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A GENERAL ONE-SIDED COMPACTNESS RESULT FOR INTERPOLATION OF BILINEAR OPERATORS

EDUARDO BRANDANI DA SILVA and DICESAR LASS FERNANDEZ

Abstract

The behavior of bilinear operators acting on the interpolation of Banach spaces in relation to compactness is analyzed, and an one-sided compactness theorem is obtained for bilinear operators interpolated by the ρ interpolation method.

1. Introduction

Multilinear operators appear naturally in several branches of classical harmonic analysis and functional analysis, including the theory of ideals of operators in Banach spaces. Recently, several singular multilinear operators have been intensively studied and the research on the bilinear Hilbert transform (see [23]) has shown the need for new results for bilinear operators; see, for example, the paper by L. Grafakos and N. Kalton [18].

A related question is: when is it possible to extend to bilinear operators the classical results for linear operators in interpolation theory? It must be observed that interpolation of bilinear operators is a classical problem in interpolation theory, and some results appear in the fundamental article of Lions and Peetre [24]. Several results on bilinear operators and interpolation were further obtained in several directions; see, for example, [19] and [26]. Some of these results have been applied in the general theory of Banach spaces, see [29], and in the theory of multilinear p-summing operators [1]. But, not all results for the linear case generalize to the bilinear case; for instance, the linear Marcinkiewicz multiplier theorem, whose natural bilinear version fails, as shown by Grafakos and Kalton in [18].

We are interested in this essay in the behavior of compactness for bilinear operators under interpolation by the real method. The study of the behavior of linear compact operators under interpolation has its origin in the classical work of M. A. Krasnoselskii, for L^p spaces. Afterwards, several authors worked on the general question of compactness of operators for interpolation of abstract

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Banach spaces. The first main authors were J. L. Lions and J. Peetre [24] and A. Calderón [3]. The proof that the real method preserves compactness with only a compact restriction in the extreme spaces was given independently in [9] and [11]. That research continued in the last years not only for more general interpolation methods (see e.g. [4], [6]) but also for the measure of non-compactness (see e.g. [33], [7], [17], [30], [5]) and entropy and approximation numbers (see e.g. [12], [13], [32], [31]).

For the real method, if $\mathbf{E} = (E_0, E_1)$, $\mathbf{F} = (F_0, F_1)$ and $\mathbf{G} = (G_0, G_1)$ are Banach couples, a classical result by Lions-Peetre asserts that if T is a bounded bilinear operator from $(E_0 + E_1) \times (F_0 + F_1)$ into $G_0 + G_1$, whose restrictions $T|_{E_k \times F_k}$ $(k = 0, 1)$ are also bounded from $E_k \times F_k$ into G_k $(k = 0, 1)$, then T is bounded from $\mathbf{E}_{\theta,p;J} \times \mathbf{F}_{\theta,q;J}$ into $\mathbf{G}_{\theta,r;J}$, where $0 < \theta < 1$ and $1/r = 1/p + 1/q - 1$. Subsequently several authors have obtained new and more general results about boundedness for the interpolation of bilinear and multilinear operators. For example, see [25], and [26].

For the multilinear case, the study on the behavior of compact operators in the interpolation spaces goes back to A. P. Calderón [3, p. 119–120]. Under an approximation hypothesis, Calderón established a one-sided type general result, but restricted to complex interpolation spaces. On the other hand, the behavior of compact multilinear operators under real interpolation functors until recently had not been investigated. In the paper [14], generalizations of Lions-Peetre compactness theorems [24, Theorem V.2.1] (the one with the same departure spaces) and [24, Theorem V.2.2] (the one with the same arriving spaces), Hayakawa's (i.e. a two-side result without approximation hypothesis) and a compactness theorem of Persson type were obtained. Similar results for more general interpolation methods were obtained in [16].

A natural question put by C. Michels in his report [27] asks whether a Cwikel type result is valid for bilinear operators? In current research, mixing ideas from [14] and [9] with new ones, a bilinear version of Cwikel's compactness theorem is obtained for the first time in the literature. The result is presented for the more general ρ method of interpolation. We call the attention that the results of this work, without the proofs, were presented in a talk at FSDONA conference held in Germany in 2011 [15]. Since that time many researchers in the area know the result, but here we present the full ideas and proofs.

2. Function parameters and interpolation spaces

We shall say that a Banach space is an *intermediate space* with respect to a Banach couple $\mathbf{E} = (E_0, E_1)$ if

$$E_0 \cap E_1 \hookrightarrow E \hookrightarrow E_0 + E_1$$

with $\hookrightarrow$ denoting bounded embeddings. By a *function parameter* ρ we shall mean a continuous and positive function on $\mathbb{R}_+$. We shall say that a function parameter ρ belongs to the class $\mathscr{B}$, if it satisfies $\rho(1) = 1$ and

$$\overline{\rho}(s) = \sup_{t>0} \frac{\rho(st)}{\rho(t)} < +\infty, \qquad s > 0.$$

Also, we shall say that a function parameter $\rho \in \mathscr{B}$ belongs to the class $\mathscr{B}^{+-}$ if it satisfies

$$\int_0^\infty \min\left(1, \frac{1}{t}\right)\overline{\rho}(t)\,\frac{dt}{t} < +\infty.$$

The function parameter $\rho_\theta(t) = t^\theta$, $0 \le \theta \le 1$, belongs to $\mathscr{B}$. It corresponds to the usual parameter θ. Further, $\rho_\theta \in \mathscr{B}^{+-}$ if $0 < \theta < 1$, but $\rho_0, \rho_1 \notin \mathscr{B}^{+-}$.

If $\rho \in \mathscr{B}^{+-}$, it may be considered an increasing parameter, and $\rho(t)/t$ a decreasing one. Furthermore, $\overline{\rho}$ may be considered non-decreasing, and $\overline{\rho}(t)/t$ non-increasing. Consequently, if $\rho \in \mathscr{B}^{+-}$ and $0 < q \le \infty$, we have

$$\|\rho^{-1}(t)\min(1, t)\|_{L^q_*} < \infty.$$

(See [20] for more information about interpolation theory and interpolation with function parameter.)

Let $\{E_0, E_1\}$ and $\{F_0, F_1\}$ be Banach couples and let $L(\{E_0, E_1\}, \{F_0, F_1\})$ be the family of all linear maps $T\colon E_0 + E_1 \to F_0 + F_1$ such that $T|_{E_k}$ is *bounded* from E_k to F_k, $k = 0, 1$.

