MATH. SCAND. 125 (2019), 67–83

UNIQUENESS OF NORM-PRESERVING EXTENSIONS OF FUNCTIONALS ON THE SPACE OF COMPACT OPERATORS

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Abstract

Godefroy, Kalton, and Saphar called a closed subspace Y of a Banach space Z an ideal if its annihilator $Y^\perp$ is the kernel of a norm-one projection P on the dual space Z^*. If Y is an ideal in Z with respect to a projection on Z^* whose range is norming for Z, then Y is said to be a strict ideal. We study uniqueness of norm-preserving extensions of functionals on the space $\mathscr{K}(X, Y)$ of compact operators between Banach spaces X and Y to the larger space $\mathscr{K}(X, Z)$ under the assumption that Y is a strict ideal in Z. Our main results are: (1) if y^* is an extreme point of B_{Y^*} having a unique norm-preserving extension to Z, and $x^{**} \in B_{X^{**}}$, then the only norm-preserving extension of the functional $x^{**} \otimes y^* \in \mathscr{K}(X, Y)^*$ to $\mathscr{K}(X, Z)$ is $x^{**} \otimes z^*$ where $z^* \in Z^*$ is the only norm-preserving extension of y^* to Z; (2) if $\mathscr{K}(X, Y)$ is an ideal in $\mathscr{K}(X, Z)$ and Y has Phelps' property U in its bidual Y^{**} (i.e., every bounded linear functional on Y admits a unique norm-preserving extension to Y^{**}), then $\mathscr{K}(X, Y)$ has property U in $\mathscr{K}(X, Z)$ whenever X^{**} has the Radon-Nikodým property.

1. Introduction

Throughout this paper, all Banach spaces will be over the scalar field $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. The closed unit ball and the unit sphere of a Banach space X will be denoted, respectively, by B_X and S_X. For a subset A of X, we denote its convex hull by $\operatorname{co}(A)$. The symbol $\mathscr{L}(X, Z)$ will stand for the space of continuous linear operators from X to another Banach space Z (over the same scalar field as X), and $\mathscr{K}(X, Z)$ for its subspace of compact operators. Whenever $\mathscr{L}$ is a subspace of $\mathscr{L}(X, Z)$, for $x^{**} \in X^{**}$ and $z^* \in Z^*$, the functional $x^{**} \otimes z^* \in \mathscr{L}^*$ is defined by

$$(x^{**} \otimes z^*)(T) = x^{**}(T^* z^*), \qquad T \in \mathscr{L}.$$

Let Z be a Banach space, and let Y be a closed subspace of Z. According to the terminology in [5], Y is said to be an *ideal* in Z if there exists a continuous

* Research supported, in part, by institutional research funding IUT20-57 of the Estonian Ministry of Education and Research.
Received 27 July 2017.
DOI: https://doi.org/10.7146/math.scand.a-112071

linear projection P on Z^* with $\ker P = Y^\perp = \{z^* \in Z^* : z^*|_Y = 0\}$ and $\|P\| = 1$. If ran P is norming for Z in the sense that

$$\|z\| = \sup_{z^* \in B_{\operatorname{ran} P}} |z^*(z)| \qquad \text{for all} \quad z \in Z,$$

then Y is called a *strict ideal*, and the ideal projection P is said to be *strict*.

It is straightforward to verify that if P is an ideal projection for Y in Z, then, for every $z^* \in Z^*$, the functional $Pz^* \in Z^*$ is a norm-preserving extension of the restriction $z^*|_Y \in Y^*$. It follows that the mapping $J_P\colon Y^* \ni y^* \mapsto Pz^* \in Z^*$ where $z^* \in Z^*$ is any extension of y^*, is a linear isometry. In particular, ran $J_P =$ ran P, and ran P is isometrically isomorphic to Y^*.

Suppose that Y is an ideal in Z with respect to an ideal projection P. Then each $z \in Z$ induces a functional $z_P \in Y^{**}$ defined by $z_P(y^*) = J_P y^*(z)$, $y^* \in Y^*$. If P is strict, the mapping $z \mapsto z_P$ is an isometry and one can identify Z with the closed subspace $Z_P = \{z_P \in Y^{**} : z \in Z\}$ of Y^{**}.

In this paper, we study uniqueness of norm-preserving extensions to $\mathcal{K}(X, Z)$ of functionals on $\mathcal{K}(X, Y)$ under the assumption that Y is a strict ideal in Z. Ideals of compact operators of this type were studied, among others, in [10].

Our first main theorem partially complements [10, Lemma 3.1] – uniqueness of norm-preserving extensions to $\mathcal{K}(X, Z)$ of functionals on $\mathcal{K}(X, Y)$ of the form $x^{**} \otimes y^*$, where $x^{**} \in B_{X^{**}}$ and y^* is an extreme point of B_{Y^*} having a unique norm-preserving extension to Z, is not covered by that result.

THEOREM 1.1. *Let X and Z be Banach spaces, and let Y be a strict ideal in Z with respect to a projection P on Z^*. Let y^* be an extreme point of B_{Y^*} having a unique norm-preserving extension to Z, and let $x^{**} \in B_{X^{**}}$. Then the only norm-preserving extension of the functional $x^{**} \otimes y^* \in \mathcal{K}(X, Y)^*$ to $\mathcal{K}(X, Z)$ is $x^{**} \otimes z^*$ where $z^* \in Z^*$ is the only norm-preserving extension of y^* to Z.*

Examples of extreme points of the dual unit ball B_{Y^*} admitting a unique norm-preserving extension to Y^{**} (and thus also to Z if Y is a strict ideal in Z) are, e.g., weak* denting points of B_{Y^*} and, in particular, weak* strongly exposed points of B_{Y^*}. Different versions of Theorem 1.1 have been proven in [10, Lemma 3.1] (see also [15, Theorem 1.1] for a simpler proof), [9, Lemma 4.3], [8, Lemma 3.4], [11, Theorem 3.7], [7, Lemma 11], and [7, Lemma 12].

Following R. R. Phelps [16], we say that a closed subspace Y of a Banach space Z has *property U* in Z if every functional $y^* \in Y^*$ has a unique norm-preserving extension to the whole space Z. For an investigation of property U, see [14]; property U for the subspace of compact operators in the corresponding space of all bounded linear operators has been studied in [17].

Our second main theorem extends [10, Theorem 3.6] (cf. also [10, Theorem 3.5] and [15, Theorem 4.5]) to the case of property U.

THEOREM 1.2. *Let X and Z be Banach spaces, and let Y be a strict ideal in Z such that $\mathcal{K}(X, Y)$ is an ideal in $\mathcal{K}(X, Z)$. Suppose that X^{**} has the Radon-Nikodým property, and let Y have property U in its bidual Y^{**}. Then $\mathcal{K}(X, Y)$ is a strict ideal having property U in $\mathcal{K}(X, Z)$.*

In [10, Theorem 3.6], Theorem 1.2 was proved under stronger assumptions and with property U replaced by strict u-ideals (for u-ideals we refer to [5] and, for a more recent study, to [12]).

