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A FAMILY OF REFLEXIVE VECTOR BUNDLES OF REDUCTION NUMBER ONE

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Abstract

A difficult issue in modern commutative algebra asks for examples of modules (more interestingly, reflexive vector bundles) having prescribed reduction number $r \geq 1$. The problem is even subtler if in addition we are interested in good properties for the Rees algebra. In this note we consider the case $r = 1$. Precisely, we show that the module of logarithmic vector fields of the Fermat divisor of any degree in projective 3-space is a reflexive vector bundle of reduction number 1 and Gorenstein Rees ring.

1. Motivation

Let R be a commutative unital Noetherian ring which is either a local ring with infinite residue field or a standard $\mathbb{N}$-graded algebra over an infinite field. Let E be a finitely generated R-module with (absolute) reduction number $\mathrm{r}(E)$, which is an important numerical invariant of E defined by means of suitable relations on Rees powers – basic concepts will be recalled in Section 2. In particular, it is known that $\mathrm{r}(E) = 0$ in the situation where E is of *linear type*, meaning as usual that the symmetric and Rees algebras of E coincide, since any such module admits no proper minimal reduction. In the standard case where $E = I$ is an R-ideal, the number $\mathrm{r}(I)$ has been widely investigated – while much still has to be done. We refer to the books [12], [20] and their references. The next situation, which is not so far from (but certainly more complicated than) the classical situation of ideals, is when E features the rigid structure of being a direct sum of ideals. Notice that E cannot be reflexive if any of the ideals in the direct sum has grade at least 2.

It is thus quite natural to consider the following motivating, difficult task: given a prescribed integer $r \geq 1$, furnish explicit examples of finitely generated reflexive R-modules E, of rank at least 2, satisfying

$$\mathrm{r}(E) = r.$$

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Reflexiveness, of course, can be achieved if for instance R is a normal domain (e.g., a polynomial ring over a field) and E is a module of second-order syzygies over R. In addition to the above conditions, we may also ask (as we will) that the module be a *vector bundle* in the sense of being locally free on the punctured spectrum of R.

The case $r = 1$, on which we focus in this paper, has attracted the interest of several authors, resulting in influential papers starting with Katz-Kodiyalam [14], where it is proved that any (non-free) torsionfree integrally closed module over a 2-dimensional regular local ring has reduction number 1 and moreover its Rees algebra is Cohen-Macaulay. This can be seen as a highly non-trivial analogue, for modules, of Huneke-Sally [11]. Further, Simis-Ulrich-Vasconcelos [19, Theorem 5.14] produced (non-reflexive) modules of reduction number $r \leq 1$ given by means of a suitable linkage via a complete intersection module assumed moreover to be a non-free vector bundle; if such a vector bundle is a reduction of the link, then $r = 1$. Results of a similar taste, but considering instead socle modules of parameter modules over 2-dimensional Cohen-Macaulay rings, are given in Hayasaka [10]. We point out that the application of linkage – socles, more precisely – to the problem in the case of ideals, together with the philosophy that such a technique tends to give rise to reduction number 1, originated in Corso-Polini-Vasconcelos [4] (see also Corso-Polini [3]).

What we do in this paper is to contribute to the problem by furnishing concretely a family of reflexive vector bundles, over a 4-dimensional polynomial ring, satisfying $r = 1$. More precisely, our Proposition 3.1 says that if k stands for an algebraically closed field of characteristic zero, then the module $T_{A/k}(\mathscr{F}_d)$ of the ambient vector fields tangent along the projective surface

$$\mathscr{F}_d = V(F_d) \subset \mathbb{P}^3_k$$

defined by the Fermat polynomial

$$F_d = x_1^d + x_2^d + x_3^d + x_4^d \in A_d \subset A = k[x_1, x_2, x_3, x_4], \qquad d \geq 2,$$

is a reflexive vector bundle of reduction number 1. This module is oftentimes denoted by $\mathrm{Der}_k(-\log \mathscr{F}_d)$ and called the *module of logarithmic derivations*, or the *tangential idealizer*, of the ideal $(F_d) \subset A$. We verify, in addition, that its Rees algebra is Gorenstein.

For the proof of Proposition 3.1 the key preparatory result is Lemma 2.1, which asserts that if the *fiber cone* $\mathbf{F}(E)$ – the special fiber of the Rees algebra $\mathbf{R}(E)$ of E – is Cohen-Macaulay of multiplicity 2, then $\mathrm{r}(E) = 1$. This lemma, which is inspired by a result from Corso-Polini-Vasconcelos [5] on the multiplicity of fiber cones of ideals, allows us to reduce the problem to the obtainment of a presentation of the standard graded k-algebra

$$\mathbf{F}(T_{A/k}(\mathscr{F}_d)) = \mathbf{R}(T_{A/k}(\mathscr{F}_d)) \otimes_A k,$$

which we will show to be, quite surprisingly, a polynomial ring in 1 indeterminate over the homogeneous coordinate ring of a Pfaffian hypersurface which, by changing signs, turns out to be the celebrated Klein quadric in $\mathbb{P}^5$.