If E and F are intermediate spaces with respect to $\{E_0, E_1\}$ and $\{F_0, F_1\}$, respectively, we say that E and F are interpolation spaces of type ρ where $\rho \in \mathscr{B}^{+-}$, if given any $T \in L(\{E_0, E_1\}, \{F_0, F_1\})$ we have

$$\|T\|_{L(E,F)} \le C\|T\|_0\overline{\rho}\left(\frac{\|T\|_1}{\|T\|_0}\right),$$

where $\|T\|_k = \|T\|_{L(E_k,F_k)}$ $(k = 0, 1)$, and $C > 0$ is a constant.

Let $\{E_0, E_1\}$ be a Banach couple. The functionals J and K are defined by

$$J(t, x) = J(t, x; \mathbf{E}) = \max\big\{\|x\|_{E_0}, t\|x\|_{E_1}\big\}, \qquad x \in E_0 \cap E_1,$$
$$K(t, x) = K(t, x; \mathbf{E}) = \inf_{x=x_0+x_1}\big\{\|x_0\|_{E_0} + t\|x_1\|_{E_1}\big\},$$

respectively, where in $K(t, x)$, $x_0 \in E_0$ and $x_1 \in E_1$. Then, we can define the following interpolation spaces.

The space $(E_0, E_1)_{\rho,q,K}$, $\rho \in \mathscr{B}^{+-}$ and $0 < q \le +\infty$, consists of all $x \in E_0 + E_1$ for which the norm

$$\|x\|_{\rho,q;K} = \big\|\big(\rho(2^n)^{-1}K(2^n, x; \mathbf{E})\big)_{n\in\mathbb{Z}}\big\|_{\ell^q(\mathbb{Z})} \text{ is finite.}$$

The space $(E_0, E_1)_{\rho,q;J}$, $\rho \in \mathscr{B}^{+-}$, consists of all $x \in E_0 + E_1$ that have a representation $x = \sum_{n=-\infty}^{\infty} u_n$, where $(u_n) \in E_0 \cap E_1$ and converges in $E_0 + E_1$, and the norm

$$\|x\|_{\rho,q;J} = \inf \left\| \left(\rho(2^n)^{-1} J(2^n, u_n; \mathbf{E})\right)_{n\in\mathbb{Z}} \right\|_{\ell^q(\mathbb{Z})},$$

is finite, where the infimum is taken over all representations $x = \sum u_n$. Besides, in the case of the interpolation space $(E_0, E_1)_{\rho,q,J}$, if $x \in E_0 \cap E_1$, then

$$\|x\|_E \leq C\|x\|_0 \overline{\rho}\left(\frac{\|x\|_1}{\|x\|_0}\right).$$

For $0 < q \leq +\infty$, the Equivalence Theorem between the J and K method holds, that is,

$$(E_0, E_1)_{\rho,q;J} = (E_0, E_1)_{\rho,q;K}.$$

Let E be an intermediate space with respect to a Banach couple $\mathbf{E} = (E_0, E_1)$ and $\rho \in \mathscr{B}^{+-}$. We say that E is an intermediate space of class $J_\rho(E_0, E_1)$ if the following embedding holds

$$(E_0, E_1)_{\rho,1;J} \hookrightarrow E,$$

and we say that E is an intermediate space of class $K_\rho(E_0, E_1)$ if the following embedding holds

$$E \hookrightarrow (E_0, E_1)_{\rho,\infty;K}.$$

We note that E is of class $J_\theta(E_0, E_1)$ if, and only if, for all $x \in E_0 \cap E_1$, we have

$$\|x\|_E \leq C\|x\|_{E_0} \overline{\rho}\left(\frac{\|x\|_{E_1}}{\|x\|_{E_0}}\right).$$

To obtain our main result, we will use the following sequence spaces. Let G be a linear space and let $(\|\cdot\|_n)_{n\in\mathbb{Z}}$ be a sequence of norms on G. For each $n \in \mathbb{Z}$, we shall denote by G_n the space G equipped with the norm $\|\cdot\|_n$: $G_n = (G, \|\cdot\|_n)$. Let ρ be any function parameter and $0 < q \leq \infty$. We shall denote by $\ell^q_\rho(G_n)$ the linear space of all sequences $(a_n)_{n\in\mathbb{Z}}$, in G, such that

$$|||(a_n)|||_{\rho,q} := \|(a_n)_{n\in\mathbb{Z}}\|_{\ell^q_\rho(G_n)} = \left[\sum_{n\in\mathbb{Z}} [\rho(2^{-n})\|a_n\|_n]^q\right]^{1/q} < +\infty.$$

If $\rho(t) = 1$ then $\ell^q_\rho(G_n)$ is denoted by $\ell^q_0(G_n)$ and if $\rho(t) = t$, it is denoted by $\ell^q_1(G_n)$. The functional $|||\cdot|||_{\rho,q}$ is a norm on $\ell^q_\rho(G_n)$.

The spaces $\ell^q_\rho(G_n)$ are related to interpolation by the following result [20]:

THEOREM 2.1. *For the above norm* $|||\cdot|||_{\rho,q}$ *and* $\rho \in \mathscr{B}^{+-}$, *one has*

$$\left(\ell_0^{q_0}(G_m), \ell_1^{q_1}(G_m)\right)_{\rho,q} = \ell_f^q(G_m),$$

where $0 < q, q_0, q_1 \leq \infty$ *and* $f(t) = 1/\rho(t^{-1})$.

For each $m \in \mathbb{Z}$, let us set $\Delta_m = \Delta_m \mathbf{E} = E_0 \cap 2^{-m} E_1$, i.e., we take Δ_m to be the space $E_0 \cap E_1$ equipped with the norm $J(2^{-m}, \cdot)$.

Given $\rho \in \mathscr{B}^{+-}$, for $f(t) = 1/\rho(t^{-1})$, every sequence $\{u_m\}$ in $\ell_f^q(\Delta_m)$ is summable in $E_0 + E_1$. Then, setting

$$\sigma(\{u_m\}) = \sum_{m=-\infty}^{\infty} u_m, \tag{2.1}$$

by Theorem 2.1, we see that the mapping $\sigma\colon \ell_f^q(\Delta_m) \to (E_0, E_1)_{\rho,q;J}$ is bounded and $(E_0, E_1)_{\rho,q;J} = \ell_f^q(\Delta_m)/\sigma^{-1}(0)$. Moreover, it may be proved that $\ell_f^q(\Delta_m) \subset \left(\ell_0^1(\Delta_m), \ell_1^1(\Delta_m)\right)_{\rho,q}$.

3. Bilinear operators and compactness

Given X, Y and Z Banach spaces and a bilinear operator $T\colon X \times Y \to Z$, the norm of T is defined by

$$\|T\|_{\mathrm{Bil}(X\times Y,Z)} = \sup\left\{\|T(x,y)\|_Z : (x,y) \in U_{X\times Y}\right\},$$

where $U_{X\times Y}$ is the unit closed ball in $X \times Y$ with respect to the norm $\|(x,y)\| = \max\{\|x\|_X, \|y\|_Y\}$. We denote by $\mathrm{Bil}(X \times Y, Z)$ the space of all bounded bilinear operators from $X \times Y$ into Z.

