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THE COMPLEX MOMENT PROBLEM: DETERMINACY AND EXTENDIBILITY

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Abstract

Complex moment sequences are exactly those which admit positive definite extensions on the integer lattice points of the upper diagonal half-plane. Here we prove that the aforesaid extension is unique provided the complex moment sequence is determinate and its only representing measure has no atom at 0. The question of converting the relation is posed as an open problem. A partial solution to this problem is established when at least one of representing measures is supported in a plane algebraic curve whose intersection with every straight line passing through 0 is at most one point set. Further study concerns representing measures whose supports are Zariski dense in $\mathbb{C}$ as well as complex moment sequences which are constant on a family of parallel "Diophantine lines". All this is supported by a bunch of illustrative examples.

There are two ways of approaching the complex moment problem (see [3]; for a recent survey of the complex moment problem see also [21]). One following an idea due to Marcel Riesz (for continuation see [13], [14], [15]) and the other via positive definite extendibility (see [28], [9]). As is well-known, positive definiteness is not sufficient for solving the complex moment problem (see [20], [3]). The present paper carries on with the study of [28] which characterizes solving the complex moment problem by extending a given sequence defined on the integer lattice points of the first quarter to a positive definite sequence on the lattice points of the upper diagonal half-plane. One may expect a relationship between the uniqueness of extending sequence on one hand and the determinacy of the resulting moment sequence. This question leads to quite a number of interesting thoughts which are exposed in this paper. Our results, which are diverse in nature, are supported by elucidative examples and lead eventually to an open problem discussed on the final pages of the paper.

1. Introduction

In this paper $\mathfrak{B}(Z)$ stands for the σ-algebra of all Borel subsets of a topological Hausdorff space Z. All measures considered in this paper are positive. We

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always tacitly assume that integrands are absolutely integrable wherever they appear. With the notation

$$\mathfrak{N} \stackrel{\text{def}}{=} \{(m,n)\colon m,n \text{ are integers such that } m \geqslant 0, n \geqslant 0\},$$
$$\mathfrak{N}_+ \stackrel{\text{def}}{=} \{(m,n)\colon m,n \text{ are integers such that } m+n \geqslant 0\},$$

we say that a sequence $\boldsymbol{\gamma} = \{\gamma_{m,n}\}_{(m,n)\in\mathfrak{N}} \subset \mathbb{C}$ is a *complex moment sequence* if there exists a Borel measure μ on $\mathbb{C}$ such that

$$\gamma_{m,n} = \int_{\mathbb{C}} z^m \bar{z}^n \, \mathrm{d}\mu(z), \quad (m,n) \in \mathfrak{N}; \tag{1}$$

recall that $\mathbb{R}$ and $\mathbb{C}$ stand for the fields of all real and complex numbers, respectively. We call the measure μ a *representing measure* for the sequence $\boldsymbol{\gamma}$. If μ in (1) is unique, then the sequence $\boldsymbol{\gamma}$ is said to be *determinate* (this is one of the three determinacy notions considered in [12]). As is easily seen, a necessary condition for $\boldsymbol{\gamma}$ to be a complex moment sequence is that $\boldsymbol{\gamma}$ is *positive definite on* $\mathfrak{N}$, that is

$$\sum_{(m,n),(p,q)\in\mathfrak{N}} \lambda_{m,n}\bar{\lambda}_{p,q}\gamma_{m+q,n+p} \geqslant 0$$

for every sequence $\{\lambda_{m,n}\}_{(m,n)\in\mathfrak{N}} \subset \mathbb{C}$ vanishing off a finite set. The above positive definiteness condition is in general not sufficient. However, it turns out that complex moment sequences are exactly those which admit positive definite extensions on $\mathfrak{N}_+$ (see [28, Theorem 1]). More precisely, $\boldsymbol{\gamma} = \{\gamma_{m,n}\}_{(m,n)\in\mathfrak{N}} \subset \mathbb{C}$ is a complex moment sequence if and only if there exists a sequence $\boldsymbol{\Gamma} = \{\Gamma_{m,n}\}_{(m,n)\in\mathfrak{N}_+} \subset \mathbb{C}$ which is *positive definite on* $\mathfrak{N}_+$, that is

$$\sum_{(m,n),(p,q)\in\mathfrak{N}_+} \lambda_{m,n}\bar{\lambda}_{p,q}\Gamma_{m+q,n+p} \geqslant 0$$

for every sequence $\{\lambda_{m,n}\}_{(m,n)\in\mathfrak{N}_+} \subset \mathbb{C}$ vanishing off a finite set, and which extends $\boldsymbol{\gamma}$, that is

$$\Gamma_{m,n} = \gamma_{m,n}, \quad (m,n) \in \mathfrak{N}.$$

Using the notation

$$\mathsf{PDE}(\boldsymbol{\gamma}) \stackrel{\text{def}}{=} \{\boldsymbol{\Gamma}\colon \boldsymbol{\Gamma} \text{ is a positive definite extension of } \boldsymbol{\gamma} \text{ on } \mathfrak{N}_+\},$$

we can simply rewrite [28, Theorem 1] as follows.

THEOREM 1. *A sequence* $\boldsymbol{\gamma} = \{\gamma_{m,n}\}_{(m,n)\in\mathfrak{N}} \subset \mathbb{C}$ *is a complex moment sequence if and only if* $\mathsf{PDE}(\boldsymbol{\gamma}) \neq \varnothing$.

The main question of this paper concerns a connection between the following two statements:

(i) $\boldsymbol{\gamma}$ is a determinate complex moment sequence on $\mathfrak{N}$,

(ii) $\mathsf{PDE}(\boldsymbol{\gamma})$ is a singleton.

