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A NOTE ON THE VAN DER WAERDEN COMPLEX

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Abstract

Ehrenborg, Govindaiah, Park, and Readdy recently introduced the van der Waerden complex, a pure simplicial complex whose facets correspond to arithmetic progressions. Using techniques from combinatorial commutative algebra, we classify when these pure simplicial complexes are vertex decomposable or not Cohen-Macaulay. As a corollary, we classify the van der Waerden complexes that are shellable.

1. Introduction

Let $V = \{x_1, \dots, x_n\}$ and suppose that $0 < k < n$. The *van der Waerden complex* of dimension k on n vertices, denoted $\mathtt{vdW}(n, k)$, is the pure simplicial complex on V whose facet set is given by

$$\mathtt{vdW}(n, k) = \langle \{x_i, x_{i+d}, x_{i+2d}, \dots, x_{i+kd}\} \mid d \in \mathbb{Z} \text{ with } 1 \leq i < i + kd \leq n\rangle.$$

In other words, the facets of $\mathtt{vdW}(n, k)$ correspond to all arithmetic progressions of length $k + 1$ whose largest element is less than or equal to n. The complexes $\mathtt{vdW}(n, k)$ were introduced by Ehrenborg, Govindaiah, Park, and Readdy [2] as part of a recent program to study the topology of complexes that arise within number theory. In particular, the work of [2] focused on the homotopy type of $\mathtt{vdW}(n, k)$.

The van der Waerden complex is a pure simplicial complex. It is known that pure simplicial complexes may have additional combinatorial and topological properties, e.g., vertex decomposable, shellable, and Cohen-Macaulay. Specifically, we have the following chain of implications (definitions are postponed until the next section):

$$\text{vertex decomposable} \implies \text{shellable} \implies \text{Cohen-Macaulay} \implies \text{pure}.$$

In general, these implications are all strict. It is natural to ask when $\mathtt{vdW}(n, k)$ has these additional properties in terms of n and k. We answer this question in this note; precisely:

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THEOREM 1.1. *Let* $0 < k < n$ *be integers. Then*

(i) $\texttt{vdW}(n, k)$ *is vertex decomposable if and only if*
- $n \leq 6$*, or*
- $n > 6$ *and* $k = 1$*, or*
- $n > 6$ *and* $\frac{n}{2} \leq k < n$.

(ii) $\texttt{vdW}(n, k)$ *is pure but not Cohen-Macaulay if and only if* $n > 6$ *and* $2 \leq k < \frac{n}{2}$.

As a corollary, we can recover a result of [5] first proved using different techniques.

COROLLARY 1.2. *Let* $0 < k < n$ *be integers. Then* $\texttt{vdW}(n, k)$ *is shellable if and only if*

- $n \leq 6$*, or*
- $n > 6$ *and* $k = 1$*, or*
- $n > 6$ *and* $\frac{n}{2} \leq k < n$.

PROOF. If k and n satisfy the above conditions, then $\texttt{vdW}(n, k)$ is vertex decomposable by Theorem 1.1, and consequently, shellable. Otherwise $\texttt{vdW}(n, k)$ is not Cohen-Macaulay by Theorem 1.1, so it cannot be shellable.

Our paper is structured as follows. We first recall the relevant background in Section 2. In Section 3 we prove Theorem 1.1 using some tools from combinatorial commutative algebra. In particular, to show that $\texttt{vdW}(n, k)$ is not Cohen-Macaulay, we will show that the Stanley-Reisner ideal of the Alexander dual of $\texttt{vdW}(n, k)$ has nonlinear first syzygies.

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2. Background

In this section we recall the relevant combinatorial and algebraic background.

Let $V = \{x_1, \ldots, x_n\}$ be a vertex set. A *simplicial complex* on V is a subset $\Delta \subseteq 2^V$ such that (a) if $F \in \Delta$ and $G \subseteq F$, then $G \in \Delta$, and (b) $\{x_i\} \in \Delta$ for all $i \in \{1, \ldots, n\}$. Elements of Δ are called *faces*, and maximal faces under inclusion are called *facets*. If $F_1, \ldots, F_s$ is a complete list of facets of Δ, we usually write $\Delta = \langle F_1, \ldots, F_s \rangle$. The *dimension* of a face F, denoted $\dim(F)$, is $\dim(F) = |F| - 1$. The *dimension of* Δ, denoted $\dim \Delta$, is $\dim \Delta = \max\{\dim(F) \mid F \text{ a facet of } \Delta\}$. A simplicial complex is *pure* if all its facets have the same dimension.