In Section 2, we describe, for a strict ideal Y in a Banach space Z with respect to a projection P on Z^*, the extreme points of B_{Y^*} having unique norm-preserving extensions to Z as the points of continuity of the formal identity operator

$$\mathrm{id}\colon (B_{Y^*}, \text{relative weak}^*) \to \big(B_{Y^*}, \text{relative } \sigma(Y^*, Z_P)\big)$$

(briefly, as weak*-to-$\sigma(Y^*, Z_P)$-PC's of B_{Y^*}), and prove Theorem 1.1. In Section 3, we provide a description of a generalisation of denting points (a partial case of this description is used in the proof of Theorem 1.1). In Section 4, we prove Theorem 1.2.

Let us recall the notion of a *slice*. Let C be a non-empty bounded subset of a Banach space Z. Given $z^* \in Z^*$, $z^* \neq 0$, and $\alpha > 0$, the set

$$S(z^*, \alpha, C) := \big\{z \in C : \operatorname{Re} z^*(z) > \sup \operatorname{Re} z^*(C) - \alpha\big\}$$

is called an (*open*) *slice* of C. If Γ is a linear subspace of Z^*, then slices of C whose defining functional comes from Γ are called Γ-*slices*. In particular, if Z happens to be a dual space, say $Z = E^*$, then slices of C whose defining functional comes from (the canonical image of) the predual E of Z are called *weak* slices*.

2. Uniqueness of norm-preserving extensions from $\mathcal{K}(X, Y)$ to $\mathcal{K}(X, Z)$ of functionals of the form $x^{**} \otimes y^*$

A result by Godefroy (see [4] or [6, p. 125, Lemma 2.14]) describes functionals in the dual unit sphere of a Banach space admitting a unique norm-preserving extension to the bidual as weak*-to-weak points of continuity of the dual unit ball. The following proposition generalises this result to the case of strict ideals.

PROPOSITION 2.1. *Let Y be a strict ideal in a Banach space Z with respect to a projection P on Z^*, and let $y^* \in S_{Y^*}$. The following assertions are equivalent:*

(i) *y^* has a unique norm-preserving extension to Z;*

(ii) *y^* is a weak*-to-$\sigma(Y^*, Z_P)$-PC of B_{Y^*}, i.e., for any net (y_α^*) in B_{Y^*},*

$$y_\alpha^* \xrightarrow[\alpha]{w^*} y^* \implies y_\alpha^* \xrightarrow[\alpha]{\sigma(Y^*,Z_P)} y^*, \tag{2.1}$$

i.e., whenever

$$y_\alpha^*(y) \underset{\alpha}{\to} y^*(y) \qquad \textit{for all} \quad y \in Y,$$

one has

$$J_P y_\alpha^*(z) = z_P(y_\alpha^*) \underset{\alpha}{\to} z_P(y^*) = J_P y^*(z) \qquad \textit{for all} \quad z \in Z.$$

REMARK 2.2. The aforecited result of Godefroy is the partial case of Proposition 2.1, where $Z = Y^{**}$ and P is the canonical projection on Y^{***} (i.e. $P = j_{Y^*}(j_Y)^*$ where $j_Y\colon Y \to Y^{**}$ and $j_{Y^*}\colon Y^* \to Y^{***}$ are canonical embeddings).

PROOF OF PROPOSITION 2.1. (i) $\Rightarrow$ (ii). Assume that y^* has a unique norm-preserving extension to Z. Then this norm-preserving extension is $J_P y^*$. Let (y_α^*) be a net in B_{Y^*} converging weak* to y^*, and let (y_β^*) be any subnet of (y_α^*). In order for (ii) to hold, it suffices to show that there is a further subnet (y_γ^*) such that $J_P y_\gamma^*(z) \to J_P y^*(z)$ for all $z \in Z$. By the weak* compactness of B_{Z^*} there are a subnet (y_γ^*) and a $z^* \in B_{Z^*}$ such that $J_P y_\gamma^* \to z^*$ weak* in Z^*. But now z^* is a norm-preserving extension of y^*, thus $z^* = J_P y^*$, and $J_P y_\gamma^* \to J_P y^*$ weak* in Z^*, as desired.

(ii) $\Rightarrow$ (i). Assume that y^* is a weak*-to-$\sigma(Y^*, Z_P)$-PC of B_{Y^*}, and let $z^* \in S_{Z^*}$ be any norm-preserving extension of y^*. It suffices to show that $z^* = J_P y^*$. Since B_{Z^*} is the weak* closure of $B_{\operatorname{ran} P}$ (this follows from the strictness of P by the separation theorem), there is a net (z_α^*) in $\operatorname{ran} P$ with $\|z_\alpha^*\| \le 1 = \|z^*\|$ for all α such that

$$z_\alpha^*(z) \underset{\alpha}{\to} z^*(z) \qquad \text{for all} \quad z \in Z.$$

Put $y_\alpha^* = z_\alpha^*|_Y$; then $J_P y_\alpha^* = P z_\alpha^* = z_\alpha^*$ and

$$y_\alpha^*(y) = z_\alpha^*(y) \underset{\alpha}{\to} z^*(y) = y^*(y) \qquad \text{for all} \quad y \in Y;$$

hence, by (2.1),

$$z_\alpha^*(z) = J_P y_\alpha^*(z) \underset{\alpha}{\to} J_P y^*(z) \qquad \text{for all} \quad z \in Z.$$

It follows that $z^*(z) = J_P y^*(z)$ for all $z \in Z$, i.e. $z^* = J_P y^*$, as desired.

The proof of Theorem 1.1 relies on the following two propositions. The first of these describes extreme points of the dual unit ball of a strict ideal having a unique norm-preserving extension to the whole space.

PROPOSITION 2.3. *Let Y be a strict ideal in a Banach space Z with respect to a projection P on Z^*, and let $y^* \in B_{Y^*}$. The following assertions are equivalent:*

(i) *y^* is an extreme point of B_{Y^*} having a unique norm-preserving extension to Z;*

(ii) *y^* is both an extreme point and a weak*-to-$\sigma(Y^*, Z_P)$-PC of B_{Y^*};*

(iii) *weak* slices of B_{Y^*} containing y^* form a neighbourhood basis for y^* in the relative $\sigma(Y^*, Z_P)$-topology of B_{Y^*};*

(iv) *whenever $\mathscr{F}$ is a finite subset of Z and $\varepsilon > 0$,*

$$y^* \notin \overline{\mathrm{co}}^{w^*}\Big(B_{Y^*} \setminus \big\{v^* \in B_{Y^*} : \max_{z\in\mathscr{F}} |z_P(v^* - y^*)| < \varepsilon\big\}\Big);$$

(v) *whenever $(y_\alpha^*)_{\alpha\in\mathscr{A}} := \big(\sum_{k=1}^{n_\alpha} \lambda_{\alpha k} y_{\alpha k}^*\big)_{\alpha\in\mathscr{A}}$ is a net of convex combinations in B_{Y^*} such that $y_\alpha^* \overset{w^*}{\underset{\alpha}{\longrightarrow}} y^*$, i.e.,*

$$\sum_{k=1}^{n_\alpha} \lambda_{\alpha k}(y_{\alpha k}^* - y^*)(y) \underset{\alpha}{\to} 0 \qquad \textit{for all} \quad y \in Y,$$

one has $y_\alpha^* \overset{\sigma(Y^*, Z_P)}{\underset{\alpha}{\longrightarrow}} y^*$; moreover,

$$\sum_{k=1}^{n_\alpha} \lambda_{\alpha k}|z_P(y_{\alpha k}^* - y^*)| \underset{\alpha}{\to} 0 \qquad \textit{for all} \quad z \in Z,$$

and thus, for every finite subset $\mathscr{F}$ of Z,

$$\sum_{k=1}^{n_\alpha} \lambda_{\alpha k} \max_{z\in\mathscr{F}} |z_P(y_{\alpha k}^* - y^*)| \underset{\alpha}{\to} 0.$$

We postpone the proof of Proposition 2.3 until the next section where a more general result will be proven (see Proposition 3.1).