We finish the paper with further remarks as well as questions and examples. In particular, we discuss about the main difficulty for generalizing Proposition 3.1 to Fermat hypersurfaces in 5 or more indeterminates. For instance, for the Fermat cubic in 5 variables, the ring $\mathbf{F}(T_{A/k}(\mathscr{F}_3))$ is Gorenstein but has multiplicity 5, so that our auxiliary lemma does not apply. Also, based on several experiments, we were led to raise the question as to whether our Proposition 3.1 is valid for any *smooth* hypersurface in $\mathbb{P}^3$.

2. Fiber cones of modules and key auxiliary result

By *ring* we always mean *commutative ring with* 1. Let R be either a Noetherian local ring with maximal ideal $\mathbf{m}$ and residue field k, or a standard graded algebra $R = \bigoplus_{i=0}^{\infty} R_i$ over a field $R_0 = k$ and with homogeneous maximal ideal $\mathbf{m} = \bigoplus_{i=1}^{\infty} R_i$. We permanently assume that k is infinite (from the next section on, we shall in fact require that $\mathrm{char}(k) = 0$). Let $E \subsetneq R^e$, $e \geq 1$, be a strict embedding of a finitely generated R-module E with $\mathrm{rank}(E) = e$ – meaning, as usual, that $K \otimes_R E \simeq K^e$, where K is the total ring of fractions of R – into the free R-module R^e. In particular, E is torsionfree over R.

We intend to briefly recall the basic definitions concerning blowup rings of modules in order to reach the concept of *fiber cone*. References for this general part are [8], [19], [21].

We write

$$S := \mathbf{S}(R^e) = \bigoplus_{n=0}^{\infty} S_n$$

for the homogeneous symmetric algebra of R^e, which may be regarded as a standard graded polynomial ring $S = R[y_1, \ldots, y_e]$ in indeterminates $y_1, \ldots, y_e$ over $S_0 = R$. In degree 1, we get the R-module $S_1 = \sum_{i=1}^{e} Ry_i$ together with the natural map $\lambda\colon R^e \to S_1$ which sends a given $v = (\alpha_1, \ldots, \alpha_e) \in R^e$ to the linear form $\lambda(v) = \sum_{i=1}^{e} \alpha_i y_i$. The *Rees algebra* of E is the graded subalgebra

$$\mathbf{R}(E) = \bigoplus_{n=0}^{\infty} E^n \subset S, \qquad E^n = [\mathbf{R}(E)]_n$$

generated over $E^0 = R$ by $\lambda(v_1), \ldots, \lambda(v_m)$ for some (any) generating set $\{v_1, \ldots, v_m\}$ of E as an R-module (in particular, $\mathbf{R}(E)$ is an integral domain if so is R). Thus, $\mathbf{R}(E)$ is generated over R by $E^1 \subset S_1$. If $\mathcal{T} \subset \mathbf{S}_R(E)$ is the ideal of R-torsion of the symmetric algebra of $E \subset R^e$, then

$$\mathbf{R}(E) = \mathbf{S}(E)/\mathcal{T}.$$

The module E is said to be *of linear type* if $\mathcal{T} = (0)$, that is, if its Rees algebra is equal to its symmetric algebra. Clearly, free modules are of linear type. If R is a domain, then the torsionfree R-module E is of linear type if and only if its symmetric algebra is a domain as well.

An R-submodule $U \subseteq E$ is said to be a *reduction* of E if $U^1 E^r = E^{r+1}$ for $r \gg 0$. The *reduction number of E with respect to U* is defined as

$$\mathrm{r}_U(E) = \min\{s \geq 0 \mid U^1 E^s = E^{s+1}\}.$$

A *minimal reduction* of E is a reduction that is minimal with respect to inclusion. Since the residue field k of R is assumed to be infinite, minimal reductions are known to exist; moreover, they also have rank e. The (*absolute*) *reduction number of E* is the integer $\mathrm{r}(E) = \min\{\mathrm{r}_U(E)\}$, where $U \subseteq E$ ranges over all minimal reductions of E. The situation of reduction number zero corresponds to the linear type case.

Still in analogy with the case of ideals, the *fiber cone* of the R-module $E \subset R^e$ is the special fiber of its Rees algebra, that is,

$$\mathbf{F}(E) = \mathbf{R}(E) \otimes_R k = \bigoplus_{n=0}^{\infty} \frac{E^n}{\mathbf{m}E^n}$$

which is *standard* graded over the field $[\mathbf{F}(E)]_0 = k$. In the present setting where k is infinite, the Krull dimension $\ell(E)$ of $\mathbf{F}(E)$ – the so-called *analytic spread* of E – is equal to $\nu(U)$ for any minimal reduction $U \subseteq E$, where $\nu(-)$ stands for minimal number of generators.

We are now ready to prove a key preparatory lemma, which may not be surprising for experts but does not seem to be available in the literature in the present context of modules. Our inspiration is Corso-Polini-Vasconcelos [5, Proposition 2.4], where a multiplicity-based criterion is given for the Cohen-Macaulayness of the fiber cone $\mathbf{F}(I) = \bigoplus_{n=0}^{\infty} I^n/\mathbf{m}I^n$ of an ideal $I \subset R$ (cf. also [2], [6], [7], [11], [13], [18]). In order to keep the very definition of Rees algebra as adopted above, we require the module E to possess a rank over R and to be embedded in a free module R^e, but specifically in the statement below, and for the sake of generality, we do not require e to be equal to the rank of E.

