infty[K_3(e^z+e^t)-(e^{2z}+e^{2t})]+\epsilon_0\sqrt{-K_0}(z+t)} \\ & \times \Phi\bigg(\frac{\epsilon_\infty K_3^2}{16} + \frac{1+\epsilon_0\sqrt{-K_0}}{2} + \frac{\epsilon_\infty K_2}{4}; \frac{1}{2}; -\epsilon_\infty\Big(e^z + e^s - \frac{1}{2}K_3\Big)^2\bigg) e^t y(t)\, dt, \end{aligned} \tag{9.1}$$

where $\Phi(a; c; x)$ is the Kummer function, admits corresponding eigensolutions defined by (3.9) *and* $(K_1)_{n,\nu}$ *satisfies the determinantal condition*

$$D_{n+1}\big((K_1)_{n,\nu}\big) = 0$$

given in (3.7).

Proof. Without loss of generality, we may consider for each integer $n \in \mathbb{N}$, and $\nu = 0, 1, \ldots, n$, the coefficients K_3, K_2, K_0 satisfying the relation (3.5). Then there are $n + 1$ of the $K_1 = (K_1)_{n,\nu}$ that are roots to the determinant $D_{n+1}(K_1) = 0$ from (3.7), from the equation

$$e^{-z} f''(z) + \big(K_4 e^{3z} + K_3 e^{2z} + K_2 e^{z} + K_1 + K_0 e^{-z}\big) f(z) = 0, \tag{9.2}$$

with $K_4 = -1$.

Suppose (9.2) admits an "eigensolution" $u(z)$. Then we define a sequence of second order partial differential operators

$$L_z := e^{-z}\bigg(\frac{\partial^2}{\partial z^2} + \ell(z)\bigg), \tag{9.3}$$

where $L_z = (L_{n,\nu})_z$ for each integer $n \geq 0$ and

$$\ell(z) := (\ell_{n,\nu})(z) = K_4 e^{4z} + K_3 e^{4z} + K_2 e^{2z} + K_1 e^{z} + K_0.$$

Let $K(z, s)$ be a function of two complex variables. Then we construct a partial differential equation which $K(z, s)$ solves:

$$L_z(K) - L_s(K) = e^{-z}\frac{\partial^2 K}{\partial z^2} - e^{-s}\frac{\partial^2 K}{\partial s^2} + [e^{-z}\ell(z) - e^{-s}\ell(s)]K. \tag{9.4}$$

Now we put

$$K(z, s) = \phi(z)\phi(s)F(\zeta), \qquad \zeta = e^{z} + e^{s}, \tag{9.5}$$

with $\phi(z) = \exp[ae^{2z} + be^{z} + d\,z]$. Here a, b, d are constants that remain to be chosen.

Substituting (9.5) into (9.4) yields

$$L_z(K) - L_s(K) = e^{-z}K_{zz} - e^{-s}K_{ss} + [e^{-z}\ell(z) - e^{-s}\ell(s)] \cdot K$$

that is,

$$
\begin{aligned}
L_z(K) - L_s(K) &= e^{-z}\frac{\partial^2 K(z,s)}{\partial z^2} - e^{-s}\frac{\partial^2 K(z,s)}{\partial s^2}\\
&= \Bigg[\left(\frac{\phi''(z)}{\phi(z)}e^{-z} - \frac{\phi''(s)}{\phi(s)}e^{-s}\right) + 2\frac{F'}{F}\left(\frac{\phi'(z)}{\phi(z)} - \frac{\phi'(s)}{\phi(s)}\right)\\
&\qquad + \frac{F''}{F}(e^z - e^s) + \sum_{j=0}^{4} K_j(e^{(j-1)z} - e^{(j-1)s})\Bigg]K(z,s),
\end{aligned}
$$

which vanishes identically if we set

$$4a^2 + K_4 = 0, \quad d^2 + K_0 = 0,$$

and

$$F''(\zeta) + (4a\zeta + 2b)F'(\zeta) + \big((K_3 + 4ab)\zeta + K_2 + 4a(1+d) + b^2\big)F(\zeta) = 0. \tag{9.6}$$

Note that $K_4 = -1$. We take $a = \epsilon_\infty/2$, $b = -\epsilon_\infty K_3/2$, $d = \epsilon_0\sqrt{-K_0}$, ($\epsilon_0^2 = \epsilon_\infty^2 = 1$). Let $z = -\epsilon_\infty(\zeta - K_3/2)^2$, $F(\zeta) = u(z)$. Then we can transform the differential equation (9.6) to

$$zu'' + (1/2 - z)u'(z) - \big(\epsilon_\infty K_3^2/16 + \epsilon_0\sqrt{-K_0}/2 + \epsilon_\infty K_2/4 + 1/2\big)u = 0,$$

which is the standard form of the confluent hypergeometric equation [16, Chap. 6].

