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A THEOREM ON CONVERGENCE TO A LÉVY PROCESS

ANDERS GRIMVALL

1. Introduction.

Let $X_{n,1},X_{n,2},\ldots,X_{n,k_n}$, $n=1,2,\ldots$, be a double sequence of random variables such that

(1) $$X_{n,1},X_{n,2},\ldots,X_{n,k_n} \text{ are independent for every } n$$

and

(2) $$\lim_{n\to\infty} \max_j P(|X_{n,j}|>\varepsilon) = 0 \quad \text{for every } \varepsilon>0 .$$

For the study of the asymptotic behaviour of the partial sums:

(3) $$S_{n,m} = \sum_{j=1}^{m} X_{n,j}, \quad m=1,2,\ldots,k_n,\ n=1,2,\ldots ,$$

it is natural to consider a sequence of random broken lines:

(4) $$\begin{aligned} Y_{n,\omega}(t) &= 0 && \text{if } 0=t_{n,0}\leqq t<t_{n,1} ,\\ &= S_{n,m}-\gamma_{n,m} && \text{if } t_{n,m}\leqq t<t_{n,m+1},\ 0<m<k_n ,\\ &= S_{n,k_n} && \text{if } t=t_{n,k_n}=1 , \end{aligned}$$

where $0=t_{n,0}<t_{n,1}<\ldots<t_{n,k_n}=1$ is a set of division time-points for each $n=1,2,\ldots$ and $\gamma_{n,m}$ are adjusting constants. $Y_n(t)$ is a random process whose sample functions are in the space $D[0,1]$ of all functions on $[0,1]$ with no discontinuities of the second kind. The space $D[0,1]$ with the Skorohod topology is a Polish space, i.e., a topological space homeomorphic with a complete separable metric space. Let P_n be the probability law governing the sample function of Y_n and hence a regular probability measure on $D[0,1]$ for every n. Thus the study of partial sums $S_{n,m}$ is reduced to the problem of convergence of the sequence $\{P_n\}$.

Suppose that $\{P_n\}$ is weakly convergent. Then $\{S_{n,k_n}, n=1,2,\ldots\}$ is necessarily convergent in law, because $S_{n,k_n}=Y_n(1)$. Suppose conversely that $\{S_{n,k_n}\}_n$ is convergent in law. Then we can find $\gamma_{n,m}$ and $t_{n,m}$ such that $\{P_n\}$ is conditionally weakly compact. This is the main result of our present paper. Prohorov [1] discussed similar problems for the following special cases:

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(a) $X_{n,1}, X_{n,2}, \ldots, X_{n,k_n}$ are identically distributed, where we can take $\gamma_{n,m}=0$ and $t_{n,m}=m/k_n$.

(b) $X_{n,1}, X_{n,2}, \ldots, X_{n,k_n}$ satisfy the Lindeberg conditions, where we can take $\gamma_{n,m}=E(S_{n,m})$ and $t_{n,m}=V(S_{n,m})/V(S_{n,k_n})$.

In the general case we are going to discuss in this paper we will use the central value and the dispersion to determine $\gamma_{n,m}$ and $t_{n,m}$.

In Section 2 we will first review the definitions and the basic properties of the central value and the dispersion following Ito [2] and then prove a few new facts which will be used in Section 3. In Section 3 we will prove our main theorem on convergence to a Lévy process and use it to show the equivalence of several definitions of the infinitely divisible laws.

2. Definitions and elementary facts about dispersion and central value.

Following [2] or [3] we adopt the following definition.

DEFINITION. The central value $\gamma=\gamma(\mu)$ of a one-dimensional probability measure μ is defined to be the unique real number γ such that

$$\int_{R^1} \arctan(x-\gamma)\,\mu(dx) = 0 .$$

The dispersion $\delta(\mu)$ is defined by

$$\delta(\mu) = -\log \iint_{R^2} e^{-|x-y|}\,\mu(dx)\mu(dy) .$$

For a realvalued random variable X with probability law μ we have the natural definitions $\gamma(X)=\gamma(\mu)$ and $\delta(X)=\delta(\mu)$.

A measure μ_1 is called a factor of μ if $\mu=\mu_1 * \mu_2$.