LEMMA 2.1. *If the standard graded k-algebra $\mathbf{F}(E)$ is Cohen-Macaulay of multiplicity* 2 (*e.g., a quadratic hypersurface ring*), *then* $\mathrm{r}(E) = 1$.

PROOF. Set $r := \mathrm{r}(E)$ and notice first that $r \geq 1$. In fact, it is well-known that the case $r = 0$ corresponds to the situation where E is of linear type, so that the special fiber ring $\mathbf{F}(E)$ would be given by the symmetric algebra of the k-vector space $E \otimes_R k$, which is simply a polynomial ring (in $\nu(E)$ indeterminates) over k. But this cannot happen, as the multiplicity of $\mathbf{F}(E)$ is not 1.

Now let U be any minimal reduction of E such that $\mathrm{r}_U(E) = r$. In particular, the dimension $\ell = \ell(E)$ of the Cohen-Macaulay graded ring $\mathbf{F}(E)$ turns out to be equal to $\nu(U)$. Therefore, since the ideal

$$U^1\mathbf{F}(E) \subset \mathbf{F}(E)$$

is generated by a homogeneous system of parameters, it must be a complete intersection in this case. It follows that the Hilbert series of $\mathbf{F}(E)$ and $\mathbf{F}(E)/U^1\mathbf{F}(E)$ are related by the equality

$$H(\mathbf{F}(E), t) = \frac{1}{(1-t)^\ell} H\left(\frac{\mathbf{F}(E)}{U^1\mathbf{F}(E)}, t\right)$$

where t is a variable and $H(\mathbf{F}(E)/U^1\mathbf{F}(E), 1) \neq 0$, which is the multiplicity of $\mathbf{F}(E)$. Notice that

$$\frac{\mathbf{F}(E)}{U^1\mathbf{F}(E)} = k \oplus \left(\bigoplus_{i=1}^{\infty} \frac{E^i}{\mathbf{m}E^i + U^1E^{i-1}}\right) = k \oplus \left(\bigoplus_{i=1}^{r} \frac{E^i}{\mathbf{m}E^i + U^1E^{i-1}}\right)$$

which yields that the h-polynomial of $\mathbf{F}(E)$ is explicitly given by

$$H\left(\frac{\mathbf{F}(E)}{U^1\mathbf{F}(E)}, t\right) = 1 + \sum_{i=1}^{r} \mathrm{length}\left(\frac{E^i}{\mathbf{m}E^i + U^1E^{i-1}}\right)t^i = 1 + \sum_{i=1}^{r} \nu(K^i)t^i,$$

where here we set $K^i := E^i/U^1E^{i-1}$, $i = 1, \ldots, r$. Evaluating at $t = 1$ we obtain that the multiplicity $f(E)$ of $\mathbf{F}(E)$ can be expressed as

$$f(E) = 1 + \sum_{i=1}^{r} \nu(K^i).$$

On the other hand, $f(E) = 2$ by hypothesis. Therefore

$$\sum_{i=1}^{r} \nu(K^i) = 1$$

which forces $r = 1$ since necessarily $K^i \neq 0$ for $i = 1, \ldots, r$.

3. Main result

We let k be an algebraically closed field of characteristic zero and $A = k[x_1, \dots, x_n]$, $n \geq 2$, be a standard graded polynomial ring over k, where as usual the indeterminates x_i's may be regarded as homogeneous coordinates of a projective space $\mathbb{P}^{n-1} = \mathbb{P}_k^{n-1}$. The irrelevant ideal is thus $\mathbf{m} = (x_1, \dots, x_n) \subset A$.

Recall that the module $T_{A/k}(\mathscr{X})$ of logarithmic vector fields of an algebraic variety $\mathscr{X} \subset \mathbb{P}^{n-1}$ – that is, vector fields defined on $\mathbb{P}^{n-1}$ and tangent along the smooth part of $\mathscr{X}$ – can be concretely regarded as the (graded) A-submodule of $D_{A/k} = \bigoplus_{i=1}^{n} A \frac{\partial}{\partial x_i} \simeq A^n$, the module of k-derivations of A, formed by the logarithmic k-derivations of the defining ideal $I_{\mathscr{X}} \subset A$ of $\mathscr{X}$. Explicitly,

$$T_{A/k}(\mathscr{X}) = \{\delta \in D_{A/k} \mid \delta(I_{\mathscr{X}}) \subset I_{\mathscr{X}}\}.$$

Thus, writing simply $\delta = (h_1, \dots, h_n) \in D_{A/k}$, we have that δ is logarithmic for $\mathscr{X}$ if and only if

$$\sum_{i=1}^{n} h_i \frac{\partial f}{\partial x_i} \in I_{\mathscr{X}}$$

for every $f \in I_{\mathscr{X}}$ (it clearly suffices to test this condition on any given generating set of $I_{\mathscr{X}}$). In particular, the Euler derivation $\varepsilon = (x_1, \dots, x_n)$ is logarithmic in virtue of Euler's identity for homogeneous polynomials. Moreover, the inclusion $I_{\mathscr{X}} D_{A/k} \subset T_{A/k}(\mathscr{X})$ yields that $\operatorname{rank}(T_{A/k}(\mathscr{X})) = n \geq 2$.