The following proposition contains a slight improvement to [15, Lemma 2.1]. Given Banach spaces X and Z, and closed subspaces $V \subset X^*$ and $Y \subset Z$, we denote by $V \otimes Y$ the linear span of the operators in $\mathscr{L}(X, Z)$ of the form $v \otimes y$ ($v \in V$, $y \in Y$) defined by $(v \otimes y)(x) = v(x)\, y$, $x \in X$.

PROPOSITION 2.4. *Let X and Z be Banach spaces, and let $V \subset X^*$ and $Y \subset Z$ be closed subspaces. Let $\mathscr{L}$ be a subspace of $\mathscr{L}(X, Z^{**})$ containing $V \otimes Y$. If $v^* \in S_{V^*}$ and $y^* \in S_{Y^*}$, then, for any norm-preserving extension $\phi \in \mathscr{L}^*$ of $v^* \otimes y^* \in (V \otimes Y)^*$, there exists a net*

$$(\phi_\alpha) = \left(\sum_{k=1}^{n_\alpha} \lambda_{\alpha k} x_{\alpha k} \otimes z^*_{\alpha k} \right)_\alpha$$

of convex combinations in $S_X \otimes S_{Z^*} = \{x \otimes z^* \in \mathscr{L}^* : x \in S_X, z^* \in S_{Z^*}\} \subset \mathscr{L}^*$ such that

(1) $\phi_\alpha \underset{\alpha}{\to} \phi$ *weak* in* $\mathscr{L}^*$;

(2) $x_\alpha := \sum_{k=1}^{n_\alpha} \lambda_{\alpha k} x_{\alpha k} \underset{\alpha}{\to} v^*$ *weak* in* V^*;

(3) $z^*_\alpha := \sum_{k=1}^{n_\alpha} \lambda_{\alpha k} z^*_{\alpha k} \underset{\alpha}{\to} z^*$ *weak* in Z^* for some norm-preserving extension $z^* \in S_{Z^*}$ of y^*.*

Moreover, if Y is a strict ideal in Z with respect to a projection P on Z^*, and $\mathscr{L} \subset \mathscr{L}(X, Z)$, then the $z^*_{\alpha k}$ can be chosen so that $z^*_{\alpha k} = J_P y^*_{\alpha k}$ for some $y^*_{\alpha k} \in S_{Y^*}$.

PROOF. The proposition without the "moreover" part was proven in [15, Lemma 2.1]. For the "moreover" part, observe that if the ideal projection P is strict and $\mathscr{L} \subset \mathscr{L}(X, Z)$, then $\|T\| = \sup\big\{|(x \otimes z^*)(T)| : x \otimes z^* \in S_X \otimes J_P(S_{Y^*})\big\}$ for all $T \in \mathscr{L}$, thus the polar of $S_X \otimes J_P(S_{Y^*}) \subset \mathscr{L}^*$ in $\mathscr{L}$ coincides with $B_{\mathscr{L}}$. Hence, by the bipolar theorem,

$$B_{\mathscr{L}^*} = B^\circ_{\mathscr{L}} = \overline{\mathrm{co}}^{w^*}\big(S_X \otimes J_P(S_{Y^*})\big) \qquad \text{in} \quad \mathscr{L}^*.$$

It follows that there is a net $(\phi_\beta)_{\beta \in \mathfrak{B}} = \big(\sum_{k=1}^{n_\beta} \lambda_{\beta k} x_{\beta k} \otimes z^*_{\beta k}\big)_{\beta \in \mathfrak{B}}$ of convex combinations in $S_X \otimes J_P(S_{Y^*})$ such that $\phi_\beta \to \phi$ weak* in $\mathscr{L}^*$.

The rest of the proof is verbatim to that of [15, Lemma 2.1].

Now we are in a position to prove Theorem 1.1.

PROOF OF THEOREM 1.1. Let $\phi \in \mathscr{K}(X, Z)^*$ be a norm-preserving extension of $x^{**} \otimes y^*$, and let $T \in \mathscr{K}(X, Z)$. We must show that $\phi(T) = (x^{**} \otimes z^*)(T)$. Since z^* is the only norm-preserving extension of its restriction $z^*|_Y = y^*$, one has $z^* = Pz^* = J_P y^*$ and thus

$$(x^{**} \otimes z^*)(T) = x^{**}(T^* z^*) = x^{**}(T^* J_P y^*).$$

Therefore, letting $\varepsilon > 0$ be arbitrary, it suffices to show that $\left|\phi(T) - x^{**}(T^* J_P y^*)\right| < 2\varepsilon$.

We follow the main idea of [15, proof of Theorem 4.1]. Since $\|(x^{**} \otimes y^*)|_{X^* \otimes Y}\| = \|x^{**} \otimes y^*\|$, there are nets (ϕ_α), (x_α), and (z^*_α) as in Proposition 2.4 where $V = X^*$, $\mathscr{L} = \mathscr{K}(X, Z)$, and $v^* = x^{**}$. By the "moreover" part of Proposition 2.4, we may assume that $z^*_{\alpha k} = J_P y^*_{\alpha k}$ where $y^*_{\alpha k} \in S_{Y^*}$. Since the operator T is compact, $T(B_X)$ is a relatively compact subset of B_Z, thus there is a finite ε-net $\mathscr{B}$ in B_Z for $T(B_X)$. For every index α and every $k \in \{1, \dots, n_\alpha\}$, let $z_{\alpha k} \in \mathscr{B}$ be such that $\|T x_{\alpha k} - z_{\alpha k}\| < \varepsilon$. Now, by Proposition 2.3, (i) $\Rightarrow$ (v),