PROPOSITION 1. *Let φ_X be the characteristic function of X. Then*

$$\delta(X) = -\log \pi^{-1}\int_{R^1} |\varphi_X(\xi)|^2 \frac{d\xi}{1+\xi^2}.$$

PROPOSITION 2. $\mu=\delta_c$ *(the δ-distribution concentrated at c) if and only if $\delta(\mu)=0$ and $\gamma(\mu)=c$.*

PROPOSITION 3. *If $X \to Y$* i.p., *then $\delta(X) \to \delta(Y)$ and $\gamma(X) \to \gamma(Y)$.*

PROPOSITION 4. *$\delta(X) \to 0$ if and only if $X-\gamma(X) \to 0$* i.p.

PROPOSITION 5. *If X and Y are independent, then $\delta(X) \leqq \delta(X+Y)$ with equality if and only if $Y=$* const. a.s.

PROPOSITION 6. *Let $\mathscr{M}$ be a set of one-dimensional probability measures and $\mathscr{M}'$ the set of all factors of probability measures in $\mathscr{M}$. Then $\{\mu'-\gamma(\mu')\}_{\mu' \in \mathscr{M}'}$ is conditionally compact if $\mathscr{M}$ is conditionally compact.*

2.2. Further properties of the central value and dispersion.

For the proof of our main result we need some more facts about central value and dispersion.

LEMMA 1. *Given $\varepsilon>0$ and $c>0$, we can find $d=d(\varepsilon,c)$ such that if $\delta(X+Y)-\delta(X)<d$ for some random variable X independent of Y with $\delta(X) \leqq c$, then $\delta(Y)<\varepsilon$.*

PROOF. First we note the following fact: If $[-a,a]$ is a compact interval and B a Borel set contained in $[-a,a]$ such that $|B| \geqq c>0$, then for some constant $K=K(c,a)>0$ and for all x and all B of the above type,

$$\int_B (1-\cos \xi x)\, d\xi \;\geqq\; K\, x^2/(1+x^2)\,. \tag{5}$$

Now let X and Y be independent random variables with characteristic functions φ_X and φ_Y, respectively. Then

$$e^{\delta(X+Y)-\delta(X)} = I_1/(I_1-I_2)\,,$$

where

$$I_1 = \int_{-\infty}^{\infty} |\varphi_X(\xi)|^2 \frac{d\xi}{1+\xi^2}, \quad I_2 = \int_{-\infty}^{\infty} |\varphi_X(\xi)|^2 (1-|\varphi_Y(\xi)|^2) \frac{d\xi}{1+\xi^2}.$$

If $\delta(X) \leqq c$ we get $\pi \geqq I_1 \geqq \pi e^{-c} = K_1(c)>0$, and

$$1-|\varphi_Y(\xi)|^2 = \int_{-\infty}^{\infty} (1-\cos \xi x)\, \mu(dx)$$

for some probability measure μ. By interchanging the order of integration,

$$I_2 = \int_{-\infty}^{\infty} \mu(dx) \int_{-\infty}^{\infty} |\varphi_X(\xi)|^2 \frac{1-\cos \xi x}{1+\xi^2}\, d\xi\,.$$

Now choose $K_2 = K_2(c)$ such that $\int_{K_2}^{\infty} d\xi/(1+\xi^2) = \frac{1}{4}K_1$. Then

$$\int_{-K_2}^{K_2} |\varphi_X(\xi)|^2 \, d\xi/(1+\xi^2) \geqq \tfrac{1}{2}K_1 .$$

Let $E = \{\xi;\ |\xi| \leqq K_2,\ |\varphi_X(\xi)|^2 \geqq \frac{1}{8}K_1/K_2\}$. Then we have

$$\int_{[-K_2, K_2]\setminus E} |\varphi_X(\xi)|^2 \, d\xi/(1+\xi^2) \leqq 2K_2 \tfrac{1}{8}K_1/K_2 = \tfrac{1}{4}K_1$$

and therefore

$$\int_E |\varphi_X(\xi)|^2 \, d\xi/(1+\xi^2) \geqq \tfrac{1}{4}K_1, \qquad |E| \geqq \tfrac{1}{4}K_1 .$$