If $\mathscr{X}$ is a hypersurface, the reflexiveness of the A-module $T_{A/k}(\mathscr{X})$ is a well-known fact, first noticed by Saito [17, Corollary 1.7] originally in the complex analytic category, and it is also known to hold in the algebraic case as well. Recently, we proved the converse of Saito's observation and thus derived the following characterization ([16, Theorem 3.1]): if $\mathscr{X}$ is an (affine or projective) embedded, proper, non-empty algebraic variety, then $T_{A/k}(\mathscr{X})$ is reflexive if and only if $\mathscr{X}$ is a hypersurface.

In the case where $n = 4$ and $\mathscr{X}$ is the Fermat surface

$$\mathscr{F}_d := V(x_1^d + x_2^d + x_3^d + x_4^d) \subset \mathbb{P}^3$$

of any given degree $d \geq 2$, we obtain Proposition 3.1 below as the main goal of this note, regarding the reduction number and blowup algebras of $T_{A/k}(\mathscr{F}_d)$.

Recall that, by a typical abuse of terminology, a finitely generated A-module is said to be a *vector bundle* if it is locally free on the Zariski-open set $\operatorname{Spec}(A) \setminus V(\mathbf{m})$, the punctured spectrum of A. Further, recall that the so-called *Klein quadric* (also dubbed *Plücker quadric*) is the hypersurface $\mathscr{K} \subset \mathbb{P}^5$ given, in homogeneous coordinates $(\lambda_1 : \ldots : \lambda_6)$ and up to changing signs, by

$$\lambda_2\lambda_4 - \lambda_1\lambda_5 - \lambda_3\lambda_6 = 0$$

which is in fact seen to be the Pfaffian of a 4×4 generic alternating matrix. This is a classical geometric object (the realization in $\mathbb{P}^5$ of the Grassmannian of lines in $\mathbb{P}^3$) and we denote its coordinate ring by $\Gamma_{\mathscr{K}}$.

PROPOSITION 3.1 ($n = 4$). *For any $d \geq 2$ the A-module $T_{A/k}(\mathscr{F}_d)$ is a reflexive vector bundle, satisfying:*

(i) $\mathbf{F}(T_{A/k}(\mathscr{F}_d)) = \Gamma_{\mathscr{K}}[u]$, *where u is an indeterminate over $\Gamma_{\mathscr{K}}$;*

(ii) $\mathrm{r}(T_{A/k}(\mathscr{F}_d)) = 1$*;*

(iii) $\mathbf{R}(T_{A/k}(\mathscr{F}_d))$ *is a Gorenstein domain.*

PROOF. By the general structure result for the module of logarithmic derivations in the homogeneous case (cf., e.g., [15, Lemma 2.2]), we can decompose $T_{A/k}(\mathscr{F}_d)$ as

$$T_{A/k}(\mathscr{F}_d) = T^0_{A/k}(\mathscr{F}_d) \oplus A\varepsilon,$$

where $T^0_{A/k}(\mathscr{F}_d) \subset T_{A/k}(\mathscr{F}_d)$ stands for the submodule formed by the derivations vanishing on the Fermat polynomial $F_d = x_1^d + x_2^d + x_3^d + x_4^d$. Setting

$$\xi_i := x_i^{d-1}, \qquad i = 1, \dots, 4,$$

the gradient ideal of F_d is $\mathfrak{J} = (\xi_1, \xi_2, \xi_3, \xi_4) \subset A$. Clearly, $T^0_{A/k}(\mathscr{F}_d)$ is isomorphic to the module of first-order syzygies of $\mathfrak{J}$. Localizing the short exact sequence

$$0 \longrightarrow T^0_{A/k}(\mathscr{F}_d) \longrightarrow A^4 \longrightarrow \mathfrak{J} \longrightarrow 0$$

at any non-maximal prime ideal $\wp \subset A$, and noticing that $\mathfrak{J}_\wp = A_\wp$ as $\mathfrak{J}$ is $\mathbf{m}$-primary, we obtain a splitting and therefore

$$(T^0_{A/k}(\mathscr{F}_d))_\wp \simeq A^3_\wp,$$

thus showing that the A-module $T_{A/k}(\mathscr{F}_d)$ (which is reflexive, by the preceding discussion) has the vector bundle property. Since furthermore $\mathfrak{J}$ is a complete intersection, the module $T^0_{A/k}(\mathscr{F}_d)$ can be realized as the cokernel of the second Koszul map of $\mathfrak{J}$, and hence there is a minimal free presentation

$$A^4 \xrightarrow{\kappa_d^{(4)}} A^6 \longrightarrow T^0_{A/k}(\mathscr{F}_d) \longrightarrow 0.$$