If the norm of T is defined using the open unit ball, denoted by $\mathring{U}_{X\times Y}$, we will have the same norm. The following lemmas will be important in what follows.

LEMMA 3.1. *Given X, Y and Z Banach spaces, if* $A \subset X$ *and* $B \subset Y$ *are dense subsets and* $T \in \mathrm{Bil}(X \times Y, Z)$, *then*

$$|||T|||_{\mathrm{Bil}(X\times Y,Z)} = \sup\left\{\|T(a,b)\|_Z : (a,b) \in A \times B \text{ with } (a,b) \in \mathring{U}_{X\times Y}\right\},$$

is a norm on $\mathrm{Bil}(X \times Y, Z)$ *and* $|||T|||_{\mathrm{Bil}(X\times Y,Z)} = \|T\|_{\mathrm{Bil}(X\times Y,Z)}$.

LEMMA 3.2. *Given Banach spaces X, Y and Z, and* $A \times B \subset X \times Y$ *a dense subset, let* $T_n \in \mathrm{Bil}(X \times Y, Z)$, $n \in \mathbb{N}$ *be a sequence of bilinear operators such that* $\lim_{n\to\infty} |||T_n|||_{\mathrm{Bil}(X\times Y,Z)} = \lambda$. *Then, there exits a sequence* $(x_n, y_n) \subset (A \times B) \cap \mathring{U}_{X\times Y}$ *such that*

$$\lim_{n\to\infty} \|T_n(x_n, y_n)\|_Z = \lambda.$$

Given Banach couples $\mathbf{E} = (E_0, E_1)$, $\mathbf{F} = (F_0, F_1)$ and $\mathbf{G} = (G_0, G_1)$, we shall denote by $\mathrm{Bil}(\mathbf{E} \times \mathbf{F}, \mathbf{G})$ the set of all bounded bilinear mappings from $(E_0 + E_1) \times (F_0 + F_1)$ to $G_0 + G_1$ such that $T|_{E_K \times F_k}$ is *bounded* from $E_k \times F_k$ into G_k, $k = 0, 1$.

Given Banach couples $\mathbf{E} = (E_0, E_1)$, $\mathbf{F} = (F_0, F_1)$, and $\mathbf{G} = (G_0, G_1)$ and intermediate spaces E, F and G respectively, we shall say that the pair $(E \times F, G)$ is a *bilinear interpolation pair of type* ρ, if for all bilinear operator T from $(E_0 + E_1) \times (F_0 + F_1)$ into $G_0 + G_1$ such that $T \colon E \times F \to G$ one has

$$\|T\|_{\mathrm{Bil}(E\times F;G)} \leq C\|T\|_{\mathrm{Bil}(E_0\times F_0,G_0)}\overline{\rho}\left(\frac{\|T\|_{\mathrm{Bil}(E_1\times F_1,G_1)}}{\|T\|_{\mathrm{Bil}(E_0\times F_0,G_0)}}\right).$$

The following result characterizes the bilinear interpolation operators which are of our interest. For the classical θ method this property was first established by Lions-Peetre [24, Th.I.4.1]. Here, we use the function parameter version from [14].

THEOREM 3.3. *Let T be a bounded bilinear operator from $(E_0 + E_1) \times (F_0 + F_1)$ into $G_0 + G_1$ whose restrictions $T|_{E_k \times F_k}$ $(k = 0, 1)$ are bounded from $E_k \times F_k$ into G_k $(k = 0, 1)$. Then, for $\rho \in \mathscr{B}^{+-}$, one has*

$$T \colon E_{\gamma,p} \times F_{\rho,q} \to G_{\rho,r},$$

where $\gamma(t) = \overline{\rho}(t^{-1})^{-1} \in \mathscr{B}^{+-}$, $1/r = 1/p + 1/q - 1$ and

$$\|T\|_{\mathrm{Bil}(E_{\gamma,p}\times F_{\rho,q},G_{\rho,r})} \leq C\|T\|_{\mathrm{Bil}(E_0\times F_0,G_0)}\overline{\rho}\left(\frac{\|T\|_{\mathrm{Bil}(E_1\times F_1,G_1)}}{\|T\|_{\mathrm{Bil}(E_0\times F_0,G_0)}}\right),$$

where $C > 0$ is a constant.

We also quote the following useful lemma, which is easy to prove.

LEMMA 3.4. *Let $\mathbf{E} = (E_0, E_1)$ and $\mathbf{F} = (F_0, F_1)$ be Banach couples and G be any Banach space. If $T \in \mathrm{Bil}(E_k \times F_k; G)$, $k = 0, 1$, then $T \in \mathrm{Bil}((E_0 \cap E_1) \times (F_0 + F_1), G)$ and $T \in \mathrm{Bil}((E_0 + E_1) \times (F_0 \cap F_1), G)$.*

For more on bilinear interpolation, see L. Maligranda [25] and M. Mastyło [26].

Given Banach spaces E, F and G, a bounded bilinear mapping T from $E \times F$ into G is *compact* if the image of the set $M = \big\{(x, y) \in E \times F : \max\{\|x\|_E, \|y\|_F\} \leq 1\big\}$ is a totally bounded subset of G.

Let us quote a few examples of bilinear compact operators.

EXAMPLE 3.5. In [28, p. 274] a compact bilinear operator is defined in $\mathrm{Bil}(H^2(\Omega) \times H^2(\Omega), H_0^2(\Omega))$, where Ω is a bounded and simply connected domain in $\mathbb{R}^2$.

EXAMPLE 3.6. Related questions on multilinear forms are studied in [8], where several examples of forms are given. Also in [21, p. 925] an interesting example of a Hilbert-Schmidt bilinear form is presented.

EXAMPLE 3.7. Even though a bilinear operator $T(x, y)$ is compact in each coordinate, it may not be a bilinear compact operator. This question has been studied in [22] and [2], where examples are also given.

Versions of Lions-Peetre Theorems about compactness of bilinear operators in interpolated spaces by the ρ method are given below. For the proofs, see [14].