$$
\begin{aligned}
\left|\phi(T) - x^{**}(T^* J_P y^*)\right| &= \lim_\alpha \left|\sum_{k=1}^{n_\alpha} \lambda_{\alpha k} J_P y^*_{\alpha k}(T x_{\alpha k}) - x^{**}(T^* J_P y^*)\right| \\
&\leq \limsup_\alpha \left|\sum_{k=1}^{n_\alpha} \lambda_{\alpha k} \big(J_P y^*_{\alpha k} - J_P y^*\big)(T x_{\alpha k})\right| \\
&\quad + \lim_\alpha \left|\left(\sum_{k=1}^{n_\alpha} \lambda_{\alpha k} x_{\alpha k} - x^{**}\right)(T^* J_P y^*)\right| \\
&\leq \limsup_\alpha \sum_{k=1}^{n_\alpha} \lambda_{\alpha k} \left|\big(J_P y^*_{\alpha k} - J_P y^*\big)(T x_{\alpha k})\right| \\
&\leq \limsup_\alpha \sum_{k=1}^{n_\alpha} \lambda_{\alpha k} \left|\big(J_P y^*_{\alpha k} - J_P y^*\big)(z_{\alpha k})\right| \\
&\quad + \limsup_\alpha \sum_{k=1}^{n_\alpha} \lambda_{\alpha k} \|J_P y^*_{\alpha k} - J_P y^*\| \|T x_{\alpha k} - z_{\alpha k}\| \\
&< \limsup_\alpha \sum_{k=1}^{n_\alpha} \lambda_{\alpha k} \left|\big(J_P y^*_{\alpha k} - J_P y^*\big)(z_{\alpha k})\right| + 2\varepsilon \\
&\leq \limsup_\alpha \sum_{k=1}^{n_\alpha} \lambda_{\alpha k} \max_{z \in \mathscr{B}} \left|\big(J_P y^*_{\alpha k} - J_P y^*\big)(z)\right| + 2\varepsilon \\
&= \limsup_\alpha \sum_{k=1}^{n_\alpha} \lambda_{\alpha k} \max_{z \in \mathscr{B}} |(y^*_{\alpha k} - y^*)(z_P)| + 2\varepsilon \\
&= 2\varepsilon.
\end{aligned}
$$

3. A description of a generalisation of denting points

Let X be a Banach space, and let τ be a locally convex topology on X. By $(X,\tau)'$ we mean the topological dual of X with respect to τ, i.e., the linear space of all τ-continuous linear functionals on X. Let $\mathscr{S}$ be a family of seminorms on X. Given $x \in X$, a finite subset $\mathscr{F}$ of $\mathscr{S}$, and an $\varepsilon > 0$, we define

$$\mathscr{U}_{\mathscr{F}}(x,\varepsilon) := \bigl\{u \in X : \max_{p\in\mathscr{F}} p(u-x) < \varepsilon\bigr\}.$$

The family $\mathscr{S}$ is said to induce the topology τ if, for every $x \in X$, the family

$$\mathfrak{B}_{\mathscr{S}}(x) := \{\mathscr{U}_{\mathscr{F}}(x,\varepsilon) : \mathscr{F} \text{ is a finite subset of } \mathscr{S} \text{ and } \varepsilon > 0\}$$

is a basis of neighbourhoods for x in τ, or, equivalently, the family $\mathfrak{B}_{\mathscr{S}}(0)$ is a basis of neighbourhoods for 0 in τ.

PROPOSITION 3.1. *Let X be a Banach space, let Γ be a linear subspace of X^*, let τ be a locally convex topology on X weaker than the norm topology such that $\Gamma \subset (X,\tau)'$, and let $\mathscr{S}$ be a family of seminorms on X inducing τ. Let C be a non-empty bounded convex subset of X, and let $x \in C$. The following assertions are equivalent:*

(i) *whenever $\mathscr{F}$ is a finite subset of $\mathscr{S}$ and $\varepsilon > 0$, there is a Γ-slice $S := S(x^*,\alpha,C)$ of C such that*

$$x \in S \subset \mathscr{U}_{\mathscr{F}}(x,\varepsilon); \tag{3.1}$$

(ii) *Γ-slices of C containing x form a neighbourhood basis for x in the relative τ-topology of C;*

(iii) *whenever $\mathscr{F}$ is a finite subset of $\mathscr{S}$ and $\varepsilon > 0$,*

$$x \notin \overline{\mathrm{co}}^{\sigma(X,\Gamma)}\bigl(C \setminus \mathscr{U}_{\mathscr{F}}(x,\varepsilon)\bigr); \tag{3.2}$$

(iv) *whenever $(x_\alpha)_{\alpha\in\mathscr{A}} := \bigl(\sum_{k=1}^{n_\alpha}\lambda_{\alpha k}x_{\alpha k}\bigr)_{\alpha\in\mathscr{A}}$ is a net of convex combinations in C such that $x_\alpha \xrightarrow{\sigma(X,\Gamma)} x$, i.e.,*

$$\sum_{k=1}^{n_\alpha}\lambda_{\alpha k}x^*(x_{\alpha k}-x) \underset{\alpha}{\to} 0 \qquad \text{for all } x^* \in \Gamma, \tag{3.3}$$

one has $x_\alpha \xrightarrow{\tau} x$; moreover,

$$\sum_{k=1}^{n_\alpha}\lambda_{\alpha k}p(x_{\alpha k}-x) \underset{\alpha}{\to} 0 \qquad \text{for all } p \in \mathscr{S}, \tag{3.4}$$

and thus, for every finite subset $\mathscr{F}$ of $\mathscr{S}$,

$$\sum_{k=1}^{n_\alpha} \lambda_{\alpha k} \max_{p\in\mathscr{F}} p(x_{\alpha k} - x) \underset{\alpha}{\to} 0. \tag{3.5}$$

If Γ separates the points of X, and C is $\sigma(X,\Gamma)$-compact, then each of the assertions (i)–(iv) *is equivalent to the assertion*

(v) *x is an extreme point of C and a $\sigma(X,\Gamma)$-to-τ-PC of C.*

REMARK 3.2. Suppose that, in Proposition 3.1, τ is the norm topology on X. In this case, if $\Gamma = X^*$, then each of the assertions (i)–(v) is equivalent to x being a denting point of C. If X happens to be a dual space, say $X = E^*$, and Γ is (the canonical image of) the predual E of X, then each of the assertions (i)–(v) is equivalent to x being a weak* denting point of C. Proposition 3.1 is probably (at least partially) known, but we could not find any reference for it.

REMARK 3.3. Proposition 2.3 follows from Proposition 3.1 by taking Y^* in the role of X, the $\sigma(Y^*, Z_P)$-topology in the role of τ, and (the canonical image of) Y in the role of Γ.

The proof of the equivalence (ii) $\Leftrightarrow$ (v) (under the additional assumptions that Γ separates the points of X, and C is $\sigma(X,\Gamma)$-compact), relies on the following partial case of [2, p. 107, Proposition 25.13].

PROPOSITION 3.4. *Let X be a Banach space, let Γ be a linear subspace of X^* separating the points of X, let C be a $\sigma(X,\Gamma)$-compact bounded convex set in X, and let $x \in C$. The following assertions are equivalent:*

(i) *x is an extreme point of C;*

(ii) *Γ-slices of C containing x form a neighbourhood basis for x in the relative $\sigma(X,\Gamma)$-topology of C.*

PROOF OF PROPOSITION 3.1. (i) $\Leftrightarrow$ (ii) is obvious.