By (5) there exists a constant $K_3 = K_3(c)$ such that

$$\begin{aligned} I_2 &\geqq \int_{-\infty}^{\infty} \mu(dx) \int_E |\varphi_X(\xi)|^2 \frac{1-\cos \xi x}{1+\xi^2} \, d\xi \\ &\geqq \int_{-\infty}^{\infty} \mu(dx) \frac{K_1}{8K_2} \frac{1}{1+K_2{}^2} \int_E (1-\cos \xi x)\, d\xi \geqq K_3 \int_{-\infty}^{\infty} \frac{x^2}{1+x^2} \mu(dx) . \end{aligned}$$

It follows that

$$\int_{-\infty}^{\infty} \frac{x^2}{1+x^2} \mu(dx) \leqq \frac{I_2}{K_3} = \frac{I_1 (e^{\delta(X+Y)-\delta(X)}-1)}{K_3\, e^{\delta(X+Y)-\delta(X)}} \leqq \frac{\pi}{K_3} (e^{\delta(X+Y)-\delta(X)}-1) .$$

Observing that

$$\int_{-\infty}^{\infty} \frac{x^2}{1+x^2} \mu(dx) \to 0 \Rightarrow \delta(Y) \to 0 ,$$

the lemma is proved.

LEMMA 2. *Given $\varepsilon > 0$ there exists a $d = d(\varepsilon) > 0$ such that, for all k and $X_1, X_2, \ldots, X_k$ independent,*

$$|\gamma(X_1 + \ldots + X_k) - \textstyle\sum_{j=1}^{k} \gamma(X_j)| < \varepsilon$$

as soon as $\delta(X_1 + \ldots + X_k) < d$.

PROOF. Assume that the lemma is false. Then for some $\varepsilon > 0$ we can find a double sequence $X_{n,1}, X_{n,2}, \ldots, X_{n,k_n}$, $n = 1, 2, \ldots$ of random variables such that:

$$X_{n,1}, X_{n,2}, \ldots, X_{n,k_n} \text{ are independent for each } n,$$
$$\delta(X_{n,1} + X_{n,2} + \ldots + X_{n,k_n}) \to 0 \text{ as } n \to \infty,$$
$$\gamma(X_{n,1}) = \gamma(X_{n,2}) = \ldots = \gamma(X_{n,k_n}) = 0,\ n = 1, 2, 3, \ldots,$$
$$|\gamma(X_{n,1} + X_{n,2} + \ldots + X_{n,k_n})| \geqq \varepsilon,\ n = 1, 2, 3, \ldots.$$

It is easy to see that we can assume that $\gamma(X_{n,1} + \ldots + X_{n,k_n}) \to \varepsilon$ as $n \to \infty$.

Let $\varphi_{n,j}$ denote the characteristic function of $X_{n,j}$. Then we have, by Proposition 4,

$$\varphi_{n,1}(\xi)\, \varphi_{n,2}(\xi) \ldots \varphi_{n,k_n}(\xi) \to e^{i\xi\varepsilon}. \tag{6}$$

Letting $\xi_0 = \pi/\varepsilon$ in (6) we get

$$\left[\int_{-\infty}^{\infty} \cos \xi_0 x\, \mu_{n,1}(dx) + i \int_{-\infty}^{\infty} \sin \xi_0 x\, \mu_{n,1}(dx)\right] \ldots$$
$$\left[\int_{-\infty}^{\infty} \cos \xi_0 x\, \mu_{n,k_n}(dx) + i \int_{-\infty}^{\infty} \sin \xi_0 x\, \mu_{n,k_n}(dx)\right] \to -1.$$

Since $\varphi_{n,j}(\xi) \rightrightarrows 1$ uniformly in j as $n \to \infty$, we have

$$\left|\sum\nolimits'_j \int_{-\infty}^{\infty} \sin \xi_0 x\, \mu_{n,j}(dx)\right| > 1$$

for all n large enough, where $\sum'$ denotes the sum over all positive or all negative terms. Further,

$$\sin \xi_0 x = \frac{\xi_0 x + H_1(x) x^2}{1 + x^2}, \quad \arctan x = \frac{x + H_2(x) x^2}{1 + x^2},$$

where $H_1(x)$ and $H_2(x)$ are bounded on the real line. By assumption

$$\sum\nolimits' \int_{-\infty}^{\infty} \xi_0 \arctan x\, \mu_{n,j}(dx) = 0.$$

Therefore

$$\left|\sum\nolimits' \int_{-\infty}^{\infty} \frac{[H_1(x) - \xi_0 H_2(x)] x^2}{1 + x^2} \mu_{n,j}(dx)\right| > 1,$$

and so we can find a constant $c>0$ such that

$$(7) \qquad \sum_{j=1}^{k_n} \int_{-\infty}^{\infty} \frac{x^2}{1+x^2} \mu_{n,j}(dx) > c > 0 \quad \text{for all } n .$$

