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ON SOME RESULTS ON H-FUNCTIONS ASSOCIATED WITH ORTHOGONAL POLYNOMIALS

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In this paper the author has obtained some results involving H-functions and Gegenbauer (Ultraspherical) classical orthogonal polynomials with the help of the known series and orthogonality-property for the polynomials. Also certain known interesting results have been obtained as particular cases of the formulae established here on specializing the parameters.

1. Introduction.

Some authors have made an attempt to unify and to extend certain results of special functions to Fourier series. Carlson and Greiman [3] have obtained a cosine series for Gegenbauer's function. MacRobert [7] and [8] has given a cosine and a sine series for the E-functions. Narain Roop [10] and Jain [6] have also obtained Fourier series for Meijer's G-functions. Recently, Parashar [9] and Shah [12] have established some results on Fourier series for H-functions.

Fox [5, p. 408] has introduced the H-function in the form of a Mellin–Barnes type integral as

$$(1.1)\qquad H_{p,q}^{m,n}\left[x \,\middle|\, \begin{matrix}\{(a_p,\alpha_p)\}\\ \{(b_q,\beta_q)\}\end{matrix}\right] = \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+\beta_j s)\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j s)}\, x^s\, ds\,,$$

where $\{(a_p,\alpha_p)\}$ denotes the set of parameters $(a_1,\alpha_1), (a_2,\alpha_2),\ldots,(a_p,\alpha_p)$ and similarly for $\{(b_q,\beta_q)\}$ and

(i) $1 \leqq m \leqq q,\ 0 \leqq n \leqq p$;

(ii) α's and β's are positive;

(iii) $p+q < 2(m+n)$, $|\arg x| < [m+n-\frac{1}{2}(p+q)]\pi$;

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(iv) further the contour L runs from $c-i\infty$ to $c+i\infty$ such that the poles of $\Gamma(b_h-\beta_h s)$, $h=1,2,\ldots,m$, lie to the right and the poles of $\Gamma(1-a_j+\alpha_j s)$, $j=1,2,\ldots,n$, lie to the left of L.

The aim of this paper is to obtain an expansion-formula (3.1) for the H-function in terms of classical orthogonal Gegenbauer polynomials using the known series. Further, the result has been utilized to evaluate the integral (3.2) involving the product of the H-function and the Gegenbauer (ultraspherical) polynomials in view of orthogonality-property for the polynomials. The H-function is a very general function. Hence many known special functions are obtainded by particular choice of parameters. A number of known and interesting particular cases have also been derived.

2. Previous results.

The following results have been employed during the course of the present investigation:

(a) The series due to Richard Askey [1]:

$$(2.1)\qquad (\sin\theta)^{2\gamma}\, C_l^{\gamma}(\cos\theta) = \sum_{k=0}^{\infty} A_{k,l}^{\gamma,\xi}\, C_{l+2k}^{\xi}(\cos\theta)\,(\sin\theta)^{2\xi};$$

where

$$A_{k,l}^{\gamma,\xi} = \frac{2^{2\xi-2\gamma}\Gamma(\xi)(l+2k+\xi)(l+2k)!\,\Gamma(l+2\gamma)\Gamma(l+k+\xi)\Gamma(k+\xi-\gamma)}{l!\,k!\,\Gamma(\gamma)\Gamma(\xi-\gamma)\Gamma(l+k+\gamma+1)\Gamma(l+2k+2\xi)}$$

and $\frac{1}{2}(\xi-1)<\gamma<\xi$, $A_{k,l}^{\gamma,\xi}>0$.

Setting $\xi=1$ in (2.1), it reduces to a known series given by Szegö [13]:

$$(2.2)\qquad (\sin\theta)^{2\gamma-1}\, C_l^{\gamma}(\cos\theta) = \sum_{k=0}^{\infty} A_{k,l}^{\gamma}\sin(l+2k+1)\theta\,,$$

for $\gamma>0$, $\gamma\neq 1,2,\ldots$, and

$$A_{k,l}^{\gamma} = \frac{2^{2-2\gamma}(l+k)!\,\Gamma(l+2\gamma)\Gamma(k-\gamma+1)}{\Gamma(\gamma)\Gamma(1-\gamma)k!\,l!\,\Gamma(l+k+\gamma+1)}\,,$$

and $C_l^{\gamma}(\cos\theta)$ [11, p. 277, (1)] is defined in the form

$$(2.3)\qquad (1-2t\cos\theta+t^2)^{-\gamma} = \sum_{l=0}^{\infty} C_l^{\gamma}(\cos\theta)\, t^l\,.$$