(i) $\Rightarrow$ (iii). Assume that (i) holds. Let $\mathscr{F}$ be a finite subset of $\mathscr{S}$ and let $\varepsilon > 0$. By (i), there is a Γ-slice $S := S(x^*, \alpha, C)$ of C satisfying (3.1). Now

$$C \setminus \mathscr{U}_{\mathscr{F}}(x,\varepsilon) \subset C \setminus S = \{u \in C : \operatorname{Re} x^*(u) \le \sup \operatorname{Re} x^*(C) - \alpha\}.$$

Since the latter set is convex and closed in the relative $\sigma(X,\Gamma)$-topology of C, one has $\overline{\mathrm{co}}^{\sigma(X,\Gamma)}(C \setminus \mathscr{U}_{\mathscr{F}}(x,\varepsilon)) \subset C \setminus S$, and (3.2) follows.

(iii) $\Rightarrow$ (iv). First observe that, for every finite subset $\mathscr{F}$ of $\mathscr{S}$, condition (3.4) implies (3.5), because

$$\sum_{k=1}^{n_\alpha} \lambda_{\alpha k} \max_{p\in\mathscr{F}} p(x_{\alpha k} - x) \le \sum_{p\in\mathscr{F}} \sum_{k=1}^{n_\alpha} \lambda_{\alpha k} p(x_{\alpha k} - x).$$

Assume that (iii) holds. Let $(x_\alpha)_{\alpha\in\mathscr{A}} := \big(\sum_{k=1}^{n_\alpha}\lambda_{\alpha k}x_{\alpha k}\big)_{\alpha\in\mathscr{A}}$ be a net of convex combinations in C satisfying (3.3). We must prove (3.4). To this end, it suffices to prove the following claim.

Claim 1. *Let $p\in\mathscr{S}$ and let $\varepsilon>0$. Put, for every $\alpha\in\mathscr{A}$,*

$$J^\alpha_{p,\varepsilon} := \big\{k\in\{1,\dots,n_\alpha\} : p(x_{\alpha k}-x)\geq\varepsilon\big\}.$$

Then $\lim_\alpha \sum_{k\in J^\alpha_{p,\varepsilon}}\lambda_{\alpha k}=0$.

Indeed, assume that Claim 1 has been proven. Let $p\in\mathscr{S}$ and $\varepsilon>0$ be arbitrary. Since C is bounded and τ is weaker than the norm topology, the seminorm p is bounded on C, i.e., $M:=\sup_{u\in C}p(u)<\infty$. By Claim 1, we can choose $\alpha_1\in\mathscr{A}$ so that $\sum_{k\in J^\alpha_{p,\varepsilon}}\lambda_{\alpha k}<\frac{\varepsilon}{2M}$ for all $\alpha>\alpha_1$. Whenever $\alpha>\alpha_1$, we have

$$\begin{aligned}\sum_{k=1}^{n_\alpha}\lambda_{\alpha k}p(x_{\alpha k}-x) &= \sum_{k\in J^\alpha_{p,\varepsilon}}\lambda_{\alpha k}p(x_{\alpha k}-x)+\sum_{k\notin J^\alpha_{p,\varepsilon}}\lambda_{\alpha k}p(x_{\alpha k}-x)\\ &< \sum_{k\in J^\alpha_{p,\varepsilon}}\lambda_{\alpha k}2M+\varepsilon<\varepsilon+\varepsilon=2\varepsilon,\end{aligned}$$

and (3.4) follows.

Claim 1 follows from the following claim.

Claim 2. *Suppose that $u_\alpha, v_\alpha\in C$, $\lambda_\alpha\in[0,1]$, and $\lambda>0$ are such that $x_\alpha=\lambda_\alpha u_\alpha+(1-\lambda_\alpha)v_\alpha$ and $\lambda_\alpha\geq\lambda$ for all $\alpha\in\mathscr{A}$. Then $u_\alpha\xrightarrow{\tau}x$.*

Indeed, assume that Claim 2 has been proven. Suppose for contradiction that $\sum_{k\in J^\alpha_{p,\varepsilon}}\lambda_{\alpha k}\nrightarrow 0$ for some $p\in\mathscr{S}$ and some $\varepsilon>0$. By passing to a subnet, we may assume that there is a $\lambda>0$ such that $\lambda_\alpha:=\sum_{k\in J^\alpha_{p,\varepsilon}}\lambda_{\alpha k}\geq\lambda$ for all $\alpha\in\mathscr{A}$. Defining, for all $\alpha\in\mathscr{A}$,

$$u_\alpha := \sum_{k\in J^\alpha_{p,\varepsilon}}\frac{\lambda_{\alpha k}}{\lambda_\alpha}x_{\alpha k}\quad\text{and}\quad v_\alpha:=\begin{cases}\displaystyle\sum_{k\notin J^\alpha_{p,\varepsilon}}\frac{\lambda_{\alpha k}}{1-\lambda_\alpha}x_{\alpha k}, & \text{if } \lambda_\alpha<1,\\ x_{\alpha 1}, & \text{if } \lambda_\alpha=1,\end{cases}$$

we have $u_\alpha, v_\alpha\in C$ and $x_\alpha=\lambda_\alpha u_\alpha+(1-\lambda_\alpha)v_\alpha$ for all $\alpha\in\mathscr{A}$; therefore $u_\alpha\xrightarrow{\tau}x$ by Claim 2. We have a contradiction, because $u_\alpha\in\operatorname{co}(C\setminus\mathscr{U}_{\{p\}}(x,\varepsilon))$ for all $\alpha\in\mathscr{A}$, but $x\notin\overline{\operatorname{co}}^\tau(C\setminus\mathscr{U}_{\{p\}}(x,\varepsilon))$ by (iii).

It remains to prove Claim 2. To this end, suppose for contradiction that there are $u_\alpha, v_\alpha\in C$, $\lambda_\alpha\in[0,1]$, and $\lambda>0$ such that $x_\alpha=\lambda_\alpha u_\alpha+(1-\lambda_\alpha)v_\alpha$ and $\lambda_\alpha\geq\lambda$ for all $\alpha\in\mathscr{A}$, but $u_\alpha\nrightarrow x$ in the τ-topology. Then $p(u_\alpha-x)\nrightarrow 0$ for some $p \in \mathscr{S}$. By passing to a subnet, we may assume that there is an $\varepsilon_1 > 0$ such that $p(u_\alpha - x) \geq \varepsilon_1$ for all $\alpha \in \mathscr{A}$. Notice that $p(v_\alpha - x) \nrightarrow 0$ because otherwise we would have

$$\begin{aligned}\liminf_\alpha p(x_\alpha - x) &\geq \liminf_\alpha \big(\lambda_\alpha p(u_\alpha - x) - (1-\lambda_\alpha) p(v_\alpha - x)\big) \\ &= \liminf_\alpha \lambda_\alpha p(u_\alpha - x) \geq \lambda \varepsilon_1 > 0.\end{aligned}$$

Hence, by passing to a subnet again, we may assume that there is an $\varepsilon_2 > 0$ such that $p(v_\alpha - x) \geq \varepsilon_2$ for all $\alpha \in \mathscr{A}$. Putting $\varepsilon_0 := \min\{\varepsilon_1, \varepsilon_2\}$, we have $x_\alpha \in \operatorname{co}(C \setminus \mathscr{U}_{\{p\}}(x, \varepsilon_0))$ for all $\alpha \in \mathscr{A}$. Since $x_\alpha \xrightarrow{\tau} x$ we have $x \in \overline{\operatorname{co}}^\tau (C \setminus \mathscr{U}_{\{p\}}(x, \varepsilon_0))$, a contradiction.