The *Alexander dual* of Δ, denoted $\Delta^\vee$, is the simplicial complex whose facets are complements of the minimal non-faces of Δ. That is, $\Delta^\vee = \{V \backslash F \mid F \notin \Delta\}$.

To any simplicial complex Δ, the *Stanley-Reisner ideal* of Δ is a monomial ideal I_Δ in the polynomial ring $R = k[x_1, \dots, x_n]$ where

$$I_\Delta = \langle x_{i_1} x_{i_2} \cdots x_{i_t} \mid \{x_{i_1}, \dots, x_{i_t}\} \notin \Delta \rangle.$$

The following result allows us to directly write out the minimal generators of the Stanley-Reisner ideal of the Alexander dual of Δ from the facets of Δ.

LEMMA 2.1 ([4, Corollary 1.5.5]). *Let* $\Delta = \langle F_1, F_2, \dots, F_s \rangle$. *Then*

$$I_{\Delta^\vee} = \langle m_{F_1^c}, \dots, m_{F_s^c} \rangle, \quad \text{where } m_{F_i^c} = \prod_{x \notin F_i} x.$$

We recall three families of pure simplicial complexes. The first family was introduced by Provan and Billera [6]; a pure simplicial complex Δ on V is *vertex decomposable* if

(i) $\Delta = \emptyset$, or $\Delta = \langle \{x_1, \dots, x_n\} \rangle$, i.e., a simplex; or

(ii) there exists a vertex $x \in V$ such that the *link* of x, i.e.,

$$\mathrm{lk}_\Delta(x) = \{H \in \Delta \mid H \cap \{x\} = \emptyset \text{ and } H \cup \{x\} \in \Delta\},$$

and the *deletion* of x, i.e., $\mathrm{del}_\Delta(x) = \{H \in \Delta \mid H \cap \{x\} = \emptyset\}$, are both vertex decomposable simplicial complexes.

The second family is the family of shellable simplicial complexes. A pure complex Δ is *shellable* if the facets of Δ can be ordered, say $F_1, \dots, F_s$, such that for all $1 \leq i < j \leq s$, there exists some $x \in F_j \setminus F_i$ and some $\ell \in \{1, \dots, j-1\}$ with $F_j \setminus F_\ell = \{x\}$.

Finally, a pure simplicial complex Δ is *Cohen-Macaulay*[1] *over k* if the minimal free resolution of $I_{\Delta^\vee}$ over $R = k[x_1, \dots, x_n]$ is linear. Recall that an ideal $I \subseteq R = k[x_1, \dots, x_n]$ has a *linear minimal free resolution* if I has a minimal free resolution of the form

$$0 \to R^{b_t}(-d-t) \to \cdots \to R^{b_2}(-d-2) \\ \to R^{b_1}(-d-1) \to R^{b_0}(-d) \to I \to 0$$

[1] One normally defines a simplicial complex Δ to be Cohen-Macaulay either in terms of the depth and dimension of R/I_Δ, or in terms of the reduced simplicial homology of Δ. Our definition uses the characterization of Cohen-Macaulay simplicial complexes due to Eagon and Reiner [1].

for some integer d, where $R(-d-i)$ denotes the polynomial ring shifted by degree $d+i$ and $R^{b_i}(-d-i) = R(-d-i) \oplus \cdots \oplus R(-d-i)$ (b_i times).

We now state some of the basic results that we require, with references to their proofs.

THEOREM 2.2. *Let Δ be a pure simplicial complex.*

(i) *If Δ is vertex decomposable, then Δ is shellable.*

(ii) *If Δ is shellable, then Δ is Cohen-Macaulay.*

(iii) *If* $\dim \Delta = 1$ *and Δ is connected, then Δ is vertex decomposable.*

PROOF. (i) is [6, Corollary 2.9]; (ii) is [7, Theorem 5.3.18]; and (iii) is [6, Theorem 3.1.2].