THEOREM 3.8. *Let E and F be Banach spaces, $\mathbf{G} = (G_0, G_1)$ a Banach couple and G a Banach space of class $J_\rho(G_0, G_1)$, $\rho \in \mathscr{B}^{+-}$. Given a bounded bilinear operator T from $E \times F$ into $G_0 + G_1$, such that $T(E \times F) \subset G_0 \cap G_1$, T is compact from $E \times F$ into G_0 and bounded from $E \times F$ into G_1, then T is also compact from $E \times F$ into G.*

THEOREM 3.9. *Let Banach couples $\mathbf{E} = (E_0, E_1)$ and $\mathbf{F} = (F_0, F_1)$, G any Banach space and $T \in \mathrm{Bil}((E_0 + E_1) \times (F_0 + F_1), G)$ be given. If $\rho \in \mathscr{B}^{+-}$, $\gamma(t) = 1/\overline{\rho}(t^{-1})$ and T is compact from $E_0 \times F_0$ into G, then T is also compact from $E \times F$ into G, where $E = (E_0, E_1)_{\gamma,p}$ and $F = (F_0, F_1)_{\rho,q}$.*

4. A one-sided bilinear compactness theorem

In this section we shall establish a bilinear version, for the ρ method, of the linear one-sided compactness theorem (see [9] and [11]), in which we assume compactness just in one of the departure spaces. We begin with a preliminary result which depends on an approximation hypothesis. This hypothesis was also used in [14] and has its origin in [10].

APPROXIMATION HYPOTHESIS. A Banach couple $\mathscr{X} = (X_0, X_1)$ satisfies the Approximation Hypothesis (AP) if there exists a sequence $\{P_n\}$ in $L(\mathscr{X}, \mathscr{X})$, with $P_n(X_0 + X_1) \subset X_0 \cap X_1$, and two other sequences $\{P_n^+\}$ and $\{P_n^-\}$ in $L(\mathscr{X}, \mathscr{X})$, such that

(AP1) the sequences are uniformly bounded in $L(\mathscr{X}, \mathscr{X})$;

(AP2) $I = P_n + P_n^+ + P_n^-$, $n \in \mathbb{N}$;

(AP3) $P_n^+ = P_n^+|_{X_0} \in L(X_0, X_1)$, $P_n^- = P_n^-|_{X_1} \in L(X_1, X_0)$, and

$$\lim_{n\to\infty} \|P_n^+\|_{L(X_0,X_1)} = \lim_{n\to\infty} \|P_n^-\|_{L(X_1,X_0)} = 0.$$

The notation X_k° $(k = 0, 1)$ stands for the closure of $X_0 \cap X_1$ in X_k, $(k = 0, 1)$, and $\overline{\mathscr{X}}_{\theta,q}$ stands for the closure of $X_0 \cap X_1$ in $(X_0, X_1)_{\theta,q}$. The following lemma, due to [10] is required.

LEMMA 4.1. *Let* $\mathbf{E} = (E_0, E_1)$ *be a Banach couple which satisfies the Approximation Hypothesis* (AP). *Then,*

$$\text{(i)} \quad \lim_{n\to\infty} \|P_n^- x\|_{E_0} = 0, \quad \text{if } x \in E_0^\circ;$$

$$\text{(ii)} \quad \lim_{n\to\infty} \|P_n^+ x\|_{E_1} = 0, \quad \text{if } x \in E_1^\circ.$$

The next theorem is the core for our main result.

THEOREM 4.2. *Let us assume that* $\mathbf{E} = (E_0, E_1)$, $\mathbf{F} = (F_0, F_1)$ *and* $\mathbf{G} = (G_0, G_1)$ *are Banach couples which satisfy the Approximation Hypothesis* (AP). *Let T be a bounded bilinear operator from* $(E_0 + E_1) \times (F_0 + F_1)$ *into* $G_0 + G_1$ *whose restrictions* $T|_{E_k \times F_k}$ $(k = 0, 1)$ *are bounded from* $E_k \times F_k$ *into* G_k $(k = 0, 1)$ *and* $T|_{E_0 \times F_0}$ *is compact.*

Given $\rho \in \mathscr{B}^{+-}$, *put* $\gamma(t) = 1/\overline{\rho}(t^{-1}) \in \mathscr{B}^{+-}$ *and* $1/r = 1/p + 1/q - 1$. *If* $E = (E_0, E_1)_{\gamma,p}$, $F = (F_0, F_1)_{\rho,q}$ *and* $G = (G_0, G_1)_{\rho,r}$, *then considering the decompositions* $I = R_n + R_n^+ + R_n^- \in L(G, G)$, $I = Q_n + Q_n^+ + Q_n^- \in L(F, F)$ *and* $I = P_n + P_n^+ + P_n^- \in L(E, E)$ *given by* (AP2), *we have that*

(1) *each of the operators* $T(P_n, Q_n)$, $R_n T(P_n, Q_n^-)$ *and* $R_n T(P_n^-, Q_n^-)$ *is compact from* $E \times F$ *to* G, *for all* n;

(2) *each of the sequence of norms*

$$\begin{aligned}
&\{\|T(P_n, Q_n^+)\|_{\mathrm{Bil}(E\times F,G)}\}, \quad \{\|R_n^- T(P_n, Q_n^-)\|_{\mathrm{Bil}(E\times F,G)}\},\\
&\{\|T(P_n^+, Q_n)\|_{\mathrm{Bil}(E\times F,G)}\}, \quad \{\|T(P_n^+, Q_n^+)\|_{\mathrm{Bil}(E\times F,G)}\},\\
&\{\|T(P_n^+, Q_n^-)\|_{\mathrm{Bil}(E\times F,G)}\}, \quad \{\|T(P_n^-, Q_n)\|_{\mathrm{Bil}(E\times F,G)}\},\\
&\{\|T(P_n^-, Q_n^+)\|_{\mathrm{Bil}(E\times F,G)}\}, \quad \{\|R_n^+ T(P_n^-, Q_n^-)\|_{\mathrm{Bil}(E\times F,G)}\}\\
\text{and } &\{\|R_n^- T(P_n^-, Q_n^-)\|_{\mathrm{Bil}(E\times F,G)}\}
\end{aligned}$$

converges to zero, when $n \to \infty$.

PROOF. *Step 1*: For $T(P_n, Q_n)$, we factorize it using the following diagram:

$$E \times F \xrightarrow{(P_n, Q_n)} (E_0 \cap E_1) \times (F_0 \cap F_1) \hookrightarrow E_j \times F_j \xrightarrow{T} G_j,$$

for $j = 0, 1$.

Since T is compact from $E_0 \times F_0$ into G_0, it follows, by Theorem 3.8, that $T \circ (P_n, Q_n)$ is compact from $E \times F$ into G.