It is well-known that the Kummer function $\Phi(a;c;z)$, where $a = \epsilon_\infty K_3^2/16 + \epsilon_0\sqrt{-K_0}/2 + \epsilon_\infty K_2/4 + 1/2$ and $c = 1/2$, is an entire solution to the confluent hypergeometric equation [16, §6.1 (2)]. Hence

$$
\begin{aligned}
F(\zeta) &= \Phi\left(\frac{\epsilon_\infty K_3^2}{16} + \frac{1+\epsilon_0\sqrt{-K_0}}{2} + \frac{\epsilon_\infty K_2}{4}; \frac{1}{2}; \frac{-\epsilon_\infty(2\zeta - K_3)^2}{4}\right)\\
&= \Phi\left(\frac{\epsilon_\infty K_3^2}{16} + \frac{1+\epsilon_0\sqrt{-K_0}}{2} + \frac{\epsilon_\infty K_2}{4}; \frac{1}{2}; -\epsilon_\infty\left(e^z + e^s - \frac{1}{2}K_3\right)^2\right).
\end{aligned}
$$

Note that $F(\zeta)$ above is a periodic function of period $2\pi i$. Consider the operator

$$
\begin{aligned}
R_z w &:= \frac{d}{dz}\left(e^{-z}\frac{d}{dz}w(z)\right)\\
&\quad + \left(K_4e^{3z} + K_3e^{2z} + K_2e^z + K_1 + \left(K_0 + \frac{1}{4}\right)e^{-z}\right)w(z) = 0, \qquad (9.7)
\end{aligned}
$$

which is self-adjoint $R_z = R_z^*$, and where

$$R_z := e^{z/2} L_z e^{-z/2}$$

is a gauge transform of L_z defined in (9.3). We define

$$T(z) := \int_\Gamma e^{(z+t)/2} K(z,t) w(t)\, dt, \tag{9.8}$$

where $w(t)$ is an eigensolution of (3.4) of the form (3.9) and Γ denotes the line segment $[0, 4\pi i]$. Applying the operator R_z to $T(z)$ and applying the gauge transform yields,

$$\begin{aligned} R_z T(z) &= \int_\Gamma R_z[e^{z/2} K(z,t)] e^{t/2} w(t)\, dt = \int_\Gamma e^{z/2} L_z[K(z,t)] e^{t/2} w(t)\, dt \\ &= e^{z/2} \int_\Gamma e^{t/2} L_t[e^{-t/2} \cdot e^{t/2} K(z,t)] w(t)\, dt = e^{z/2} \int_\Gamma R_t[e^{t/2} K(z,t)] w(t)\, dt, \end{aligned} \tag{9.9}$$

where the K_t assumes the same form as (9.7) with z replaced by t. But then integration-by-parts twice yields

$$\begin{aligned} &\int_\Gamma R_t[e^{t/2} K(z,t)] w(t)\, dt \\ &\quad = \int_\Gamma \left\{ \frac{d}{dt}\left[e^{-t} \frac{d}{dt}\big(e^{t/2} K(z,t)\big)\right] w(t) \right. \\ &\qquad\quad \left. + \left[\sum_{j=1}^4 K_j e^{jt} + \left(K_0 + \frac{1}{4}\right)\right] \big(e^{t/2} K(z,t)\big) w(t) \right\} dt \\ &\quad = e^{-t} \frac{d}{dt}\big(e^{t/2} K(z,t)\big) w(t) \Big|_\Gamma - \int_\Gamma \left[e^{-t} \frac{d}{dt}\big(e^{t/2} K(z,t)\big) w(t) \right] w'(t)\, dt \\ &\qquad\quad + \int_\Gamma \left[\sum_{j=1}^4 K_j e^{jt} + \left(K_0 + \frac{1}{4}\right)\right] \big(e^{t/2} K(z,t)\big) w(t)\, dt \\ &\quad = 0 - \big(e^{t/2} K(z,t) w(t)\big) e^{-t} w'(t) \Big|_\Gamma + \int_\Gamma \big(e^{t/2} K(z,s) w(t)\big)\big(e^{-t} w'(t)\big)' dt \\ &\qquad\quad + \int_\Gamma \left[\sum_{j=1}^4 K_j e^{jt} + \left(K_0 + \frac{1}{4}\right)\right] \big(e^{t/2} K(z,t)\big) w(t)\, dt \\ &\quad = 0 + \int_\Gamma \big(e^{t/2} K(z,t) w(t)\big) \cdot R_t[w(t)]\, dt = 0, \end{aligned}$$