Substituting $l=0$ and replacing γ by $1-s$ in (2.2), we obtain a known Fourier series due to MacRobert [8]:

$$(2.4)\qquad \frac{(\pi)^{\frac{1}{2}}}{2}\frac{\Gamma(2-s)}{\Gamma(\frac{3}{2}-s)}(\sin\theta)^{1-2s} = \sum_{r=0}^{\infty}\frac{(s)_r}{(2-s)_r}\sin(2r+1)\theta, \qquad 0\leqq\theta\leqq\pi,\ \mathrm{Re}(s)\leqq\tfrac{1}{2}\,.$$

(b) The expansion-formula [11, p. 283, (37)]:

$$C_l^\gamma(\cos\theta) = \sum_{k=0}^{l} \frac{(\gamma)_k(\gamma)_{l-k}}{k!\,(l-k)!}\cos(l-2k)\theta$$
$$= \sum_{k=0}^{l} \frac{(-l)_k(\gamma)_k(\gamma)_l}{k!\,l!\,(1-\gamma-l)_k}\cos(l-2k)\theta\,. \tag{2.5}$$

(c) The orthogonality-property for Gegenbauer polynomials [11, p. 281, (27) and (28)]:

If $\operatorname{Re}(\gamma) > -\frac{1}{2}$, then

$$\int_{-1}^{1}(1-x^2)^{\gamma-\frac{1}{2}}C_l^\gamma(x)C_m^\gamma(x)\,dx = \begin{cases} 0 & \text{if } m \neq l\,, \\ \dfrac{(2\gamma)_l\Gamma(\frac{1}{2})\Gamma(\gamma+\frac{1}{2})}{l!\,(\gamma+l)\Gamma(\gamma)}, & \text{if } m = l\,. \end{cases} \tag{2.6}$$

(d) Legendre's duplication formula [11, p. 24, (2)]:

$$(\pi)^{\frac{1}{2}}\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma(z+\tfrac{1}{2})\,. \tag{2.7}$$

3. Main results.

The main results to be established are contained in the following Expansion formula (i) and Integral (ii).

(i) *Expansion-formula*:

$$\sum_{u=0}^{\infty} H_{p+3,\,q+3}^{m+2,\,n+1}\left[2^2 z \,\middle|\, \begin{matrix}(2-\xi-u,1),\ \{(a_p,\alpha_p)\},\ (1,1),\ (l+u+2,1)\\ (l+2,2),\ \{(b_q,\beta_q)\},\ (2-\xi,1)\end{matrix}\right]\cdot \tag{3.1}$$
$$\cdot\frac{2^{2\xi-2}\Gamma(\xi)(l+2u+\xi)(l+2u)!\,\Gamma(l+u+\xi)}{l!\,u!\,\Gamma(l+2u+2\xi)}\,C_{l+2u}^{\xi}(\cos\theta)\,(\sin\theta)^{2\xi}$$
$$= \sin^2\theta\sum_{k=0}^{l}\frac{(-l)_k}{k!\,l!}\,H_{p+3,\,q+3}^{m+2,\,n+1}\,\cdot$$
$$\cdot\left[\frac{z}{\sin^2\theta}\,\middle|\,\begin{matrix}(l+1,1),\ \{(a_p,\alpha_p)\},\ (1,1),\ (1,1)\\ (1+k,1),\ (1+l,1),\ \{(b_q,\beta_q)\},\ (l-k+1,1)\end{matrix}\right]\cos(l-2k)\theta\,,$$

where $0 \leqq \theta \leqq \pi$ and

$$\operatorname{Re}[l+2(1-a_i/\alpha_i)] > 0, \quad i = 0,1,\ldots,n;$$
$$\operatorname{Re}(1+b_h/\beta_h) > 0, \quad h = 1,2,\ldots,m\,,$$

$$\sum_1^p\alpha_j - \sum_1^q\beta_j \leqq 0, \quad \sum_1^n\alpha_j - \sum_{n+1}^p\alpha_j + \sum_1^m\beta_j - \sum_{m+1}^q\beta_j \equiv \lambda > 0,$$
$$|\arg z| < \tfrac{1}{2}\lambda\pi\,.$$