Step 2: We want to show that

$$\lim_{n\to\infty} \|T \circ (P_n, Q_n^+)\|_{\mathrm{Bil}(E\times F,G)} = 0.$$

Since

$$\begin{aligned}\|T(P_n, Q_n^+)\|_{\mathrm{Bil}(E\times F,G)} &\le C\|T(P_n, Q_n^+)\|_0 \overline{\rho}\left(\frac{\|T(P_n, Q_n^+)\|_1}{\|T(P_m, Q_n^+)\|_0}\right) \\ &\le C\overline{\rho}(\|T\|_1)\frac{\overline{\rho}\big(1/\|T(P_n, Q_n^+)\|_0\big)}{1/\|T(P_n, Q_n^+)\|_0},\end{aligned}$$

where $\|\cdot\|_k = \|\cdot\|_{\mathrm{Bil}(E_k\times F_k,G_k)}$ $(k = 0, 1)$, it is enough to show that $\|T(P_n, Q_n^+)\|_0 \to 0$, as $n \to \infty$.

Aiming for a contradiction, let us suppose that $\|T(P_n, Q_n^+)\|_{\mathrm{Bil}(E_0\times F_0,G_0)}$ converges to $\lambda > 0$. Since $\{P_n\}$ and $\{Q_n^+\}$ are uniformly bounded in $E_0 \times F_0$, by Lemma 3.2 there exists a sequence $\{(a_n, b_n)\} \in U_{E_0} \times U_{F_0}$ and a subsequence $\{n'\}$, such that $\|T(P_{n'}, Q_{n'}^+)\|_{\mathrm{Bil}(E_0\times F_0,G_0)} \to \lambda > 0$ and

$$\|T(P_{n'}a_{n'}, Q_{n'}^+b_{n'})\|_{G_0} \to \lambda \quad \text{as } n' \to \infty.$$

By the compactness assumption on $T: E_0 \times F_0 \to G_0$ we may assume, passing the another subsequence if necessary, that $\{T(P_{n'}a_{n'}, Q_{n'}^+b_{n'})\}$ converges to some element b in G_0, so that $\|b\|_{G_0} = \lambda$. But

$$\begin{aligned}&\|T(P_{n'}a_{n'}, Q_{n'}^+b_{n'})\|_{G_0+G_1} \\ &\quad\le C\|T\|_{\mathrm{Bil}((E_0+E_1)\times(F_0+F_1),G_0+G_1)}\|P_{n'}a_{n'}\|_{E_0+E_1}\|Q_{n'}^+b_{n'}\|_{F_0+F_1} \\ &\quad\le C\|P_{n'}a_{n'}\|_{E_0}\|Q_{n'}^+b_{n'}\|_{F_1} \\ &\quad\le C\|P_{n'}\|_{L(E_0,F_0)}\|a_{n'}\|_{E_0}\|Q_{n'}^+\|_{L(F_0,F_1)}\|b_{n'}\|_{F_0}.\end{aligned}$$

Since $\lim_{n'\to\infty}\|Q_{n'}^+\|_{L(F_0,F_1)} = 0$, it follows that $T(P_{n'}a_{n'}, Q_{n'}^+b_{n'}) \to 0$ in $G_0 + G_1$, as $n \to \infty$. Consequently $b = 0$, and $\lambda = 0$, which is not the case.

Step 3: For the compactness of $R_nT(P_n, Q_n^-)$, for all n, let us consider the diagram

$$E_k \times F_k \xrightarrow{T(P_n,Q_n^-)} G_k \xrightarrow{R_n} G_0 \cap G_1 \hookrightarrow G,$$

for $k = 0, 1$.

Thus, by Theorem 3.9, $R_nT(P_n, Q_n^-): E \times F \to G$ is compact.

Step 4: For $R_n^- T(P_n, Q_n^-)$, since

$$
\begin{aligned}
&\|R_n^- T(P_n, Q_n^-)\|_{\mathrm{Bil}(E\times F,G)} \\
&\qquad \le C\|R_n^- T(P_n, Q_n^-)\|_0 \overline{\rho}\left(\frac{\|R_n^- T(P_n, Q_n^-)\|_1}{\|R_n^- T(P_m, Q_n^-)\|_0}\right) \\
&\qquad \le C\overline{\rho}\big(\|R_n^- T(P_n, Q_n^-)\|_1\big)\frac{\overline{\rho}\big(1/\|R_n^- T(P_n, Q_n^-)\|_0\big)}{1/\|R_n^- T(P_n, Q_n^-)\|_0} \\
&\qquad \le C\overline{\rho}(\|T\|_1)\frac{\overline{\rho}\big(1/\|R_n^- T(P_n, Q_n^-)\|_0\big)}{1/\|R_n^- T(P_n, Q_n^-)\|_0},
\end{aligned}
$$

where $\|\cdot\|_k = \|\cdot\|_{\mathrm{Bil}(E_k\times F_k,G_k)}$, $k = 0, 1$, and by the fact that the sequences $\{R_n^-\}, \{P_n\}$ and $\{Q_n^-\}$ are uniformly bounded, it is enough to prove that $\|R_n^- T(P_n, Q_n^-)\|_0 \to 0$, for $n \to \infty$.

From properties of the ρ-method (see [12] and [13]), one has that $E = (E_0, E_1)_{\gamma,p} = (E_0^\circ, E_1)_{\gamma,p}$ and $F = (F_0, F_1)_{\rho,q} = (F_0^\circ, F_1)_{\rho,q}$, then by Lemma 3.1 it follows that $\|R_n^- T(P_n, Q_n^-)\|_0 = |||R_n^- T(P_n, Q_n^-)|||_{\mathrm{Bil}(E_0^\circ\times F_0^\circ,G_0)}$. Let us suppose that $|||R_n^- T(P_n, Q_n^-)|||_0$ does not converge to zero. Then, there exists a subsequence $\{R_{n'}^- T(P_{n'}, Q_{n'}^-)\}$ such that $|||R_{n'}^- T(P_{n'}, Q_{n'}^-)|||_{G_0} \to \lambda_1 > 0$. By the Lemma 3.2, there exists a sequence (x_n, y_n), with $x_n \in U_{E_0} \cap E_1$ and $y_n \in U_{F_0} \cap F_1$, such that $\|R_{n'}^- T(P_{n'}x_{n'}, Q_{n'}^- y_{n'})\|_{G_0} \to \lambda_1 > 0$ for $n' \to \infty$.

Since the sequence $\{(P_{n'}x_{n'}, Q_{n'}^- y_{n'})\}$ is bounded in $E_0 \times F_0$, the compactness of $T\colon E_0\times F_0 \to G_0$ guarantees another subsequence $\{T(P_{n''}x_{n''}, Q_{n''}^- y_{n''})\}$ which converges for some $z_0 \in G_0$. Then, for n'' being great enough, one has

$$\|R_{n''}^- z_0\|_{G_0} > \frac{\lambda_1}{2}.$$

However, since $(x_n, y_n) \in (E_0 \cap E_1) \times (F_0 \cap F_1)$, then $\{T(P_{n''}x_{n''}, Q_{n''}^- y_{n''})\} \subset G_0 \cap G_1$, for all n''. Given that $z_0 = \lim_{n''\to\infty} T(P_{n''}x_{n''}, Q_{n''}^- y_{n''})$ then, $z_0 \in \overline{G_0 \cap G_1}^{\,0}$, therefore by the Lemma 3.4 one has $\lim\limits_{n''\to\infty} \|R_{n''}^- z_0\|_{G_0} = 0$, which is a contradiction.