EXAMPLE 2.3. We show that both $\mathtt{vdW}(5,2)$ and $\mathtt{vdW}(6,2)$ are vertex decomposable. Not only do these examples illuminate our definitions, we require these special arguments for these complexes to prove Theorem 1.1. We begin with

$$\Delta = \mathtt{vdW}(5,2) = \langle \{x_1,x_2,x_3\}, \{x_2,x_3,x_4\}, \{x_3,x_4,x_5\}, \{x_1,x_3,x_5\}\rangle.$$

We form the deletion and link of x_5:

$$\mathrm{del}_\Delta(x_5) = \langle \{x_1,x_2,x_3\}, \{x_2,x_3,x_4\}\rangle \quad \text{and} \quad \mathrm{lk}_\Delta(x_5) = \langle \{x_3,x_4\}, \{x_1,x_3\}\rangle.$$

Now $\mathrm{lk}_\Delta(x_5)$ is vertex decomposable by Theorem 2.2(iii). Let $\Gamma = \mathrm{del}_\Delta(x_5)$ and form the link and deletion with respect to x_4:

$$\mathrm{del}_\Gamma(x_4) = \langle \{x_1,x_2,x_3\}\rangle \quad \text{and} \quad \mathrm{lk}_\Gamma(x_5) = \langle \{x_2,x_3\}\rangle.$$

Both of these complexes are simplicies, so $\mathrm{del}_\Delta(x_5)$ is vertex decomposable, and consequently, so is $\mathtt{vdW}(5,2)$

The proof for the complex

$$\Delta = \mathtt{vdW}(6,2) = \langle \{x_1,x_2,x_3\}, \{x_2,x_3,x_4\}, \{x_3,x_4,x_5\}, \\ \{x_4,x_5,x_6\}, \{x_1,x_3,x_5\}, \{x_2,x_4,x_6\}\rangle$$

is similar. We form the deletion and link of x_6. In particular,

$$\mathrm{del}_\Delta(x_6) = \langle \{x_1,x_2,x_3\}, \{x_2,x_3,x_4\}, \{x_3,x_4,x_5\}, \{x_1,x_3,x_5\}\rangle = \mathtt{vdW}(5,2),$$

and

$$\mathrm{lk}_\Delta(x_6) = \langle \{x_4,x_5\}, \{x_2,x_4\}\rangle.$$

We just showed that $\mathtt{vdW}(5,2) = \mathrm{del}_\Delta(x_6)$ is vertex decomposable, and $\mathrm{lk}_\Delta(x_6)$ is vertex decomposable by Theorem 2.2(iii). So, $\mathtt{vdW}(6,2)$ is vertex decomposable.

We complete this section with some results about the first syzygy module of a monomial ideal. Let I be a monomial ideal of $R = k[x_1, \dots, x_n]$ whose unique set of minimal generators are $G(I) = \{m_1, \dots, m_s\}$. Let $d_i = \deg(m_i)$ for $i = 1, \dots, s$, and let e_{m_i} denote the basis element of the shifted R-module $R(-d_i)$. We can then construct the following degree zero R-module homomorphism

$$\varphi\colon M = R(-d_1) \oplus R(-d_2) \oplus \cdots \oplus R(-d_s) \longrightarrow I$$

where $e_{m_i} \mapsto m_i$ for $i = 1, \dots, s$. The *first syzygy module of I* is then

$$\mathrm{Syz}^1_R(I) = \{(F_1, \dots, F_s) \in M \mid \varphi(F_1, \dots, F_s) = F_1 m_1 + \cdots + F_s m_s = 0\},$$

i.e., $\mathrm{Syz}^1_R(I) = \ker(\varphi)$. The module $\mathrm{Syz}^1_R(I)$ is a finitely generated R-module; in fact:

THEOREM 2.4 ([3, Corollary 4.13]). *Let $I \subseteq R = k[x_1, \dots, x_n]$ be a monomial ideal with minimal generators $G(I) = \{m_1, \dots, m_s\}$. Then*

$$\mathrm{Syz}^1_R(I) = \langle \sigma_{j,i} e_{m_i} - \sigma_{i,j} e_{m_j} \mid 1 \le i < j \le s \rangle, \quad \text{where } \sigma_{i,j} = \frac{m_i}{\gcd(m_i, m_j)}.$$

The set of generators in the above result may not be a minimal set of generators. However, some subset of these generators is a minimal set of generators. The first syzygy module is *generated by linear first syzygies* if there is some subset $T \subseteq \{\sigma_{j,i} e_{m_i} - \sigma_{i,j} e_{m_j} \mid 1 \le i < j \le s\}$ that generates $\mathrm{Syz}^1_R(I)$, and for all $\sigma_{j,i} e_{m_i} - \sigma_{i,j} e_{m_j} \in T$, $\deg \sigma_{i,j} = \deg \sigma_{j,i} = 1$.