On the other hand it follows from (6) that

$$|\varphi_{n,1}(\xi)|^2 \, |\varphi_{n,2}(\xi)|^2 \cdots |\varphi_{n,k_n}(\xi)|^2 \rightrightarrows 1 \quad \text{as } n \to \infty .$$

Therefore

$$\sum_{j=1}^{k_n} (1-|\varphi_{n,j}(\xi)|^2) = \sum_{j=1}^{k_n} \int_{-\infty}^{\infty} (1-\cos \xi x) \, \tilde{\mu}_{n,j}(dx) \rightrightarrows 0 \quad \text{as } n \to \infty$$

($\tilde{\mu}_{n,j}$ denotes the symmetrization of $\mu_{n,j}$), and from this

$$\int_{-1}^{1} \sum_{j=1}^{k_n} (1-|\varphi_{n,j}(\xi)|^2) \, d\xi \to 0 ,$$

which implies

$$(8) \qquad \sum_{j=1}^{k_n} \int_{-\infty}^{\infty} \frac{x^2}{1+x^2} \tilde{\mu}_{n,j}(dx) \to 0 \quad \text{as } n \to \infty .$$

Comparing (7) and (8) we can see that Lemma 2 is proved, when we have verified:

Lemma 3. *If μ is a one-dimensional probability measure with $\gamma(\mu)=0$ and if $\tilde{\mu}$ denotes the symmetrization of μ, we have*

$$\int_{-\infty}^{\infty} \frac{x^2}{1+x^2} \tilde{\mu}(dx) \geqq \frac{1}{16} \int_{-\infty}^{\infty} \frac{x^2}{1+x^2} \mu(dx) .$$

Proof. We can assume that $m \geqq 0$, where m is a median of μ, that is, $\mu(-\infty, m] \geqq \frac{1}{2}$ and $\mu[m,\infty) \geqq \frac{1}{2}$. First we note that

$$I = \int_{-\infty}^{\infty} \frac{x^2}{1+x^2} \tilde{\mu}(dx) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{(x-y)^2}{1+(x-y)^2} \mu(dx) \mu(dy)$$

$$\geqq \int_{x>0} \int_{y \leqq 0} \frac{x^2}{1+x^2} \mu(dx) \mu(dy) + \int_{x \leqq 0} \int_{y \geqq 0} \frac{x^2}{1+x^2} \mu(dx) \mu(dy)$$

$$\geqq \mu(-\infty, 0] \int_{x>0} \frac{x^2}{1+x^2} \mu(dx) + \mu[0,\infty) \int_{x \leqq 0} \frac{x^2}{1+x^2} \mu(dx) ,$$

and from this also

$$I \geqq \tfrac{1}{2} \int_{-\infty}^{\infty} \frac{(x-m)^2}{1+(x-m)^2}\,\mu(dx)\,.$$

If $\mu(-\infty, 0] \geqq \frac{1}{16}$, it follows immediately that

$$I \geqq \frac{1}{16} \int_{-\infty}^{\infty} \frac{x^2}{1+x^2}\,\mu(dx)\,,$$

so it is enough to consider the case $\mu(-\infty, 0] \leqq \frac{1}{16}$. Then

$$\int_{x \leqq 0} \frac{x^2}{1+x^2}\,\mu(dx) \leqq \frac{16}{15} I$$

and

$$\int_{2m}^{\infty} \frac{x^2}{1+x^2}\,\mu(dx) \leqq 4 \int_{2m}^{\infty} \frac{(x-m)^2}{1+(x-m)^2}\,\mu(dx) \leqq 8I\,.$$