(iv) $\Rightarrow$ (iii). Assume that (iv) holds. Suppose for contradiction that there exist a finite subset $\mathscr{F}$ of $\mathscr{S}$ and an $\varepsilon > 0$ such that $x \in \overline{\operatorname{co}}^{\sigma(X,\Gamma)}(C \setminus \mathscr{U}_{\mathscr{F}}(x, \varepsilon))$. Then there exists a net $(x_\alpha)_{\alpha\in\mathscr{A}} := \big(\sum_{k=1}^{n_\alpha} \lambda_{\alpha k} x_{\alpha k}\big)_{\alpha\in\mathscr{A}}$ of convex combinations in C such that $x_\alpha \xrightarrow{\sigma(X,\Gamma)} x$ and $\max_{p\in\mathscr{F}} p(x_{\alpha k} - x) \geq \varepsilon$ for all $\alpha \in \mathscr{A}$ and all $k \in \{1, \ldots, n_\alpha\}$. But now we have $\sum_{k=1}^{n_\alpha} \lambda_{\alpha k} \max_{p\in\mathscr{F}} p(x_{\alpha k} - x) \geq \varepsilon$ for all $\alpha \in \mathscr{A}$. This contradicts (3.5).

(iii) $\Rightarrow$ (i). Assume that (iii) holds. Let $\mathscr{F}$ be a finite subset of $\mathscr{S}$ and let $\varepsilon > 0$. We need to show that there exists a Γ-slice $S := S(x^*, \alpha, C)$ of C satisfying (3.1).

By the assumption (iii), we have (3.2), thus, by the Hahn-Banach separation theorem, there are $x^* \in \Gamma$ and $\beta > 0$ such that

$$\operatorname{Re} x^*(x) - \beta > \sup \operatorname{Re} x^*\big(\overline{\operatorname{co}}^{\sigma(X,\Gamma)}(C \setminus \mathscr{U}_{\mathscr{F}}(x, \varepsilon))\big).$$

Put $\alpha := \sup \operatorname{Re} x^*(C) - \operatorname{Re} x^*(x) + \beta$; then

$$\operatorname{Re} x^*(x) = \sup \operatorname{Re} x^*(C) + \beta - \alpha > \sup \operatorname{Re} x^*(C) - \alpha,$$

thus $x \in S(x^*, \alpha, C)$.

It remains to show that $S(x^*, \alpha, C) \subset \mathscr{U}_{\mathscr{F}}(x, \varepsilon)$. Let $u \in S(x^*, \alpha, C)$. Suppose for contradiction that $u \notin \mathscr{U}_{\mathscr{F}}(x, \varepsilon)$, i.e., $u \in C \setminus \mathscr{U}_{\mathscr{F}}(x, \varepsilon)$. Then

$$\begin{aligned}\operatorname{Re} x^*(u) &\leq \sup \operatorname{Re} x^*\big(C \setminus \mathscr{U}_{\mathscr{F}}(x, \varepsilon)\big) \leq \sup \operatorname{Re} x^*\big(\overline{\operatorname{co}}^{\sigma(X,\Gamma)}(C \setminus \mathscr{U}_{\mathscr{F}}(x, \varepsilon))\big) \\ &< \operatorname{Re} x^*(x) - \beta = \sup \operatorname{Re} x^*(C) - \alpha < \operatorname{Re} x^*(u),\end{aligned}$$

a contradiction.

For the rest of the proof, assume that Γ separates the points of X, and that C is $\sigma(X, \Gamma)$-compact.

(ii) $\Rightarrow$ (v). Assume that (ii) holds. Then x is clearly a $\sigma(X, \Gamma)$-to-τ-PC of C. It remains to show that x is an extreme point of C. This follows from

Proposition 3.4, because, since $\Gamma \subset (X, \tau)'$, the $\sigma(X, \Gamma)$-topology is weaker than τ and, thus, by (ii), also the assertion (ii) of Proposition 3.4 holds.

(v) $\Rightarrow$ (ii) is obvious by Proposition 3.4.

4. Property U for $\mathscr{K}(X, Y)$ in $\mathscr{K}(X, Z)$

Let us fix some more notation, point out some observations, and agree in some conventions.

Let X and Z be Banach spaces. Recall that, for $x^{**} \in X^{**}$ and $z^* \in Z^*$, the functional $x^{**} \otimes z^* \in \mathscr{K}(X, Z)^*$ is defined by $(x^{**} \otimes z^*)(T) = x^{**}(T^* z^*)$, $T \in \mathscr{K}(X, Z)$. Define

$$B_{X^{**}} \otimes B_{Z^*} = \{x^{**} \otimes z^* : x^{**} \in B_{X^{**}}, z^* \in B_{Z^*}\} \subset \mathscr{K}(X, Z)^*.$$

Observe that $B_{X^{**}} \otimes B_{Z^*}$ is a weak* closed subset of $\mathscr{K}(X, Z)^*$.

Let us make the convention that, unless explicitly stated otherwise, whenever considering topological properties (such as, e.g., compactness and Borelness) of subsets of the set $B_{X^{**}} \otimes B_{Z^*} \subset B_{\mathscr{K}(X,Z)^*}$, the topology we have in mind is the relative weak* topology of this set.

Since, for every $T \in \mathscr{K}(X, Z)$, there is some $\phi \in C := B_{X^{**}} \otimes B_{Z^*} \subset B_{\mathscr{K}(X,Z)^*}$ such that $\operatorname{Re} \phi(T) = \|T\|$, by the Hahn-Banach separation theorem, it quickly follows that $\overline{\text{co}}^{w^*}(C) = B_{\mathscr{K}(X,Z)^*}$. Thus, for every $f \in S_{\mathscr{K}(X,Z)^*}$, as a consequence of the Riesz representation theorem, there is a regular Borel probability measure μ on $\overline{C}^{w^*} = C$ such that $f(T) = \int_C \phi(T)\, d\mu(\phi)$ for every $T \in \mathscr{K}(X, Z)$.

The proof of Theorem 1.2 relies on the following Theorem 4.1 (cf. [13, Theorem 1.2]) which shows that, under the assumptions of Theorem 1.2, whenever a functional $f \in S_{\mathscr{K}(X,Z)^*}$ is represented as an integral with respect to a regular Borel probability measure on $B_{X^{**}} \otimes B_{Z^*}$ as above, the ideal projection for $\mathscr{K}(X, Y)$ in $\mathscr{K}(X, Z)$ when applied to f "passes under the integral sign". The proof of Theorem 4.1, in turn, relies on Theorem 4.2 below. We remark that the proof of the prototype [10, Theorem 3.6] of Theorem 1.2 relies on the Feder-Saphar description [3, Theorem 1] of the dual of $\mathscr{K}(X, Z)$, which can also be derived from Theorem 4.2 (see [13, Corollary 2.2]).