The construction of $\mathrm{Syz}^1_R(I)$ is the first step in the construction of the minimal free resolution of I. In particular, we have the following fact.

THEOREM 2.5. *If I is a monomial ideal with a linear resolution, then* $\mathrm{Syz}^1_R(I)$ *is generated by linear first syzygies.*

3. Proof of the main theorem

We prove Theorem 1.1 in this section. To do so, we require the following two lemmas about the facets of $\mathtt{vdW}(n, k)$. Given a facet $F = \{x_i, x_{i+d}, x_{i+2d}, \dots, x_{i+kd}\} \in \mathtt{vdW}(n, k)$, we call d the *increment of F*. Note that every facet has an associated increment.

LEMMA 3.1. *Suppose $n \ge 7$. Let $F \in \mathtt{vdW}(n, 2)$ be any facet such that its increment is the largest possible odd integer d. If $G \in \mathtt{vdW}(n, 2)$ is any other facet with increment $d' \ne d$, then $|F \cap G| \le 1$.*

PROOF. Because $n \geq 7$, the complex $\mathtt{vdW}(n, 2)$ contains the facet $\{1, 4, 7\}$. Thus the largest odd increment d satisfies $d \geq 3$. Let $F = \{x_a, x_{a+d}, x_{a+2d}\}$ be any facet whose increment is d and let $G = \{x_b, x_{b+d'}, x_{b+2d'}\}$ be any other facet whose increment is $d' \neq d$.

It is immediate that $F \neq G$, so $|F \cap G| \leq 2$. So suppose $|F \cap G| = 2$. Since $a < a + d < a + 2d$ and $b < b + d' < b + 2d'$, we have the following possible cases:

(a) $a = b$ and $a + d = b + d'$
(b) $a = b$ and $a + d = b + 2d'$
(c) $a = b$ and $a + 2d = b + d'$
(d) $a = b$ and $a + 2d = b + 2d'$
(e) $a = b + d'$ and $a + d = b + 2d'$
(f) $a = b + d'$ and $a + 2d = b + 2d'$
(g) $a + d = b$ and $a + 2d = b + d'$
(h) $a + d = b$ and $a + 2d = b + 2d'$
(i) $a + d = b + d'$ and $a + 2d = b + 2d'$.

Cases (a), (d), (e), (g) and (i) all imply $d = d'$, so we can eliminate those cases. For cases (b) and (h), we would have $d = 2d'$, which implies that the odd integer d is even, so this case cannot happen. Finally, for cases (c) and (f), we would have $2d = d'$. But $d \geq 3$ is the largest odd increment, so the largest increment of $\mathtt{vdW}(n, 2)$ is either d or $d + 1$. But $d' = 2d > d + 1$, so this is not a valid increment, and consequently, this case cannot happen.

Therefore, it must be the case that $|F \cap G| \leq 1$.

We now prove a similar lemma, but now we do not require the increment to be odd.

LEMMA 3.2. *Suppose $n \geq 7$ and $2 < k < \frac{n}{2}$. Let $F \in \mathtt{vdW}(n, k)$ be any facet whose increment d is the largest possible. If $G \in \mathtt{vdW}(n, k)$ is any other facet with increment $d' \neq d$, then $|F \cap G| \leq k - 1$.*

PROOF. Since $k < \frac{n}{2}$, we have $\{x_1, x_3, \ldots, x_{1+2k}\} \in \mathtt{vdW}(n, k)$. If $F \in \mathtt{vdW}(n, k)$ has the largest possible increment d, we must therefore have $d \geq 2$.

Let $F = \{x_a, x_{a+d}, \ldots, x_{a+kd}\}$ be a facet with increment d, and suppose $G = \{x_b, x_{b+d'}, \ldots, x_{b+kd'}\}$ is a facet with increment $d' \neq d$. Since the facets are distinct, we must have $|F \cap G| \leq k$.