Regarding ${\kappa_d}^{(4)}$ as a 6×4 matrix (taken in canonical bases), we can write

$$
{\kappa_d}^{(4)} = \begin{pmatrix} 0 & -\xi_3 & 0 & -\xi_2 \\ 0 & 0 & -\xi_3 & \xi_1 \\ 0 & \xi_1 & \xi_2 & 0 \\ -\xi_2 & \xi_4 & 0 & 0 \\ \xi_1 & 0 & \xi_4 & 0 \\ \xi_3 & 0 & 0 & \xi_4 \end{pmatrix},
$$

so that the symmetric algebra of $T^0_{A/k}(\mathscr{F}_d)$ can be expressed as the quotient ring $A[\mathbf{t}]/\mathfrak{L}$, where

$$
\mathbf{t} = \{t_1, \ldots, t_6\}
$$

is a set of 6 indeterminates over A and $\mathfrak{L}$ is the ideal generated by the entries of the matrix product $(\mathbf{t}) \cdot {\kappa_d}^{(4)}$. Since the A-module $T^0_{A/k}(\mathscr{F}_d)$ has rank $4-1=3$, it is locally free on $\operatorname{Spec}(A) \setminus V(\mathfrak{a})$, where

$$
\mathfrak{a} := I_3({\kappa_d}^{(4)}) \subset A,
$$

which is the 3rd Fitting ideal of $T^0_{A/k}(\mathscr{F}_d)$ and hence must be non-zero. This puts us in a position to apply a standard device (cf., e.g., [15, Lemma 2.11]) in order to conclude that the A-torsion of the symmetric algebra $\mathbf{S}(T^0_{A/k}(\mathscr{F}_d))$ is the saturation $(0)\colon \mathfrak{a}^\infty = \bigcup_{t \geq 1} (0)\colon \mathfrak{a}^t$, which by a routine verification is seen to be the principal ideal of $\mathbf{S}(T^0_{A/k}(\mathscr{F}_d))$ generated by the image, modulo $\mathfrak{L}$, of the quadratic polynomial

$$
Q := t_2 t_4 - t_1 t_5 - t_3 t_6 \in k[\mathbf{t}] \subset A[\mathbf{t}].
$$

Therefore, $\mathbf{R}(T^0_{A/k}(\mathscr{F}_d)) = A[\mathbf{t}]/(\mathfrak{L}\colon \mathfrak{a}^\infty) = A[\mathbf{t}]/(\mathfrak{L}, Q)$ and hence $\mathbf{R}(T_{A/k}(\mathscr{F}_d))$ can be written as $A[\mathbf{t}, u]/(\mathfrak{L}, Q)$ where u is a new indeterminate (see also Remark 3.3). Now, since clearly

$$
\mathfrak{L} \subset (x_1, x_2, x_3, x_4)(\mathbf{t}),
$$

we have

$$
\mathbf{F}(T_{A/k}(\mathscr{F}_d)) = (k[\mathbf{t}]/(Q))[u] = \Gamma_{\mathscr{K}}[u]
$$

thus giving (i). Together with Lemma 2.1, this yields $\mathrm{r}(T_{A/k}(\mathscr{F}_d)) = 1$, which proves (ii).

In order to check (iii), we notice first that by [21, Proposition 8.1] the dimension of the Rees algebra $\mathbf{R}(T_{A/k}(\mathscr{F}_d))$ (which is obviously a domain) is equal to $\dim(A) + \operatorname{rank}(T_{A/k}(\mathscr{F}_d)) = 8$. Thus, setting $B := A[\mathbf{t}, u]$, we have

$$
\operatorname{height}(\mathfrak{L}, Q) = \dim(B) - 8 = 3.
$$

Now, we consider a minimal free resolution of the Rees ideal $(\mathfrak{L}, Q) \subset B$, which is seen to possess the shape

$$0 \longrightarrow B \longrightarrow B^5 \longrightarrow B^5 \longrightarrow (\mathfrak{L}, Q) \longrightarrow 0$$

and hence $B/(\mathfrak{L}, Q)$ has projective dimension 3 over B. This yields that $(\mathfrak{L}, Q)$ is a perfect ideal. Moreover, its Cohen-Macaulay type (the last Betti number in the resolution) is 1, as needed.

REMARK 3.2 (On char(k)). It is probably true that this proposition remains valid in the situation where k has positive characteristic provided that it does not divide d. However, mainly in order to keep our result working safely for every $d \geq 2$, we focused on the characteristic zero setting.

REMARK 3.3 (Avoiding saturation). In the case of $T_{A/k}(\mathscr{F}_d)$, with $n = 4$ as in Proposition 3.1, the Rees equations may be computed quite naively by means of the very definition of Rees algebra instead of the saturation technique used in our proof. First, we embed $\mathbf{R}(T^0_{A/k}(\mathscr{F}_d))$ in the graded polynomial ring $S := \mathbf{S}(A^4) = A[y_1, y_2, y_3, y_4]$, so that we can express it as the A-subalgebra

$$\mathbf{R}(T^0_{A/k}(\mathscr{F}_d)) = A[\lambda_1, \dots, \lambda_6] \subset S,$$

where the linear forms $\lambda_i := \lambda(\delta_i) \in S_1$, $i = 1, \dots, 6$, correspond to the Koszul syzygies $\delta_1, \dots, \delta_6$ of $\mathfrak{J}$, in accordance with the general recipe recalled in Section 2. Explicitly,

$$\lambda_1 = -\xi_4 y_1 + \xi_1 y_4, \quad \lambda_2 = -\xi_4 y_2 + \xi_2 y_4, \quad \lambda_3 = -\xi_4 y_3 + \xi_3 y_4,$$
$$\lambda_4 = -\xi_3 y_1 + \xi_1 y_3, \quad \lambda_5 = -\xi_3 y_2 + \xi_2 y_3, \quad \lambda_6 = -\xi_2 y_1 + \xi_1 y_2,$$