Step 5: To prove that

$$
\begin{aligned}
\lim_{n\to\infty} \|T(P_n^+, Q_n)\| &= \lim_{n\to\infty} \|T(P_n^+, Q_n^+)\| = \lim_{n\to\infty} \|T(P_n^+, Q_n^-)\| \\
&= \lim_{n\to\infty} \|T(P_n^-, Q_n)\| = \lim_{n\to\infty} \|T(P_n^-, Q_n^+)\| = 0,
\end{aligned}
$$

it is enough to follow Step 2.

Step 6: For compactness of $R_n T(P_n^-, Q_n^-)$, we factorize it using the following diagram:

$$E_j \times F_j \xrightarrow{(P_n^-, Q_n^-)} E_j \times F_j \xrightarrow{T} G_j \xrightarrow{R_n} G_0 \cap G_1 \hookrightarrow G,$$

for $j = 0, 1$.

Since T is compact from $E_0 \times F_0$ into G_0, it follows by Theorem 3.9 that $R_n T(P_n^-, Q_n^-)$ is compact from $E \times F$ into G.

Step 7: For $R_n^+ T(P_n^-, Q_n^-)$, we have

$$\begin{aligned}
&\|R_n^+ T(P_n^-, Q_n^-)\|_{\mathrm{Bil}(E\times F, G)} \\
&\quad \le C\|R_n^+ T(P_n^-, Q_n^-)\|_0 \overline{\rho}\left(\frac{\|R_n^+ T(P_n^-, Q_n^-)\|_1}{\|R_n^+ R_n^+ T(P_n^-, Q_n^-)\|_0}\right) \\
&\quad \le C\overline{\rho}\big(\|R_n^+ T(P_n, Q_n^-)\|_1\big)\frac{\overline{\rho}\big(1/\|R_n^+ T(P_n^-, Q_n^-)\|_0\big)}{1/\|R_n^+ T(P_n^-, Q_n^-)\|_0} \\
&\quad \le C\frac{\overline{\rho}\big(1/\|T\|_0\big)}{1/\|T\|_0}\overline{\rho}\big(\|R_n^+ T(P_n^-, Q_n^-)\|_1\big).
\end{aligned}$$

Since the sequences $\{R_n^+\}$, $\{P_n^-\}$ and $\{Q_n^-\}$ are uniformly bounded, it is enough to prove that $\|R_n^+ T(P_n^-, Q_n^-)\|_1 \to 0$. Considering the diagram

$$E_1 \times F_1 \xrightarrow{(P_n^-, Q_n^-)} E_0 \times F_0 \xrightarrow{T} G_0 \xrightarrow{R_n^+} G_1,$$

we have

$$\|R_n^+ T(P_n^-, Q_n^-)\|_1 \le \|R_n^+\|_{L(G_0,G_1)}\|T\|_0\|P_n^-\|_{L(E_1,E_0)}\|Q_n^-\|_{L(F_1,F_0)}.$$

By (AP3) we obtain the convergence to zero.

Step 8: For $R_n^- T(P_n^-, Q_n^-)$ it is enough to follow the reasoning in Step 4. The proof is complete.

Now, our main goal will be dealt with. We shall state a bilinear version of Cwikel's compactness theorem.

THEOREM 4.3. *Let* $\mathbf{E} = (E_0, E_1)$*,* $\mathbf{F} = (F_0, F_1)$ *and* $\mathbf{G} = (G_0, G_1)$ *be Banach couples. Let* $T \in \mathrm{Bil}(\mathbf{E} \times \mathbf{F}, \mathbf{G})$ *be given, such that the restriction* $T|_{E_0\times F_0}$ *is compact from* $E_0 \times F_0$ *into* G_0*. Then, given* $\rho \in \mathscr{B}^{+-}$*,* T *is compact from* $\mathbf{E}_{\gamma,p} \times \mathbf{F}_{\rho,q}$ *into* $\mathbf{G}_{\rho,r}$*, where* $\gamma(t) = 1/\overline{\rho}(t^{-1})$ *and* $1/r = 1/p + 1/q - 1$.

PROOF. For each $m \in \mathbb{Z}$, let $\Delta_m\mathbf{E} = E_0 \cap 2^{-m}E_1$ be the space $E_0 \cap E_1$ equipped with the norm $J(2^{-m}, \cdot)$, and $\Delta_m\mathbf{F}$ defined in a similar way. Let $\Sigma_m\mathbf{G} = G_0 + 2^{-m}G_1$ be the space $G_0 + G_1$ equipped with the norm $K(2^{-m}, \cdot)$.

Let $\sigma\colon \ell^1_k(\Delta_m\mathbf{E}) \to E_k$, $k = 0, 1$ be the operator given in (2.1) (and the same for the space $\ell^1_k(\Delta_m\mathbf{F})$) and let $j\colon G_k \to \ell^\infty_k(\Sigma_m\mathbf{G})$, $k = 0, 1$ be the map given by

$$j\colon b \longrightarrow (b) = (\dots, b, b, b, b, \dots).$$

To prove that the bounded bilinear mapping

$$T\colon \mathbf{E}_{\gamma,p} \times \mathbf{F}_{\rho,q} \to \mathbf{G}_{\rho,r}$$

is compact, if $\widetilde{T} = j \circ T \circ (\sigma, \sigma)$, it is enough to show that

$$\widetilde{T}\colon \big(\ell^1_0(\Delta_m\mathbf{E}), \ell^1_1(\Delta_m\mathbf{E})\big)_{\gamma,p} \times \big(\ell^1_0(\Delta_m\mathbf{F}), \ell^1_1(\Delta_m\mathbf{F})\big)_{\rho,q} \longrightarrow \big(\ell^\infty_0(\Sigma_m\mathbf{G}), \ell^\infty_1(\Sigma_m\mathbf{G})\big)_{\rho,r}$$

is compact. Indeed, from Theorem 2.1 one has that

$$\begin{aligned} \big(\ell^1_0(\Delta_m\mathbf{E}), \ell^1_1(\Delta_m\mathbf{E})\big)_{\gamma,p} &= \ell^p_{f_1}(\Delta_m\mathbf{E}),\\ \big(\ell^1_0(\Delta_m\mathbf{F}), \ell^1_1(\Delta_m\mathbf{F})\big)_{\rho,q} &= \ell^q_{f_2}(\Delta_m\mathbf{F})\\ \text{and}\quad \big(\ell^\infty_0(\Sigma_m\mathbf{G}), \ell^\infty_1(\Sigma_m\mathbf{G})\big)_{\rho,r} &= \ell^q_{f_2}(\Sigma_m\mathbf{G}),\end{aligned}$$

where $f_1(t) = 1/\gamma(t^{-1})$ and $f_2(t) = 1/\rho(t^{-1})$. Since $(E_0, E_1)_{\gamma,p;J} = \ell^p_{f_1}(\Delta_m\mathbf{E})/\sigma^{-1}(0)$, $(F_0, F_1)_{\rho,q;J} = \ell^q_{f_2}(\Delta_m\mathbf{E})/\sigma^{-1}(0)$ and j is a metric injection, then it will follow that the mapping T is compact.