Since

$$\tfrac{1}{2}\pi\,\frac{1}{16} \geqq \int_{x<0} \arctan(-x)\,\mu(dx) = \int_{x>0} \arctan x\,\mu(dx)$$

$$\geqq \tfrac{1}{2}\arctan m \geqq \tfrac{1}{4}\arctan 2m\,,$$

we first get $m \leqq \frac{1}{4}$ and then we have

$$\begin{aligned}
\int_{x \leqq 0} \frac{x^2}{1+x^2}\,\mu(dx) &\geqq \int_{x \leqq -2m} \frac{x^2}{1+x^2}\,\mu(dx) \\
&\geqq \int_{x \leqq -2m} m \arctan(-x)\,\mu(dx) \\
&\geqq \frac{3}{4} \int_{x<0} m \arctan(-x)\,\mu(dx) \\
&= \frac{3}{4} \int_{x>0} m \arctan x\,\mu(dx) \\
&\geqq \frac{3m}{4} \int_{0}^{2m} \arctan x\,\mu(dx) \\
&\geqq \frac{3m}{4} \int_{0}^{2m} \frac{1}{4m}\,\frac{x^2}{1+x^2}\,\mu(dx) = \frac{3}{16} \int_{0}^{2m} \frac{x^2}{1+x^2}\,\mu(dx)\,.
\end{aligned}$$

Therefore

$$\int_0^{2m} \frac{x^2}{1+x^2}\,\mu(dx) \leqq \frac{16}{3} \int_{x\leqq 0} \frac{x^2}{1+x^2}\mu(dx) \leqq 6I\,,$$

and so we have

$$\int_{-\infty}^{\infty} \frac{x^2}{1+x^2}\mu(dx) = \int_{x\leqq 0} + \int_0^{2m} + \int_{x\geqq 2m} \frac{x^2}{1+x^2}\,\mu(dx) \leqq 16I\,.$$

LEMMA 4. *Given $\varepsilon>0$ and $c>0$ we can find a $d=d(\varepsilon,c)>0$ such that $\delta(X+Y)-\delta(X)<\varepsilon$ for all pairs (X,Y) of independent random variables satisfying $\delta(Y)<d$ and $\delta(X)<c$.*

PROOF. Use Proposition 1.

3.1. A theorem on convergence to a Lévy process.

Let $X_{n,1},X_{n,2},\ldots,X_{n,k_n}$, $n=1,2,\ldots$, be a double sequence of random variables subject to conditions (1) and (2). Assume further that $X_{n,j}$ has distribution $\mu_{n,j}$ and that

(9) $$\mu_{n,1} * \mu_{n,2} * \cdots * \mu_{n,k_n} \xrightarrow{w} \mu\,,$$

where $\delta(\mu)>0$. Define for fixed $\omega\in(\Omega,\mathscr{B},P)$ and $n\in N$ a broken line

$$Y_{n,\omega}(t) = \begin{cases} 0 & \text{if } 0\leqq t<t_{n,1} \\ S_{n,m}-\sum_{j=1}^{m}\gamma(X_{n,j})+t_{n,m}\sum_{j=1}^{k_n}\gamma(X_{n,j}) & \text{if } t_{n,m}\leqq t<t_{n,m+1} \\ S_{n,k_n} & \text{if } t=1\,. \end{cases}$$

We take

$$t_{n,m} = \frac{\delta(\mu_{n,1}*\cdots*\mu_{n,m})}{\delta(\mu_{n,1}*\cdots*\mu_{n,k_n})}\,,$$

$n=1,2,\ldots$. Then $Y_{n,\omega}(t)$ defines a distribution P_n in the function space $D[0,1]$, which is assumed to have the Skorohod topology ([4, p. 109]). Without restriction we can assume that $D[0,1]$ is a complete separable metric space ([4, p. 113]). We are going to prove

THEOREM 1. *The sequence $\{P_n\}_n$ of distributions in $D[0,1]$ defined above is weakly conditionally compact and every limit distribution P corresponds stochastically to a process $X_t(\omega)$, $t\in[0,1]$, continuous in probability and with independent increments (a Lévy process). Moreover, $X_1(\omega)$ has probability law μ.*

3.2. Prohorov's lemma.

We now consider an arbitrary double sequence $\tilde{X}_{n,j}$, $j=1,2,\ldots,k_n$, $n=1,2,\ldots$, of random variables, and a family of division time-points

$t_{n,j}, j=1,2,\ldots,k_n, n=1,2,\ldots$. We construct the random broken lines with vertices $(t_{n,m}, \sum_{j=1}^{m}\tilde{X}_{n,j})$. In Prohorov [1, p. 193] the following lemma is proved.