since both $e^{t/2}K(z,t)$, $w(t)$ and $w'(t)$ return to the same values after a $4\pi i$ shift, and that $R_t[w(t)] \equiv 0$. Combining this with (9.8) shows that $T(z)$ is a solution to the equation (9.9). We have thus proved, up to a non-zero constant, that the right-hand side of the (9.1), which we denote by $\tilde{T}(z)$ is a solution to (9.2). Since $\sqrt{-K_0} \in \mathbb{Z}$, the $\tilde{T}(z)$ and moreover its derivative $\tilde{T}'(z)$ are both periodic of period $4\pi i$. It follows from standard Sturm-Liouville theory (see e.g. [15, §2.2, (2.2.1)]) that there is a sequence of real eigenvalues $\lambda_{n,\nu}$, $\nu = 0, 1, 2, \ldots$, and that $\lambda_{n,\nu} \to \infty$ as $\nu \to \infty$. It is known that corresponding to each $\lambda_{n,\nu}$ there can be at most one such eigenfunction satisfying the boundary condition. Since both $\mathrm{BH}_{n,\nu}$ and $\tilde{T}$ satisfy the same equation (9.2) with the same boundary condition $f(0) = f(4\pi i)$ and $f'(0) = f'(4\pi i)$, we have that $\mathrm{BH}_{n,\nu}$ and $\tilde{T}$ can differ by at most a non-zero constant. This completes the proof. We note that for each non-negative integer n, only the first $n+1$ solutions of the form (3.9) corresponding to the eigenvalues $\lambda_{n,\nu}$, $0 \le \nu \le n$, are eigensolutions to the integral equation (9.1) here.

10. Comments and conclusions

We would like to point out that it is known that the Lamé equation is a periodic version of a limiting case of the *Heun equation* [36] and our equation PBHE (1.2) is a periodic version of the *biconfluent Heun equation* (5.3) [36], which is sometimes called the *rotating harmonic oscillator* [32]. Despite the long history of BHE (see e.g. [38], [13]), our understanding of the equation is still far from satisfactory [32, §6]. As far as the authors are aware, the PBHE first appears in Turbiner's study of *quasi-exact solvable* differential operators related to considerations of the Lie algebra $sl(2)$ [40, Eqn. VII, Table 1]. This paper appears to be the first serious study of the PBHE from the Hill's equation viewpoint using differential Galois theory and Nevanlinna's value distribution. In particular, we have shown that the Liouvillian solutions of the PBHE are precisely those solutions which have finite exponent of convergence of zeros (non-oscillatory). We then show that these Liouvillian solutions exhibits novel orthogonality relations.

We have just learnt, in the final stage of preparation of this paper, that our orthogonality results for PBHE can be explained with a new theory of *jointly orthogonal polynomials* proposed recently by Felder and Willwacher [18]; see also [4]. Their theory also covers orthogonality for the Lamé and Whittaker-Hill equations. However, our PBHE and orthogonality weight is more general than what is contained in [18] (i.e., $K_3 = 0$ in (1.2)). On the other hand, it is tempting to think that our orthogonality results for the PBHE are a simple "pull-back" of those in [18]. We argue that there are major differences between those differential equations with rational potentials and their periodic counterparts. First, usually, both the orthogonality and integral equations results for the periodic equations are much more elegant than their "rational counterparts". Second, sometimes, certain results only exist for equations with periodic potentials. For example, although the *double-confluent Heun equation* (DHE) [36] can only have asymptotic expansions for solutions at the origin $x = 0$ which is an irregular singularity, Luo and the first author of this paper have obtained in [12] both the general and (anti-)periodic entire solutions for a periodic counterpart of the DHE, namely the Whittaker-Hill equation (1.8). An additional advantage is that the periodic equation allows us to utilize the far-reaching classical Floquet theory [29], [15]. In the case of Fredholm integral equations of the second kind with periodic symmetric kernel, the boundary condition is also much simpler than their rational kernel counterparts [31].

Finally we recall that Bank [5] suggested an algorithm involving the construction of what he called *approximate square-roots* to find explicit representations of non-oscillatory solutions to the (1.3). We would like to point out that Bank's algorithm is essentially a special case of the case (1) of Kovacic's algorithm that we have applied in this paper. We shall pursue this matter in a subsequent project.