THEOREM 4.1. *Let X and Z be Banach spaces, and let Y be a strict ideal in Z with respect to a projection π on Z^* such that*

- *X^{**} and Y^* have the Radon-Nikodým property;*
- *$\mathscr{K}(X, Y)$ is an ideal in $\mathscr{K}(X, Z)$ with respect to a projection P on $\mathscr{K}(X, Z)^*$ such that*

$$P(x^{**} \otimes z^*) = x^{**} \otimes \pi z^* \quad \textit{for all } x^{**} \in X^{**} \textit{ and all } z^* \in Z^*. \quad (4.1)$$

Let μ be a regular Borel (with respect to the relative weak* topology) probability measure on $C := B_{X^{**}} \otimes B_{Z^*} \subset B_{\mathscr{K}(X,Z)^*}$. Then there is a Borel set $C' \subset C$ such that

(a) $\int_{C\setminus C'} |\phi(S)|\, d\mu(\phi) = 0$ *for all* $S \in \mathscr{K}(X, Y)$*;*

(b) *for every* $T \in \mathscr{K}(X, Z)$*, the function* $C \ni \phi \mapsto (P\phi)(T)\, \chi_{C'}(\phi) \in \mathbb{K}$ *is measurable;*

(c) *defining* $f \in \mathscr{K}(X, Z)^*$ *by* $f(T) = \int_C \phi(T)\, d\mu(\phi)$*,* $T \in \mathscr{K}(X, Z)$*, one has*
$$(Pf)(T) = \int_{C'} (P\phi)(T)\, d\mu(\phi), \quad T \in \mathscr{K}(X, Z).$$

The proof of Theorem 4.1 relies on the following result. We omit its proof, because it is almost verbatim to that of [13, Theorem 2.1] with some obvious changes. We remark that the proof reduces to an application of a theorem of Edgar [1, Theorem 4.3.11].

THEOREM 4.2. *Let X and Z be Banach spaces, and let Y be a closed subspace of Z. Suppose that Y^* (respectively, X^{**}) has the Radon-Nikodým property. Let μ be a regular Borel (with respect to the relative weak* topology) probability measure on $C := B_{X^{**}} \otimes B_{Z^*} \subset B_{\mathscr{K}(X,Z)^*}$. Denote by $\mathscr{C}$ (respectively, $\mathscr{D}$) the collection of compact subsets A of C with the following property:*

- *there is a norm compact set $Y_A^* \subset S_{Y^*}$ (respectively, $X_A^{**} \subset S_{X^{**}}$) such that, for every $\phi \in A$, there are $y^* \in Y_A^*$ and $x^{**} \in B_{X^{**}}$ (respectively, $y^* \in B_{Y^*}$ and $x^{**} \in X_A^{**}$) with $\phi|_{\mathscr{K}(X,Y)} = x^{**} \otimes y^*$.*

Then there are pairwise disjoint Borel sets $C_j \subset C$, $j \in \{0\} \cup \mathbb{N}$ (respectively, $D_i \subset C$, $i \in \{0\} \cup \mathbb{N}$) such that $C = \bigcup_{j=0}^{\infty} C_j$ (respectively, $C = \bigcup_{i=0}^{\infty} D_i$), where $\int_{C_0} |\phi(S)|\, d\mu(\phi) = 0$ (respectively, $\int_{D_0} |\phi(S)|\, d\mu(\phi) = 0$) for all $S \in \mathscr{K}(X, Y)$, and $C_j \in \mathscr{C}$, $j \in \mathbb{N}$ (respectively, $D_i \in \mathscr{D}$, $i \in \mathbb{N}$).

PROOF OF THEOREM 4.1. For all $j, i \in \{0\} \cup \mathbb{N}$, put $E_{ji} := C_j \cap D_i$ where the sets C_j and D_i are as in Theorem 4.2. Put $C' = \bigcup_{j,i=1}^{\infty} E_{ji}$ and, for every $n \in \mathbb{N}$,
$$\widehat{C}_n := \bigcup_{j,i=1}^{n} E_{ji} \quad \text{and} \quad \widetilde{C}_n := C' \setminus \widehat{C}_n.$$

Choose an increasing sequence of indices $(k_n)_{n=1}^{\infty}$ so that $\mu\big(\widetilde{C}_{k_n}\big) < 1/n$ for every $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, let $A_n \subset S_{Y^*}$ and $B_n \subset S_{X^{**}}$ be a finite $1/n$-net, respectively, for $\bigcup_{j=1}^{k_n} Y_{C_j}^*$ and $\bigcup_{i=1}^{k_n} X_{D_i}^{**}$ where the sets $Y_{C_j}^*$ and $X_{D_i}^{**}$ are as in Theorem 4.2.

Let $T \in S_{\mathscr{K}(X,Z)}$. By Goldstine's theorem (or by the bipolar theorem), there is a net $(S_\alpha)_{\alpha\in\mathscr{A}}$ in $B_{\mathscr{K}(X,Y)}$ such that $S_\alpha \xrightarrow[\alpha]{\sigma(\mathscr{K}(X,Z),\operatorname{ran} P)} T$. Choose an increasing sequence of indices $(\alpha_n)_{n=1}^{\infty}$ so that, whenever $n \in \mathbb{N}$, for each $\alpha \succeq \alpha_n$, one has $\big|(u^{**} \otimes J_\pi v^*)(S_\alpha - T)\big| < 1/n$ for all $v^* \in A_n$ and all $u^{**} \in B_n$.

Now let $n \in \mathbb{N}$ be fixed, and suppose that $\phi \in \widehat{C}_{k_n}$, i.e., $\phi \in E_{ji} = C_j \cap D_i$ for some $j, i \in \{1, \dots, k_n\}$. Then there are $y_0^* \in Y_{C_j}^*$, $x^{**} \in B_{X^{**}}$, $y^* \in B_{Y^*}$, and $x_0^{**} \in X_{D_i}^{**}$ such that $\phi|_{\mathscr{K}(X,Y)} = x^{**} \otimes y_0^* = x_0^{**} \otimes y^*$, and thus

$$P\phi = x^{**} \otimes J_\pi y_0^* = x_0^{**} \otimes J_\pi y^*.$$

One has $x^{**} = \gamma x_0^{**}$ and $y^* = \gamma y_0^*$ for some $\gamma \in \mathbb{K}$ with $|\gamma| \le 1$. Choosing $v^* \in A_n$ and $u^{**} \in B_n$ so that $\|y_0^* - v^*\| < 1/n$ and $\|x_0^{**} - u^{**}\| < 1/n$, one has, whenever $\alpha \succeq \alpha_n$,

$$\begin{aligned}
|(P\phi)(T) - \phi(S_\alpha)| = |(P\phi)(T - S_\alpha)| &= |(\gamma x_0^{**} \otimes J_\pi y_0^*)(T - S_\alpha)| \\
&\le \Big|\big((x_0^{**} - u^{**}) \otimes J_\pi y_0^*\big)(T - S_\alpha)\Big| \\
&\quad + \Big|\big(u^{**} \otimes J_\pi (y_0^* - v^*)\big)(T - S_\alpha)\Big| \\
&\quad + \big|(u^{**} \otimes J_\pi v^*)(S_\alpha - T)\big| \\
&< \frac{2}{n} + \frac{2}{n} + \frac{1}{n} = \frac{5}{n}.
\end{aligned}$$

It follows that $\phi(S_{\alpha_n}) \xrightarrow[n\to\infty]{} (P\phi)(T)$ for each $\phi \in C'$; thus the function $C \ni \phi \mapsto (P\phi)(T)\, \chi_{C'}(\phi) \in \mathbb{K}$ is measurable.