LEMMA 5. *The sequence $\{\tilde{P}_n\}_n$ of distributions in $D[0,1]$ defined by the above-mentioned random broken lines is conditionally compact and every limit distribution for $\{\tilde{P}_n\}_n$ corresponds stochastically to a continuous process with independent increments if*

(i) $\theta_n = \max_j (t_{n,j} - t_{n,j-1}) \to 0$ *as* $n \to \infty$,

(ii) $\max_{|\Delta| \leqq d} P\{|\sum_{t_{n,j}\in\Delta}\tilde{X}_{n,j}| > \lambda\} \to 0$ *as* $d \to 0$,

uniformly in n for every fixed $\lambda > 0$,

(iii) $\max_{\Delta} P\{|\sum_{t_{n,j}\in\Delta}\tilde{X}_{n,j}| > \lambda\} \to 0$

uniformly in n as $\lambda \to \infty$. The maximum in (ii) *and* (iii) *is taken over all Δ of form $(t_{n,j}, t] \subset [0,1]$.*

REMARK. The proof in [1] is not completely correct because Lemma 2.4 on p. 182 is false. But using a similar theorem given by Billingsley in [4, p. 125], we can easily correct Prohorov's proof, and proceeding in this way we automatically get a proof in the case when $D[0,1]$ has the Skorohod topology. (The topology used by Prohorov is in fact equivalent to the Skorohod topology.)

3.3. Proof of Theorem 1.

We now return to the theorem on convergence to a Lévy process stated in Section 3.1. First we prove that the Prohorov broken line $\tilde{Y}_{n,\omega}(t)$ determined by the double sequence $\tilde{X}_{n,j}, j=1,2,\ldots,k_n, n=1,2,\ldots$, defined by

$$\tilde{X}_{n,j} = X_{n,j} - \gamma(X_{n,j}),$$

and the time-points $t_{n,m} = \delta(\mu_{n,1} * \ldots * \mu_{n,m})/\delta(\mu_{n,1} * \ldots * \mu_{n,k_n})$ satisfies the conditions in Lemma 5.

(i) Since $\delta(\mu_{n,1} * \ldots * \mu_{n,k_n}) \to \delta(\mu) > 0$ as $n \to \infty$, condition (i) follows immediately from Lemma 4 and the fact that $\delta(\mu_{n,j}) \to 0$ uniformly in j as $n \to \infty$.

(ii) Using Lemma 1 we get

$$\sup_{|\Delta| \leqq d} \delta\left(\sum_{t_{n,j}\in\Delta} \tilde{X}_{n,j}\right) \to 0 \quad \text{uniformly in } n \text{ as } d \to \infty .$$

From Lemma 2 we have

$$\sup_{|\Delta| \leqq d} \left|\gamma\left(\sum_{t_{n,j}\in\Delta} \tilde{X}_{n,j}\right)\right| \to 0 \quad \text{uniformly in } n \text{ as } d \to 0 ,$$

and so by Proposition 4,

$$\sup_{|\Delta|\leqq d} P\left\{\left|\sum_{t_{n,j}\in\Delta} \tilde{X}_{n,j}\right| > \lambda\right\} \to 0 \quad \text{uniformly in } n \text{ as } d \to 0 .$$

(iii) We can choose n_0 so large that

$$\left|\sum_{t_{n,j}\in\Delta} \gamma(X_{n,j}) - \gamma\left(\sum_{t_{n,j}\in\Delta} X_{n,j}\right)\right| \leqq 1$$

for all Δ with $|\Delta| \leqq n_0^{-1}$. Since $\mathscr{M} = \{\mu_{n,1} * \ldots * \mu_{n,k_n}\}_n$ is conditionally compact, the family

$$\mathscr{M}' = \{\mu_{n,m} * \mu_{n,m+1} * \ldots * \mu_{n,p} - \sum_{j=m}^{p} \gamma(\mu_{n,j}) \; : \; 1 \leqq m \leqq p \leqq k_n, \\ t_{n,m}, t_{n,m+1}, \ldots, t_{n,p} \in \Delta, \; |\Delta| \leqq n_0^{-1}\}$$

is also conditionally compact by Proposition 6. Since $D[0,1]$ is a complete separable metric space, $\mathscr{M}'$ is tight. Therefore, given $\varepsilon > 0$, we can choose λ so large that

$$P\left\{\left|\sum_{t_{n,j}\in\Delta} \tilde{X}_{n,j}\right| > \tfrac{1}{2}\lambda n_0^{-1}\right\} \leqq \tfrac{1}{2}\varepsilon n_0^{-1} \quad \text{for all } \Delta, \; |\Delta| \leqq n_0^{-1} .$$