thus yielding Klein's equation $\lambda_2\lambda_4 = \lambda_1\lambda_5 + \lambda_3\lambda_6$, which is the only non-linear minimal relation on the λ_i's. In fact, the kernel of the natural A-algebra surjection

$$A[t_1, \dots, t_6] \longrightarrow A[\lambda_1, \dots, \lambda_6]$$

is seen to be generated by the quadratic form $Q = t_2t_4 - t_1t_5 - t_3t_6$ and the linear forms generating $\mathfrak{L}$. Therefore, $(\mathfrak{L}, Q)$ is the Rees ideal of $T_{A/k}(\mathscr{F}_d)$ in the sense that it defines $\mathbf{R}(T_{A/k}(\mathscr{F}_d))$ in the ring $A[t_1, \dots, t_6, u]$, as needed. Of course, the saturation method has the advantage of giving to the Rees ideal a theoretic well-structured shape, which in particular tends to point more accurately to diverse clues on numerical invariants and associated primes, for instance.

REMARK 3.4. In case we do not have any information about a free resolution of the Rees ideal $(\mathfrak{L}, Q) \subset A[\mathbf{t}, u]$, we can prove that it is Cohen-Macaulay, at least, by resorting to the graded analogue of [19, Example 4.17] since the module $T_{A/k}(\mathscr{F}_d)$ is a reflexive vector bundle over a 4-dimensional graded polynomial ring, satisfying

$$\nu(T_{A/k}(\mathscr{F}_d)) = 7 = \operatorname{rank}(T_{A/k}(\mathscr{F}_d)) + 3.$$

Moreover we point out that, in general, if R is a Noetherian local ring and E is a finitely generated reflexive vector bundle over R with rank $e \geq 1$ and $\nu(E) \geq e+2$, then $\ell(E) \geq e+2$. This follows from [19, Proposition 4.1(b)].

The following question arises: in order to get precisely $\ell(E) = e+2$, do we need to require that $\nu(E) = e+2$? The answer is *no*. Indeed, for the module $T_{A/k}(\mathscr{F}_d) \subset A^4$ we have

$$\begin{aligned}\nu(T_{A/k}(\mathscr{F}_d)) = 7 &> \operatorname{rank}(T_{A/k}(\mathscr{F}_d)) + 2 \\ &= 6 = \dim(\Gamma_{\mathscr{K}}[u]) = \ell(T_{A/k}(\mathscr{F}_d)).\end{aligned}$$

Furthermore, we note that

$$\underbrace{\mathrm{r}(T_{A/k}(\mathscr{F}_d))}_{=1} < \underbrace{\ell(T_{A/k}(\mathscr{F}_d)) - \operatorname{rank}(T_{A/k}(\mathscr{F}_d))}_{=2} < \underbrace{\dim(A) - 1}_{=3}$$

thus illustrating that the inequalities given in [19, Theorem 4.2] may be simultaneously strict.

REMARK 3.5 (The cases $n = 2$ and $n = 3$). It is easy to get rid of the first low-dimensional situations, which, as we expect, feature a much simpler behavior: $T_{A/k}(\mathscr{F}_d)$ is of linear type in these cases. In order to check this, we employ below some of the notations and general facts used in the proof of Proposition 3.1.

(i) Assume that $A = k[x_1, x_2]$. Since $A/\mathfrak{J}$ has projective dimension 2 (over A) and $T^0_{A/k}(\mathscr{F}_d)$ is the syzygy module of $\mathfrak{J}$, we obtain that $T^0_{A/k}(\mathscr{F}_d)$ is free (of rank 1). In virtue of the equality $T_{A/k}(\mathscr{F}_d) = T^0_{A/k}(\mathscr{F}_d) \oplus A\varepsilon$, we get that $T_{A/k}(\mathscr{F}_d)$ is free as well, hence of linear type, and $\mathbf{F}(T_{A/k}(\mathscr{F}_d)) = k[t_1, t_2]$.

(ii) Now assume that $A = k[x_1, x_2, x_3]$. The module $T^0_{A/k}(\mathscr{F}_d)$ of first-order syzygies of $\mathfrak{J}$ is 3-generated in this case, and the second-order syzygy matrix $\kappa_d^{(3)}$ of $\mathfrak{J}$ is simply the transpose of the row-matrix $(-\xi_2\ \xi_1\ \xi_3) = (-x_2^{d-1}\ x_1^{d-1}\ x_3^{d-1})$. The symmetric algebra

$$\mathbf{S}(T^0_{A/k}(\mathscr{F}_d)) = \frac{A[t_1, t_2, t_3]}{(-\xi_2 t_1 + \xi_1 t_2 + \xi_3 t_3)}$$

is seen to be an integral domain, and hence so is $\mathbf{S}(T_{A/k}(\mathscr{F}_d)) = \mathbf{S}(T^0_{A/k}(\mathscr{F}_d))[u]$, where u is a new indeterminate. It follows that $T_{A/k}(\mathscr{F}_d)$ is a non-free module of linear type, and $\mathbf{F}(T_{A/k}(\mathscr{F}_d)) = k[t_1, t_2, t_3, u]$.

4. Further remarks, examples and questions

We finish by considering further aspects as well as illustrations and questions that are of interest in regard of this paper.