Letting, again, $n \in \mathbb{N}$ be fixed and $\alpha \succeq \alpha_n$, one has

$$\begin{aligned}
&\left|\int_{C'} (P\phi)(T)\, d\mu(\phi) - f(S_\alpha)\right| \le \int_{C'} |(P\phi)(T) - \phi(S_\alpha)|\, d\mu(\phi) \\
&\quad = \int_{\widehat{C}_{k_n}} |(P\phi)(T) - \phi(S_\alpha)|\, d\mu(\phi) + \int_{\widetilde{C}_{k_n}} |(P\phi)(T) - \phi(S_\alpha)|\, d\mu(\phi) \\
&\quad < \frac{5}{n} + \frac{2}{n} = \frac{7}{n},
\end{aligned}$$

and it follows that

$$(Pf)(T) = \lim_\alpha (Pf)(S_\alpha) = \lim_\alpha f(S_\alpha) = \int_{C'} (P\phi)(T)\, d\mu(\phi).$$

Proof of Theorem 1.2. Let π be the ideal projection for Y in Z, and let P be an ideal projection for $\mathscr{K}(X, Y)$ in $\mathscr{K}(X, Z)$.

For the strictness of P, it suffices to prove (4.1). Since Y has property U in Y^{**}, by [18, Theorem 15] (or see [6, p. 126]), Y^* has the Radon-Nikodým property. An argument from [8, proof of Proposition 4.1, (b) $\Rightarrow$ (c)] shows that $B_{Y^*} = \overline{\mathrm{co}}^{\|\cdot\|}(w^*\text{-}\operatorname{str.exp} B_{Y^*})$ (i.e., B_{Y^*} is the norm closed convex hull of its weak* strongly exposed points). A standard argument (see, e.g., [10, proof of Theorem 3.4]) now yields (4.1). Indeed, let $x^{**} \in B_{X^{**}}$ and $z^* \in B_{Z^*}$. Putting $y^* = z^*|_Y$, it suffices to show that

$$J_P(x^{**} \otimes y^*) = x^{**} \otimes J_\pi y^*, \tag{4.2}$$

because in this case one would have

$$\begin{aligned} P(x^{**} \otimes z^*) &= J_P\big((x^{**} \otimes z^*)|_{\mathscr{K}(X,Y)}\big) = J_P(x^{**} \otimes y^*) = x^{**} \otimes J_\pi y^* \\ &= x^{**} \otimes \pi z^*. \end{aligned}$$

Whenever $v^* \in w^*\text{-}\operatorname{str.exp} B_{Y^*}$, the functional $x^{**} \otimes v^* \in \mathscr{K}(X, Y)^*$ has a unique norm-preserving extension to $\mathscr{K}(X, Z)$ (this follows from Theorem 1.1 and the sentence following it), thus this norm-preserving extension must be $x^{**} \otimes J_\pi v^*$, i.e. $J_P(x^{**} \otimes v^*) = x^{**} \otimes J_\pi v^*$. By linearity, the latter remains true for $v^* \in \mathrm{co}(w^*\text{-}\operatorname{str.exp} B_{Y^*})$, and, since $y^* \in \overline{\mathrm{co}}^{\|\cdot\|}(w^*\text{-}\operatorname{str.exp} B_{Y^*})$, by continuity, also (4.2) holds.

Let $f \in S_{\mathscr{K}(X,Z)^*}$ be such that $\|Pf\| = \|f\| = 1$. For proving that $\mathscr{K}(X, Y)$ has property U in $\mathscr{K}(X, Z)$, it suffices to show that $Pf = f$. As explained in the beginning of the section, there is a regular Borel (with respect to the relative weak* topology) probability measure on $C := B_{X^{**}} \otimes B_{Z^*} \subset \mathscr{K}(X, Z)^*$ representing f, i.e., $f(T) = \int_C \phi(T)\, d\mu(\phi)$ for all $T \in \mathscr{K}(X, Z)$. Now we can apply Theorem 4.1. Letting the set C' be as in Theorem 4.1, one has $\mu(C \setminus C') = 0$, because

$$\begin{aligned} 1 = \|Pf\| = \|f|_{\mathscr{K}(X,Y)}\| &= \sup_{S \in B_{\mathscr{K}(X,Y)}} |f(S)| = \sup_{S \in B_{\mathscr{K}(X,Y)}} \left| \int_C \phi(S)\, d\mu(\phi) \right| \\ &\le \sup_{S \in B_{\mathscr{K}(X,Y)}} \int_C |\phi(S)|\, d\mu(\phi) = \sup_{S \in B_{\mathscr{K}(X,Y)}} \int_{C'} |\phi(S)|\, d\mu(\phi) \le \mu(C'). \end{aligned}$$

Put $C_1 := \{\phi \in C' : \|P\phi\| = 1\}$. Then $\mu(C' \setminus C_1) = 0$ (the function $C \ni \phi \mapsto \|P\phi\|$ is measurable since it is lower semicontinuous), because otherwise one would have $\int_{C' \setminus C_1} \|P\phi\|\, d\mu(\phi) < \mu(C' \setminus C_1)$ and thus

$$\begin{aligned} \|Pf\| &\le \sup_{S \in B_{\mathscr{K}(X,Y)}} \int_{C'} \big|\phi(S)\big|\, d\mu(\phi) \\ &\le \int_{C_1} \|P\phi\|\, d\mu(\phi) + \int_{C' \setminus C_1} \|P\phi\|\, d\mu(\phi) \\ &< \mu(C_1) + \mu(C' \setminus C_1) = 1. \end{aligned}$$

For every $\phi \in C_1$, one has $P\phi = \phi$. Indeed, let $\phi = x^{**} \otimes z^* \in C_1$, where $x^{**} \in B_{X^{**}}$, $z^* \in B_{Z^*}$. One has

$$1 = \|P\phi\| \leq \|\phi\| = \|x^{**}\| \, \|z^*\| \leq 1$$

and, by (4.1),

$$1 = \|P\phi\| = \|x^{**} \otimes \pi z^*\| = \|x^{**}\| \, \|\pi z^*\|,$$

thus $\|\pi z^*\| = \|z^*\| = 1$. Since Y has property U in Y^{**} and Y is a strict ideal in Z, also Y has property U in Z, and therefore $\pi z^* = z^*$. It follows that $P\phi = \phi$.

Now, by Theorem 4.1, for any $T \in \mathscr{K}(X, Z)$,

$$\begin{aligned}(Pf)(T) &= \int_{C'} (P\phi)(T)\, d\mu(\phi) = \int_{C_1} (P\phi)(T)\, d\mu(\phi) \\ &= \int_{C_1} \phi(T)\, d\mu(\phi) = \int_{C} \phi(T)\, d\mu(\phi) = f(T).\end{aligned}$$

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