Now, we need verify that the Banach couples $(\ell^1_0(\Delta_m X), \ell^1_1(\Delta_m X))$, where either $X = \mathbf{E}$ or $X = \mathbf{F}$, and $(\ell^\infty_0(\Sigma_m\mathbf{G}), \ell^\infty_1(\Sigma_m\mathbf{G}))$ satisfy the Approximation Hypothesis (AP).

For each $n \in \mathbb{N}$, let us consider the cutting operators P_n, P_n^+ and P_n^-, defined on $\ell^1_0(\Delta_m X) + \ell^1_m(\Delta_m X)$ by

$$\begin{aligned} P_n(u_m) &= \{\dots, 0, 0, u_{-n}, u_{-n+1}, \dots, u_0, \dots, u_{n-1}, u_n, 0, 0, \dots\},\\ P_n^+(u_m) &= \{\dots, 0, 0, u_{n+1}, u_{n+2}, \dots\},\\ P_n^-(u_m) &= \{\dots, u_{-n-2}, u_{-n-1}, 0, 0, \dots\}.\end{aligned}$$

We see that the identity operator I on $\ell^1_k(\Delta_m X)$ ($k = 0, 1$), may be written as $I = P_n + P_n^+ + P_n^-$ and P_n, P_n^+ and P_n^- are uniformly bounded, with norm 1. Moreover, $P_n^+\colon \ell^1_1(\Delta_m) \to \ell^1_0(\Delta_m)$ and $P_n^-\colon \ell^1_0(\Delta_m) \to \ell^1_1(\Delta_m)$, and their norms are bounded by $2^{-(n+1)}$. Also, $P_n\colon \ell^1_0(\Delta_m) + \ell^1_1(\Delta_m) \to \ell^1_0(\Delta_m) \cap \ell^1_1(\Delta_m)$ and its norm is bounded by 2^n from $\ell^1_0(\Delta_m)$ to $\ell^1_1(\Delta_m)$ and from $\ell^1_1(\Delta_m)$ to $\ell^1_0(\Delta_m)$. Hence the Banach couple $(\ell^1_0(\Delta_m X), \ell^1_1(\Delta_m X))$ verifies the Approximation Hypothesis (AP). With a similar reasoning, we verify that the couple $(\ell_0^\infty(\Sigma_m\mathbf{G}), \ell_1^\infty(\Sigma_m\mathbf{G}))$ also satisfies the Approximation Hypothesis (AP) and, if $I = R_n + R_n^+ + R_n^-$ we have the same boundedness for the norms of R_n, R_n^+ and R_n^- in $(\ell_0^\infty(\Sigma_m\mathbf{G}), \ell_1^\infty(\Sigma_m\mathbf{G}))$.

Now, let us consider the following decomposition

$$\begin{aligned}
\widetilde{T} &= \widetilde{T}(I, I) = \widetilde{T}(P_n + P_n^+ + P_n^-, Q_n + Q_n^+ + Q_n^-) \\
&= \widetilde{T}(P_n, Q_n) + \widetilde{T}(P_n, Q_n^+) + \widetilde{T}(P_n, Q_n^-) + \widetilde{T}(P_n^+, Q_n) + \widetilde{T}(P_n^+, Q_n^+) \\
&\quad + \widetilde{T}(P_n^+, Q_n^-) + \widetilde{T}(P_n^-, Q_n) + \widetilde{T}(P_n^-, Q_n^+) + \widetilde{T}(P_n^-, Q_n^-) \\
&= \widetilde{T}(P_n, Q_n) + \widetilde{T}(P_n, Q_n^+) + (R_n + R_n^+ + R_n^-)\widetilde{T}(P_n, Q_n^-) \\
&\quad + \widetilde{T}(P_n^+, Q_n) + \widetilde{T}(P_n^+, Q_n^+) + \widetilde{T}(P_n^+, Q_n^-) \\
&\quad + \widetilde{T}(P_n^-, Q_n) + \widetilde{T}(P_n^-, Q_n^+) + (R_n + R_n^+ + R_n^-)\widetilde{T}(P_n^-, Q_n^-) \\
&= \widetilde{T}(P_n, Q_n) + \widetilde{T}(P_n, Q_n^+) + R_n\widetilde{T}(P_n, Q_n^-) + R_n^+\widetilde{T}(P_n, Q_n^-) \\
&\quad + R_n^-\widetilde{T}(P_n, Q_n^-) + \widetilde{T}(P_n^+, Q_n) + \widetilde{T}(P_n^+, Q_n^+) + \widetilde{T}(P_n^+, Q_n^-) \\
&\quad + \widetilde{T}(P_n^-, Q_n) + \widetilde{T}(P_n^-, Q_n^+) + R_n\widetilde{T}(P_n^-, Q_n^-) + R_n^+\widetilde{T}(P_n^-, Q_n^-) \\
&\quad + R_n^-\widetilde{T}(P_n^-, Q_n^-).
\end{aligned}$$