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Appendix A. Kovacic's algorithm

Let us fix the notation. For

$$r = \frac{s}{t}, \qquad s, t \in \mathbb{C}[x],$$

(1) Denote by Γ' be the set of (finite) poles of r, i.e., $\Gamma' = \{c \in \mathbb{C} : t(c) = 0\}$.

(2) Denote by $\Gamma = \Gamma' \cup \{\infty\}$.

(3) By the order $o(r_c)$ of r at $c \in \Gamma'$, we mean the multiplicity of c as a pole of r.

(4) By the order $o(r_\infty)$ of r at ∞, we mean the order of the pole of $r(1/x)$ at $x = 0$. That is $o(r_\infty) = \deg(t) - \deg(s)$.

We list only the case (1) out of the four cases in the original Kovacic algorithm here. We refer either to Kovacic's original article [27], or to [1] (see also [35]) for the full algorithm.

The first case of four sub-cases. In this case $\big[\sqrt{r}\,\big]_c$ and $\big[\sqrt{r}\,\big]_\infty$ means the Laurent series of $\sqrt{r}$ at c and the Laurent series of $\sqrt{r}$ at ∞ respectively. Furthermore, we define $\epsilon(p)$ as follows: if $p \in \Gamma$, then $\epsilon(p) \in \{+, -\}$. Finally, the complex numbers α_c^+, α_c^-, α_∞^+, α_∞^- will be defined in the first step. If the differential equation has no poles, then it can only fall into this case.

Step 1. Search for each $c \in \Gamma' \cup \{\infty\}$ the corresponding situation as follows:

- If $o(r_c) = 0$, then
$$\big[\sqrt{r}\,\big]_c = 0, \qquad \alpha_c^\pm = 0.$$
- If $o(r_c) = 1$, then
$$\big[\sqrt{r}\,\big]_c = 0, \qquad \alpha_c^\pm = 1.$$
- If $o(r_c) = 2$, and $r = \cdots + b(x-c)^{-2} + \cdots$, then
$$\big[\sqrt{r}\,\big]_c = 0, \qquad \alpha_c^\pm = \frac{1 \pm \sqrt{1+4b}}{2}.$$
- If $o(r_c) = 2v \geq 4$, and $r = (a(x-c)^{-v} + \cdots + d(x-c)^{-2})^2 + b(x-c)^{-(v+1)} + \cdots$, then
$$\big[\sqrt{r}\,\big]_c = a(x-c)^{-v} + \cdots + d(x-c)^{-2}, \qquad \alpha_c^\pm = \frac{1}{2}\left(\pm\frac{b}{a} + v\right).$$
- If $o(r_\infty) > 2$, then
$$\big[\sqrt{r}\,\big]_\infty = 0, \qquad \alpha_\infty^+ = 0, \qquad \alpha_\infty^- = 1.$$
- If $o(r_\infty) = 2$ and $r = \cdots + bx^{-2} + \cdots$, then
$$\big[\sqrt{r}\,\big]_\infty = 0, \qquad \alpha_\infty^\pm = \frac{1 \pm \sqrt{1+4b}}{2}.$$
- If $o(r_\infty) = -2v \leq 0$, and $r = (ax^v + \cdots + d)^2 + bx^{v-1} + \cdots$, then
$$\big[\sqrt{r}\,\big]_\infty = ax^v + \cdots + d, \qquad \alpha_\infty^\pm = \frac{1}{2}\left(\pm\frac{b}{a} - v\right).$$

Step 2. Find $D \neq \emptyset$ defined by
$$D = \left\{ m \in \mathbb{Z}_+ : m = \alpha_\infty^{\epsilon_\infty} - \sum_{c\in\Gamma'} \alpha_c^{\epsilon_c},\ \forall (\epsilon_p)_{p\in\Gamma} \right\}.$$

If $D = \emptyset$, then we should start with the case (2). Now if $\#D > 0$, then for each $m \in D$ we search for $\omega \in \mathbb{C}(x)$ such that

$$\omega = \epsilon(\infty)\big[\sqrt{r}\,\big]_\infty + \sum_{c\in\Gamma'}\big(\epsilon(c)\big[\sqrt{r}\,\big]_c + \alpha_c^{\epsilon(c)}(x-c)^{-1}\big).$$

Step 3. For each $m \in D$, search for a monic polynomial P_m of degree m with

$$P_m'' + 2\omega P_m' + (\omega' + \omega^2 - r)P_m = 0.$$

If one is successful, then $y_1 = P_m e^{\int \omega}$ is a solution of the differential equation. Else, case (1) cannot hold.

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