Suppose that $|F \cap G| = k$. Since $|G| = k + 1 > 3$, there must be $x_{b+id'}$, $x_{b+(i+1)d'} \in G$, i.e., two consecutive terms of the arithmetic progression in G such that

$$a + \ell d = b + id' \quad \text{and} \quad a + jd = b + (i + 1)d' \quad \text{for some } \ell < j.$$

But these two equations imply that $(j-\ell)d = d'$, i.e., $d' \geq d$, contradicting the fact that d is the largest increment. So $|F \cap G| \leq k-1$.

We now prove Theorem 1.1.

PROOF OF THEOREM 1.1. We break the proof into cases depending on $0 < k < n$.

Case 1: $k = 1$ and $1 < n$. In this case $\texttt{vdW}(n, 1)$ is vertex decomposable by Theorem 2.2 (iii) because

$$\texttt{vdW}(n, 1) = \langle \{x_i, x_j\} \mid 1 \leq i < j \leq n \rangle,$$

is a connected one-dimensional simplicial complex.

Case 2: $\frac{n}{2} \leq k < n$. If $1 = k < 2$, then $\texttt{vdW}(2, 1)$ is vertex decomposable by the previous case. We now proceed by induction on n. If $k = n-1$, then $\texttt{vdW}(n, n-1) = \langle \{x_1, x_2, x_3, \ldots, x_n\} \rangle$ is a simplex, and hence, vertex decomposable.

So suppose that $\frac{n}{2} \leq k < n-1$. Every facet of $\texttt{vdW}(n, k)$ must have increment $d = 1$ since $\frac{n}{2} \leq k$. So

$$\begin{aligned}\Delta &= \texttt{vdW}(n, k)\\ &= \langle \{x_1, x_2, \ldots, x_{k+1}\}, \{x_2, x_3, \ldots, x_{k+2}\}, \ldots, \{x_{n-k}, \ldots, x_n\} \rangle.\end{aligned}$$

We form the link and deletion of x_n:

$$\mathrm{del}_\Delta(x_n) = \texttt{vdW}(n-1, k) \quad \text{and} \quad \mathrm{lk}_\Delta(x_n) = \langle \{x_{n-k}, \ldots, x_{n-1}\} \rangle.$$

Since $\frac{n-1}{2} < k < n-1$, by induction $\texttt{vdW}(n-1, k)$ is vertex decomposable. Because $\mathrm{lk}_\Delta(x_n)$ is a simplex, we can now conclude that $\texttt{vdW}(n, k)$ is vertex decomposable if $\frac{n}{2} \leq k < n$.

Case 3: $0 < k < n \leq 6$. The only n and k in this case not covered by Case 1 or 2 is $(n, k) = (5, 2)$ or $(6, 2)$. We now use Example 2.3 to complete this case.

Case 4: $n > 6$ and $2 \leq k < \frac{n}{2}$. Let $I = I_{\texttt{vdW}(n,k)^\vee}$ be the Stanley-Reisner ideal of the Alexander dual of $\texttt{vdW}(n, k)$. We will show that $\mathrm{Syz}^1_R(I)$ cannot be generated by linear first syzygies. It will then follow by Theorem 2.5 that I does not have a linear minimal free resolution, and consequently, $\texttt{vdW}(n, k)$ is a simplicial complex that is pure but not Cohen-Macaulay.

If $\texttt{vdW}(n, k) = \langle F_1, \ldots, F_s \rangle$, then by Lemma 2.1,

$$I = \left\langle m_{F_i^c} = \prod_{x \notin F_i} x \;\middle|\; i = 1, \ldots, s \right\rangle.$$

Since the complex is pure, this ideal is generated by s monomials all of degree $n-k-1$.