Since the mapping

$$\ell_0^1(\Delta_m\mathbf{E}) \times \ell_0^1(\Delta_m\mathbf{F}) \xrightarrow{(\sigma,\sigma)} E_0 \times F_0 \xrightarrow{T} G_0 \xrightarrow{j} \ell_0^\infty(\Sigma_m\mathbf{G})$$

is compact, and the sequence spaces satisfy (AP), Theorem 4.2 may be applied, and it follows that each one of the operators $\widetilde{T}(P_n, Q_n)$, $R_n\widetilde{T}(P_n, Q_n^-)$ and $R_n\widetilde{T}(P_n^-, Q_n^-)$ are compact from $\ell_{f_1}^p(\Delta_m\mathbf{E}) \times \ell_{f_2}^q(\Delta_m\mathbf{F})$ to $\ell_{f_2}^r(\Sigma_m\mathbf{G})$, for all n; and that each one of the sequence of norms

$$\begin{gathered}
\{\|\widetilde{T}(P_n, Q_n^+)\|\},\ \ \{\|R_n^-\widetilde{T}(P_n, Q_n^-)\|\},\ \ \{\|\widetilde{T}(P_n^+, Q_n)\|\}, \\
\{\|\widetilde{T}(P_n^+, Q_n^+)\|\},\ \ \{\|\widetilde{T}(P_n^+, Q_n^-)\|\},\ \ \{\|\widetilde{T}(P_n^-, Q_n)\|\}, \\
\{\|\widetilde{T}(P_n^-, Q_n^+)\|\},\ \ \{\|R_n^+\widetilde{T}(P_n^-, Q_n^-)\|\}\ \text{ and }\ \{\|R_n^-\widetilde{T}(P_n^-, Q_n^-)\|\}
\end{gathered}$$

converges to zero in $\mathrm{Bil}(\ell_{f_1}^p(\Delta_m\mathbf{E}) \times \ell_{f_2}^q(\Delta_m\mathbf{F}), \ell_{f_2}^r(\Sigma_m\mathbf{G}))$, when $n \to \infty$.

Thus, to prove that

$$\begin{aligned}
\widetilde{T}\colon \big(\ell_0^1(\Delta_m\mathbf{E}), \ell_1^1(\Delta_m\mathbf{E})\big)_{\gamma,p} &\times \big(\ell_0^1(\Delta_m\mathbf{F}), \ell_1^1(\Delta_m\mathbf{F})\big)_{\rho,q} \\
&\longrightarrow \big(\ell_0^\infty(\Sigma_m\mathbf{G}), \ell_1^\infty(\Sigma_m\mathbf{G})\big)_{\rho,r}
\end{aligned}$$

is also compact, it only remains to control $R_n^+T(P_n, Q_n^-)\colon \ell_{f_1}^p(\Delta_m\mathbf{E}) \times \ell_{f_2}^q(\Delta_m\mathbf{F}) \to \ell_{f_2}^r(\Sigma_m\mathbf{G})$.

We have

$$
\begin{aligned}
&\big\| R_n^+ \widetilde{T}(P_n, Q_n^-) \big\|_{\mathrm{Bil}(\ell^p_{f_1}(\Delta_m \mathbf{E}) \times \ell^q_{f_2}(\Delta_m \mathbf{F}), \ell^r_{f_2}(\Sigma_m \mathbf{G}))} \\
&\quad \le C \| R_n^+ \widetilde{T}(P_n, Q_n^-) \|_0 \overline{\rho} \left(\frac{\| R_n^+ \widetilde{T}(P_n, Q_n^-) \|_1}{\| R_n^+ \widetilde{T}(P_m, Q_n^-) \|_0} \right) \\
&\quad \le C \overline{\rho} (\| R_n^+ \widetilde{T}(P_n, Q_n^-) \|_1) \frac{\overline{\rho}\big(1/\| R_n^+ \widetilde{T}(P_n, Q_n^-) \|_0\big)}{1/\| R_n^+ \widetilde{T}(P_n, Q_n^-) \|_0} \\
&\quad \le C \frac{\overline{\rho}\big(1/\| \widetilde{T} \|_0\big)}{1/\| \widetilde{T} \|_0} \overline{\rho}\big(\| R_n^+ \widetilde{T}(P_n, Q_n^-) \|_1\big),
\end{aligned}
$$

where $\|\cdot\|_i = \|\cdot\|_{\mathrm{Bil}(\ell^1_i(\Delta_m \mathbf{E}) \times \ell^1_i(\Delta_m \mathbf{F}), \ell^\infty_i(\Sigma_m \mathbf{G}))}$, $i = 0, 1$.

By the fact that the sequences $\{R_n^+\}$, $\{P_n\}$ and $\{Q_n^-\}$ are uniformly bounded, it is enough to prove that $\| R_n^+ \widetilde{T}(P_n, Q_n^-) \|_1 \to 0$. Considering the diagram

$$
\begin{aligned}
\ell^1_1(\Delta_m \mathbf{E}) \times \ell^1_1(\Delta_m \mathbf{F}) \xrightarrow{(P_n, Q_n^-)} \ell^1_0(\Delta_m \mathbf{E}) \times \ell^1_0(\Delta_m \mathbf{F}) \\
\xrightarrow{\widetilde{T}} \ell^\infty_0(\Sigma_m \mathbf{G}) \xrightarrow{R_n^+} \ell^\infty_1(\Sigma_m \mathbf{G}),
\end{aligned}
$$

we have by (AP3) that

$$
\begin{aligned}
\| R_n^+ \widetilde{T}(P_n, Q_n^-) \|_1 &\le \| R_n^+ \|_{L(\ell^\infty_0(\Sigma_m \mathbf{G}), \ell^\infty_1(\Sigma_m \mathbf{G}))} \| \widetilde{T} \|_0 \\
&\qquad \cdot \| P_n \|_{L(\ell^1_1(\Delta_m \mathbf{E}), \ell^1_0(\Delta_m \mathbf{E}))} \| Q_n^- \|_{L(\ell^1_1(\Delta_m \mathbf{F}), \ell^1_0(\Delta_m \mathbf{F}))} \\
&\le 2^n 2^{-(n+1)} \| \widetilde{T} \|_0 2^{-(n+1)},
\end{aligned}
$$

then we obtain the convergence to zero.

The proof is complete.

As a corollary, the bilinear version of Cwikel's theorem for the classical θ method is obtained.

COROLLARY 4.4. *Let* $\mathbf{E} = (E_0, E_1)$, $\mathbf{F} = (F_0, F_1)$ *and* $\mathbf{G} = (G_0, G_1)$ *be Banach couples. Let* $T \in \mathrm{Bil}(\mathbf{E} \times \mathbf{F}, \mathbf{G})$ *be given, such that the restriction* $T|_{E_0 \times F_0}$ *is compact from* $E_0 \times F_0$ *into* G_0. *Then, given* $0 < \theta < 1$, T *is compact from* $\mathbf{E}_{\theta,p} \times \mathbf{F}_{\theta,q}$ *into* $\mathbf{G}_{\theta,r}$, *where* $1/r = 1/p + 1/q - 1$.

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DEPARTMENT OF MATHEMATICS
MARINGA UNIVERSITY
AVENIDA COLOMBO 5790
MARINGÁ-PR
87020-900
BRAZIL
E-mail: ebsilva@uem.br

INSTITUTE OF MATHEMATICS - IMECC
UNIVERSITY OF CAMPINAS - UNICAMP
RUA SÉRGIO BUARQUE DE HOLANDA 651
CAMPINAS-SP
13083-859
BRAZIL
E-mail: dicesar@ime.unicamp.br