If now Δ is an arbitrary subinterval of $[0,1]$ of form $(t_{n,j}, t]$, we can for all large enough n divide it into at most $2n_0$ intervals of form $(t_{n,k}, t]$, each one of length at most n_0^{-1}. Therefore taking λ large enough,

$$P\left\{\left|\sum_{t_{n,j}\in\Delta} \tilde{X}_{n,j}\right| > \lambda\right\} < \varepsilon \quad \text{for all } \Delta \text{ and all } n ,$$

which proves (iii).

Lemma 5 can now be applied. We see that $\{\tilde{P}_n\}_n$ is conditionally compact and every limit distribution $\tilde{P}$ corresponds stochastically to a Lévy process $X_t(\omega)$, $t \in [0,1]$. Let $\{P_q\}$ be a subsequence converging weakly to $\tilde{P}$. If π_1 is the projection, which takes $Y(\cdot) \in D[0,1]$ into $Y(1)$, then π_1 is continuous and therefore,

$$\tilde{P}_q \xrightarrow{w} \tilde{P} \Rightarrow \tilde{P}_q \pi_1^{-1} \to \tilde{P}\pi_1^{-1} .$$

But $\tilde{P}_q \pi_1^{-1}$ is the prohability law of $\sum_{j=1}^{k_q} \tilde{X}_{q,j}$, that is,

$$\tilde{P}_q \pi_1^{-1} = \mu_{q,1} * \mu_{q,2} * \ldots * \mu_{q,k_q} - \sum_{j=1}^{k_q} \gamma(\mu_{q,j}) ,$$

and $\tilde{P}\pi_1^{-1}$ is the probability law of $\tilde{X}_1(\omega)$. Since

$$\mu_{q,1} * \mu_{q,2} * \ldots * \mu_{q,k_q} \xrightarrow{w} \mu \quad \text{as } q \to \infty ,$$

there exists a constant c such that

$$\sum_{j=1}^{k_q} \gamma(\mu_{q,j}) \to c \quad \text{as } q \to \infty , \tag{10}$$

and $\tilde{X}_1(\omega)$ has the probability law $\mu - c$.

From (10) it follows that $\{\sum_{j=1}^{k_n} \gamma(X_{n,j})\}_n$ is bounded. Therefore we can easily see that Lemma 5 can also be applied to the random broken lines $Y_{n,\omega}(t)$ in our theorem. In this case also every limit process $X_t(\omega)$, $t \in [0,1]$, is such that $X_1(\omega)$ has the probability law μ.

REMARK 1. The sequence $\{P_n\}_n$ in the theorem is not convergent in general, as is easily seen.

REMARK 2. Since Lévy's decomposition of the sample path of a Lévy process can be proved directly [3, Section 1.7], the theorem in this note can be used to prove the equivalence of the following characterizations of an infinitely divisible distribution.

(i) There exists a family $\{\mu_n\}_n$ of distributions such that $\mu = \mu_n^{n*}$ for all n.

(ii) There exists a family $\{\mu_{n,k}\}_{n,k}$ of distributions such that

$$\mu_{n,1} * \mu_{n,2} * \cdots * \mu_{n,k_n} \xrightarrow{w} \mu \quad \text{and} \quad \mu_{n,j}[-\varepsilon, +\varepsilon] \to 0$$

uniformly in j as $n \to \infty$ for every $\varepsilon > 0$.

(iii) There exists a Lévy process $X_t(\omega)$, $t \in [0,1]$, such that μ is the probability law of $X_1(\omega)$.

(iv) The characteristic function $\varphi_\mu(z)$ of μ can be written in the form

$$\varphi_\mu(z) = \exp\{imz - \tfrac{1}{2}vz^2 + \textstyle\int_{-\infty}^{\infty} (e^{iuz} - 1 - iuz/(1+u^2)n(du)\}$$

where $v \geqq 0$, $n(du) \geqq 0$, and $\int_{-\infty}^{\infty}(u^2/(1+u^2))\, n(du) < \infty$.

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UNIVERSITY OF GOTHENBURG, SWEDEN
AND
UNIVERSITY OF AARHUS, DENMARK