REMARK 4.1 (On the case $n \geq 5$). We just want to comment on the main difficulty for extending Proposition 3.1 to the higher dimensional case

$$A = k[x_1, \ldots, x_n], \qquad n \geq 5.$$

The proof could begin in a totally analogous manner. Indeed, we have $T_{A/k}(\mathscr{F}_d) = T^0_{A/k}(\mathscr{F}_d) \oplus A\varepsilon$, where $T^0_{A/k}(\mathscr{F}_d)$ is the module of the derivations vanishing on the Fermat polynomial $F_d = x_1^d + \ldots + x_n^d$. Setting again $\xi_i = x_i^{d-1}$, $i = 1, \ldots, n$, the gradient ideal of F_d is the complete intersection $\mathfrak{J} = (\xi_1, \ldots, \xi_n) \subset A$, which yields that $T^0_{A/k}(\mathscr{F}_d)$ has a minimal free presentation

$$\bigwedge^3 A^n \xrightarrow{\kappa_d{}^{(n)}} \bigwedge^2 A^n \longrightarrow T^0_{A/k}(\mathscr{F}_d) \longrightarrow 0.$$

Regarding $\kappa_d{}^{(n)}$ as a matrix (taken in canonical bases), each of its $\binom{n}{3}$ column-vectors can be written with exactly 3 non-zero coordinates; more precisely, its transpose has the form

$$(0, \ldots, -\xi_j, \ldots, \xi_i, \ldots, \xi_t, \ldots, 0) \in A^{\binom{n}{2}}, \qquad \text{with} \quad i < j < t.$$

Thus $\mathbf{S}(T^0_{A/k}(\mathscr{F}_d)) = A[\mathbf{t}]/\mathfrak{L}$, where $\mathbf{t}$ is a set of $\binom{n}{2}$ indeterminates over A, and

$$\mathfrak{L} = I_1((\mathbf{t}) \cdot \kappa_d{}^{(n)}).$$

Since the A-module $T^0_{A/k}(\mathscr{F}_d)$ has rank $n-1$, it is locally free on the Zariski-open set $\mathrm{Spec}(A) \setminus V(\mathfrak{a})$, where

$$\mathfrak{a} := I_{\binom{n}{2}-(n-1)}(\kappa_d{}^{(n)}) \subset A,$$

so that $\mathbf{S}(T^0_{A/k}(\mathscr{F}_d)) = A[\mathbf{t}]/\mathfrak{L}$ has A-torsion given by $(\mathfrak{L}\colon \mathfrak{a}^\infty)/\mathfrak{L}$, and hence $\mathfrak{L}\colon \mathfrak{a}^\infty$ is the Rees ideal of $T_{A/k}(\mathscr{F}_d)$ in the ring $A[\mathbf{t}, u]$. But now the crucial obstacle comes: at least for $n = 5$ (and $d = 3$), the k-algebra $\mathbf{F}(T_{A/k}(\mathscr{F}_3)) = A[\mathbf{t}, u]/(\mathfrak{L}\colon \mathfrak{a}^\infty) \otimes_A (A/\mathbf{m})$ – which does *not* define a hypersurface in this case – has multiplicity greater than 2, so that Lemma 2.1 does not apply. This is detailed in Example 4.2 below.

EXAMPLE 4.2. Let $A = k[x_1, x_2, x_3, x_4, x_5]$ and consider the Fermat cubic 3-fold $\mathscr{F}_3 \subset \mathbb{P}^4$. In this case, the ideal $\mathfrak{L}$ defining $\mathbf{S}(T_{A/k}(\mathscr{F}_3))$ in the polynomial ring $A[\mathbf{t}, u]$ in $\binom{5}{2} + 1 = 11$ indeterminates $t_1, \ldots, t_{10}, u$ over A is generated by $\binom{5}{3} = 10$ linear forms arising from the Koszul syzygies of $\mathfrak{J}$. If $\mathfrak{a} = I_6(\kappa_3^{(5)})$ then the Rees ideal of $T_{A/k}(\mathscr{F}_3)$ is given by

$$\mathfrak{L}\colon \mathfrak{a}^\infty = (\mathfrak{L}, Q_1, \ldots, Q_5)$$

where

$$Q_1 = t_2t_5 - t_1t_6 - t_4t_{10}, \quad Q_2 = t_2t_8 - t_1t_9 - t_3t_{10}, \quad Q_3 = t_3t_5 - t_1t_7 - t_4t_8,$$
$$Q_4 = t_3t_6 - t_2t_7 - t_4t_9, \quad Q_5 = t_6t_8 - t_5t_9 - t_7t_{10},$$

so that

$$\mathbf{F}(T_{A/k}(\mathscr{F}_3)) = \frac{k[\mathbf{t}, u]}{(Q_1, \ldots, Q_5)}$$

which, by a computation with [9] (or with its first version [1]), is seen to have multiplicity 5 and hence Lemma 2.1 does not apply. On the other hand, we note that this fiber cone is Gorenstein, of dimension

$$\ell(T_{A/k}(\mathscr{F}_3)) = 8.$$

As it is well-known, we can alternatively obtain that $\mathbf{F}(T_{A/k}(\mathscr{F}_3))$ has multiplicity 5 by observing that its defining ideal is generated by the Pfaffians of a suitable skew-symmetric matrix, to wit,

$$\begin{pmatrix} 0 & t_1 & t_2 & t_3 & t_4 \\ -t_1 & 0 & t_{10} & t_8 & t_5 \\ -t_2 & -t_{10} & 0 & t_9 & t_6 \\ -t_3 & -t_8 & -t_9 & 0 & t_7 \\ -t_4 & -t_5 & -t_6 & -t_7 & 0 \end{pmatrix}.$$

We believe that all the features discussed in this example are unaffected if we replace $\mathscr{F}_3$ by the Fermat hypersurface $\mathscr{F}_d \subset \mathbb{P}^4$ of arbitrary degree $d \geq 2$.