(ii) *Integral*:

(3.2)

$$\sum_{k=0}^{l} \frac{(-l)_k}{k!l!} \int_0^{\pi} H^{m+2,\,n+1}_{p+3,\,q+3}\left[\frac{z}{\sin^2\theta}\,\middle|\,\begin{matrix}(l+1,1),\ \{(a_p,\alpha_p)\},\ (1,1),\ (1,1)\\ (1+k,1),\ (1+l,1),\ \{(b_q,\beta_q)\},\ (l-k+1,1)\end{matrix}\right]\cdot$$

$$\cdot \sin^2\theta\cos(l-2k)\theta C^{\xi}_{l+2\gamma}(\cos\theta)\,d\theta$$

$$=\frac{\pi}{2}\frac{\Gamma(l+\gamma+\xi)}{l!\gamma!\Gamma(\xi)} H^{m+2,\,n+1}_{p+3,\,q+3}\left[2^2z\,\middle|\,\begin{matrix}(2-\xi-\gamma,1),\ \{(a_p,\alpha_p)\},\ (1,1),\ (l+\gamma+2,1)\\ (l+2,2),\ \{(b_q,\beta_q)\},\ (2-\xi,1)\end{matrix}\right],$$

where $0\leqq\theta\leqq\pi$ and $\gamma=0,1,2,\ldots$.

Proof of (i). To establish (3.1), expressing the H-function in Mellin–Barnes type integral (1.1) on the left of (3.1), the expression becomes

(3.3) $$\sum_{u=0}^{\infty}\frac{1}{2\pi i}\int_L$$

$$\frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_js)\Gamma(l+2-2s)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_js)\Gamma(\xi+u-1+s)\,2^{2s}z^s}{\prod_{j=m+1}^{q}\Gamma(1-b_j+\beta_js)\Gamma(\xi-1+s)\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_js)\Gamma(1-s)\Gamma(l+u+2-s)}\,ds\cdot$$

$$\cdot\left\{\frac{2^{2\xi-2}\Gamma(\xi)(l+2u+\xi)(l+2u)!\,\Gamma(l+u+\xi)}{l!u!\Gamma(l+2u+2\xi)}C^{\xi}_{l+2u}(\cos\theta)\,(\sin\theta)^{2\xi}\right\},$$

valid under the conditions given in (3.1), and the poles of $\Gamma(l+2-2s)$ lie to the right and those of $\Gamma(\xi+u-1+s)$ lie to the left of the contour L.

On changing the order of summation and integration in view of [2, p. 500] and the conditions given in (3.1), it reduces to

(3.4) $$\frac{1}{2\pi i}\int_L\frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_js)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_js)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+\beta_js)\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_js)}z^s\cdot$$

$$\cdot\left\{\sum_{u=0}^{\infty}\frac{2^{2s+2\xi-2}\Gamma(l+2-2s)\Gamma(\xi+u-1+s)\Gamma(\xi)(l+2u+\xi)(l+2u)!\,\Gamma(l+u+\xi)}{\Gamma(\xi-1+s)\Gamma(1-s)\Gamma(l+u+2-s)l!u!\Gamma(l+2u+2\xi)}\cdot\right.$$

$$\left.\cdot\,C^{\xi}_{l+2u}(\cos\theta)\,(\sin\theta)^{2\xi}\right\}ds\,.$$

Replacing γ by $1-s$ in (2.1) and then using this in (3.4), we get

$$(3.5)\qquad \frac{1}{2\pi i}\int_L$$

$$\frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)z^s}{\prod_{j=m+1}^{q}\Gamma(1-b_j+\beta_j s)\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j s)}\{(\sin\theta)^{2-2s}C_l^{1-s}(\cos\theta)\}\,ds\,.$$