We first consider the case that $3 \leq k < \frac{n}{2}$. Let F be any facet with the largest increment d. Since $n > 6$, we know that $d \geq 3$. Now take another facet G with increment $d' \neq d$. We know that

$$\frac{m_{G^c}}{\gcd(m_{F^c}, m_{G^c})} e_{m_{F^c}} - \frac{m_{F^c}}{\gcd(m_{F^c}, m_{G^c})} e_{m_{G^c}}$$

is a (possibly non-minimal) generator of $\mathrm{Syz}^1_R(I)$ by Theorem 2.4. Moreover, this generator is not a linear first syzygy because Lemma 3.2 tells us that $|F \cap G| \leq k-1$, which implies that

$$\deg\left(\frac{m_{G^c}}{\gcd(m_{F^c}, m_{G^c})}\right) \geq 2 \quad \text{and} \quad \deg\left(\frac{m_{F^c}}{\gcd(m_{F^c}, m_{G^c})}\right) \geq 2.$$

To see why, m_{F^c} and m_{G^c} are squarefree monomials, so

$$\begin{aligned}\deg(\gcd(m_{F^c}, m_{G^c})) &= |F^c \cap G^c| = |(F \cup G)^c| = n - |F \cup G| \\ &= n - |F| - |G| + |F \cap G| \\ &\leq n - (k+1) - (k+1) + (k-1) = n-k-3.\end{aligned}$$

Since $\deg(m_{F^c}) = \deg(m_{G^c}) = n-k-1$, the result follows.

Now suppose that $\mathrm{Syz}^1_R(I)$ is generated by linear first syzygies. So, in particular there are facets $H_1, \ldots, H_t \in \{F_1, \ldots, F_s\}$, not necessarily distinct, so that we can write

$$\begin{aligned}&\frac{m_{G^c}}{\gcd(m_{F^c}, m_{G^c})} e_{m_{F^c}} - \frac{m_{F^c}}{\gcd(m_{F^c}, m_{G^c})} e_{m_{G^c}} \\ &\quad = \sum_{i=1}^{t} A_i \left(\frac{m_{H_i^c}}{\gcd(m_{H_i^c}, m_{H_{i+1}^c})} e_{m_{H_{i+1}^c}} - \frac{m_{H_{i+1}^c}}{\gcd(m_{H_i^c}, m_{H_{i+1}^c})} e_{m_{H_i^c}} \right), \qquad (3.1)\end{aligned}$$

where each $\frac{m_{H_i^c}}{\gcd(m_{H_i^c}, m_{H_{i+1}^c})} e_{m_{H_{i+1}^c}} - \frac{m_{H_{i+1}^c}}{\gcd(m_{H_i^c}, m_{H_{i+1}^c})} e_{m_{H_i^c}}$ is a linear first syzygy.

Note that if the facet H has increment d, the largest possible increment, and

$$\frac{m_{H^c}}{\gcd(m_{H^c}, m_{K^c})} e_{m_{K^c}} - \frac{m_{K^c}}{\gcd(m_{H^c}, m_{K^c})} e_{m_{H^c}}$$

is any linear first syzygy involving H, then K must also have increment d. Indeed, if the increment of K is $d' \neq d$, then we could again use Lemma 3.2 to show that

$$\deg\left(\frac{m_{H^c}}{\gcd(m_{H^c}, m_{K^c})}\right) \geq 2 \quad \text{and} \quad \deg\left(\frac{m_{K^c}}{\gcd(m_{H^c}, m_{K^c})}\right) \geq 2,$$

contradicting the fact we have a linear first syzygy.

Because $e_{m_{F^c}}$ appears on both sides of (3.1), at least one of the H_is must be F. In the light of discussion in the previous paragraph, we are forced to have

$$\frac{m_{G^c}}{\gcd(m_{F^c}, m_{G^c})} e_{m_{F^c}} = \sum A_{H,K} \left(\frac{m_{H^c}}{\gcd(m_{H^c}, m_{K^c})} e_{m_{K^c}} - \frac{m_{K^c}}{\gcd(m_{H^c}, m_{K^c})} e_{m_{H^c}} \right),$$

where all the H and K have increment d. That is, all the linear first syzygies involving a facet with increment d must appear together. But this means that

$$0 = \varphi\left(\frac{m_{G^c}}{\gcd(m_{F^c}, m_{G^c})} e_{m_{F^c}} \right) = \frac{m_{G^c}}{\gcd(m_{F^c}, m_{G^c})} m_{F^c} \neq 0,$$

which is false. Here, φ is the R-module homomorphism used to define $\mathrm{Syz}^1_R(I)$.

The proof for $k = 2$ is similar. The only difference is that F is picked to be any facet with the largest odd increment, and we use Lemma 3.1 instead of Lemma 3.2.

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