QUESTION 4.3. If $n = 5$, as in the example above, what is the reduction number r of $T_{A/k}(\mathscr{F}_3)$? Is it possible that $r = 1$ as in the case $n = 4$? Since we have checked that $\mathbf{F}(T_{A/k}(\mathscr{F}_3))$ is Gorenstein, hence Cohen-Macaulay, the following observation seems to be relevant: by the proof of Lemma 2.1, the degree of the h-polynomial of the fiber cone is precisely the reduction number, which implies that it does not depend on the choice of the minimal reduction of the module. Thus, since $\mathbf{F}(T_{A/k}(\mathscr{F}_3))$ has multiplicity 5, it follows (again by the proof of Lemma 2.1) an equality

$$\sum_{i=1}^{r} \nu\left(\frac{T_{A/k}(\mathscr{F}_3)^i}{U^1 T_{A/k}(\mathscr{F}_3)^{i-1}}\right) = 4$$

for *any* given minimal reduction $U \subset T_{A/k}(\mathscr{F}_3)$. In particular, $r = 1$ if and only if there exists a minimal reduction U satisfying $\nu(T_{A/k}(\mathscr{F}_3)^1/U^1) = 4$, but we have been unable to verify whether this is the case. Naturally, a harder question is: for any given $n \geq 5$ and $d \geq 2$, how can we compute the reduction number of $T_{A/k}(\mathscr{F}_d)$?

QUESTION 4.4. Assume that $n \geq 6$. Is the fiber cone $\mathbf{F}(T_{A/k}(\mathscr{F}_3))$ Gorenstein or at least Cohen-Macaulay? What is its multiplicity? Is its defining ideal still generated by Pfaffians? In other words, with a view to understand the fiber cone for arbitrary n, it is of interest to realize how naturally generic skew-symmetric matrices occur in the problem.

REMARK 4.5 (On the general smooth case). It seems plausible to guess that Proposition 3.1 is valid for any smooth surface $\mathscr{S} = V(F) \subset \mathbb{P}^3$ (of degree at least 2). Naturally, each ξ_i could be replaced by the ith partial derivative of the polynomial F, so that, by smoothness, the gradient ideal $\mathfrak{J} \subset A$ of F possesses a Koszul resolution as well. It is thus clear that such steps are completely analogous. The problem however is on the effective side and concerns precisely the determination of the Rees ideal $\mathfrak{L}\colon \mathfrak{a}^\infty$ of $T_{A/k}(\mathscr{S})$ in the ring $A[\mathbf{t}, u]$, which is fundamental for the obtainment of the defining ideal of the fiber cone in $k[\mathbf{t}, u]$. In each of the several examples of smooth divisors $\mathscr{S} = V(F) \subset \mathbb{P}^3$ that we have considered, there is an identification

$$\mathbf{F}(T^0_{A/k}(\mathscr{S})) \simeq \Gamma_{\mathscr{K}}$$

induced simply by a projective change of coordinates $\sigma \in \mathrm{PGL}_k(6)$. More precisely, we verified that $\mathbf{F}(T_{A/k}(\mathscr{S})) = k[\mathbf{t}, u]/(Q')$, where Q' is the polynomial obtained from the Pfaffian $Q = t_2t_4 - t_1t_5 - t_3t_6$ after the action of σ on the t_i's. We furnish below an illustration of this behavior.

EXAMPLE 4.6. We consider the smooth quartic surface $\mathscr{S} \subset \mathbb{P}^3$ defined by

$$F = x_1^4 + x_1^3x_2 + x_2^3x_3 + x_3^4 + x_3^3x_4 + x_4^4.$$

Here we have $\mathbf{F}(T_{A/k}(\mathscr{S})) = (k[\mathbf{t}]/(Q'))[u]$, where

$$Q' = t_2t_4 - t_1t_5 - t_3\left(t_6 - \frac{4}{9}t_5\right)$$

which obviously can be obtained from $Q = t_2t_4 - t_1t_5 - t_3t_6$ by the projective change of coordinates given by $t_6 \mapsto t_6 - 4/9t_5$ (the variables $t_1, \ldots, t_5$ are kept fixed). We computed this example with [1], and by means of a careful analysis of the computation we realized that the structural difference between Q and Q' is a consequence of the choice of bases in the presentation of $T^0_{A/k}(\mathscr{F}_d)$

(see the proof of Proposition 3.1). In addition, the same analysis suggests that the general case of a smooth hypersurface in $\mathbb{P}^3$ of degree at least 2 (as asked in Remark 4.5 above) could be carried out first by regarding the partial derivatives of its defining equation as "variables", as they generate a parameter ideal anyway, so that the argument (in particular, as to the structure of the Rees ideal) presumably would follow in parallel to the case of Fermat divisors treated in this paper.

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