By virtue of (2.5) for $\gamma=1-s$ and then change of the order of integration and summation (which is permitted), the expression (3.5) takes the form

$$(3.6)\qquad \sin^2\theta\sum_{k=0}^{l}\frac{(-l)_k}{k!\,l!}\cdot$$

$$\cdot\left\{\frac{1}{2\pi i}\int_L\frac{\Gamma(1+k-s)\Gamma(1+l-s)\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+\beta_j s)\Gamma(-l+k+s)\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j s)}\cdot\right.$$

$$\left.\cdot\frac{\Gamma(-l+s)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)}{\Gamma(1-s)\Gamma(1-s)}\,\frac{z^s}{(\sin^2\theta)^s}\,ds\right\}\cos(l-2k)\theta\,,$$

which yields the expression on the right of (3.1) on interpreting (1.1).

PROOF OF (ii) To prove (3.2), we multiply both sides of (3.1) by $C_{l+2\gamma}^{\xi}(\cos\theta)$ and integrate with respect to θ over $(0,\pi)$, and then change the order of integration and summation (which is easily seen to be justified), we obtain

$$(3.7)\qquad \sum_{u=0}^{\infty}H_{p+3,\,q+3}^{m+2,\,n+1}\left[2^2z\,\middle|\,\begin{matrix}(2-\xi-u,1),\,\{(a_p,\alpha_p)\},\,(1,1),\,(l+u+2,1)\\(l+2,2),\,\{(b_q,\beta_q)\},\,(2-\xi,1)\end{matrix}\right]\cdot$$

$$\cdot\frac{2^{2\xi-2}\Gamma(\xi)\,(l+2u+\xi)\,(l+2u)!\,\Gamma(l+u+\xi)}{l!\,u!\,\Gamma(l+2u+2\xi)}\int_0^{\pi}(\sin\theta)^{2\xi}C_{l+2u}^{\xi}(\cos\theta)\,C_{l+2\gamma}^{\xi}(\cos\theta)\,d\theta$$

$$=\sum_{k=0}^{l}\frac{(-l)_k}{k!\,l!}\int_0^{\pi}H_{p+3,\,q+3}^{m+2,\,n+1}\left[\frac{z}{\sin^2\theta}\,\middle|\,\begin{matrix}(l+1,1),\,\{(a_p,\alpha_p)\},\,(1,1),\,(1,1)\\(1+k,1),\,(1+l,1),\,\{(b_q,\beta_q)\},\,(l-k+1,1)\end{matrix}\right]\cdot$$

$$\cdot\sin^2\theta\cos(l-2k)\theta\,C_{l+2\gamma}^{\xi}(\cos\theta)\,d\theta\,.$$

Now we make use of (2.6) with $x=\cos\theta$ and (2.7) on the left of (3.7). This ultimately yields the right hand side of (3.2).

4. Particular cases.

It may be noted that on account of the generalized nature of the H-function, several new interesting results can be derived with proper choice of parameters. Hence the formulae established in this paper are of general character.

Three particular cases should be mentioned here:

(i) In (3.1) and (3.2), setting $l=0, \xi=1$ and using (2.2), we obtain the known results due to Parashar [9, p. 1083, (1.3), and p. 1084, (2.6)].

(ii) By taking $l=0, \xi=1$ and $\alpha_j=\beta_h=1$, $j=1,2,\ldots,p$, $h=1,2,\ldots,q$ etc., in (3.1) and (3.2), the known results on series and integrals involving Meijer's G-functions given by Jain [6] and Narain Roop [10] can be obtained.

(iii) In (3.1), using the known relation [4, p. 215, (2)]

$$H^{p,1}_{q+1,p}\left[x \,\middle|\, \begin{matrix}(1,1), \{(b_q,1)\} \\ \{(a_p,1)\}\end{matrix}\right] = G^{p,1}_{q+1,p}\left(x \,\middle|\, \begin{matrix}1, b_1, \ldots, b_q \\ a_1, \ldots, a_p\end{matrix}\right)$$

$$= E(p;\, a_r\!:\, q;\, b_s\!:\, x)\,,$$

where E is MacRobert's E-function, we obtain the Fourier series of the E-function due to MacRobert [8].

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