According to Proposition 5, if $\boldsymbol{\gamma}$ has a representing measure μ such that $\mu(\{0\}) > 0$, then $\mathsf{PDE}(\boldsymbol{\gamma})$ is infinite. We will show that (i) implies (ii) provided the representing measure of $\boldsymbol{\gamma}$ vanishes at $\{0\}$ (see Theorem 6). The implication (ii) $\Rightarrow$ (i) holds when $\boldsymbol{\gamma}$ has a representing measure supported in a real algebraic set belonging to a distinguished class of plane algebraic curves (see Theorem 22 and Corollary 23). It is an open problem whether (ii) implies (i) in full generality (see Section 6).

We recall that in view of [28, Prop. 6] a sequence $\boldsymbol{\Gamma} = \{\Gamma_{m,n}\}_{(m,n)\in\mathfrak{N}_+} \subset \mathbb{C}$ is positive definite if and only if there exist two Borel measures μ on $\mathbb{C}^* \overset{\text{def}}{=} \mathbb{C}\setminus\{0\}$ and ν on $\mathbb{T}$ (the unit circle centered at the origin) such that

$$\Gamma_{m,n} = \int_{\mathbb{C}^*} z^m \bar{z}^n \,\mathrm{d}\mu(z) + \delta_{m+n,0} \int_{\mathbb{T}} z^m \bar{z}^n \,\mathrm{d}\nu(z), \quad (m,n) \in \mathfrak{N}_+, \tag{2}$$

where $\delta_{k,\ell} = 1$ if $k = \ell$ and $\delta_{k,\ell} = 0$ otherwise. If (2) holds, we say that (μ, ν) is a *representing* pair for $\boldsymbol{\Gamma}$. If such a pair is unique, then $\boldsymbol{\Gamma}$ is called *determinate*. Depending on circumstances, we will identify a Borel measure μ on $\mathbb{C}^*$ with a Borel one on $\mathbb{C}$ vanishing on $\{0\}$.

In this paper the notation $\boldsymbol{\gamma} = \{\gamma_{m,n}\}_{(m,n)\in\mathfrak{N}}$ will be used interchangeably with $\boldsymbol{\gamma}\colon \mathfrak{N} \to \mathbb{C}$; the same applies to $\boldsymbol{\Gamma}$ and $\mathfrak{N}_+$.

2. Determinacy from extendibility

In this section we investigate the interplay between the determinacy of a complex moment sequence $\boldsymbol{\gamma}$ and special properties of the set $\mathsf{PDE}(\boldsymbol{\gamma})$ including those related to its cardinality.

As shown in Lemma 2 a representing measure for a complex moment sequence can be retrieved from a representing pair for its positive definite extension. Below, δ_a stands for the Dirac measure at a understood as the Borel measure on $\mathbb{C}$ of total mass 1 at the point $a \in \mathbb{C}$.

LEMMA 2. *Suppose $\boldsymbol{\gamma}$ is a complex moment sequence. Then the following assertions hold:*

(i) *if (μ, ν) is a representing pair for some $\boldsymbol{\Gamma} \in \mathsf{PDE}(\boldsymbol{\gamma})$, then $\mu + \nu(\mathbb{T})\delta_0$ is a representing measure for $\boldsymbol{\gamma}$,*

(ii) *if μ is a representing measure for $\boldsymbol{\gamma}$ and $\mu(\{0\}) = 0$, then $(\mu, 0)$ is a representing pair for some $\boldsymbol{\Gamma} \in \mathsf{PDE}(\boldsymbol{\gamma})$.*

Proof. (i) Since for every $(m, n) \in \mathfrak{N}$, the second term in (2) coincides with the integral of $z^m \bar{z}^n$ over $\mathbb{C}$ with respect to the measure $\nu(\mathbb{T})\delta_0$, we get (i).

(ii) Since, by our assumption, the function $\mathbb{C}^* \ni z \to z^m \bar{z}^n \in \mathbb{C}$ is absolutely integrable for every $(m, n) \in \mathfrak{N}_+$, we can define the sequence $\boldsymbol{\Gamma} = \{\Gamma_{m,n}\}_{(m,n)\in\mathfrak{N}_+}$ by (2) with $\nu = 0$. Then $\boldsymbol{\Gamma} \in \mathsf{PDE}(\boldsymbol{\gamma})$ and $(\mu, 0)$ is a representing pair for $\boldsymbol{\Gamma}$.

From now on φ will denote the continuous function

$$\varphi: \mathbb{T} \to \mathbb{T}, \quad \varphi(z) = z^2, \quad z \in \mathbb{T}. \tag{3}$$

If $\boldsymbol{\gamma}$ is a complex moment sequence, then an extension $\boldsymbol{\Gamma} \in \mathsf{PDE}(\boldsymbol{\gamma})$ is called *quasi-determinate* if for any two representing pairs (μ_1, ν_1) and (μ_2, ν_2) for $\boldsymbol{\Gamma}$ we have

$$\mu_1 = \mu_2 \quad \text{and} \quad \nu_1 \circ \varphi^{-1} = \nu_2 \circ \varphi^{-1},$$

where $\nu_j \circ \varphi^{-1}$ is the transport of the measure ν_j via φ given by

$$(\nu_j \circ \varphi^{-1})(\sigma) \overset{\text{def}}{=} \nu_j(\varphi^{-1}(\sigma)), \quad \sigma \in \mathfrak{B}(\mathbb{T}), \quad j = 1, 2. \tag{4}$$

This notion appears to be very natural as the following result shows.

Now we are ready to clarify the role played by determinacy in the question of uniqueness of positive definite extensions. This is in a sense the basic statement.

Theorem 3. *Let $\boldsymbol{\gamma}: \mathfrak{N} \to \mathbb{C}$ be a complex moment sequence. Then the following conditions are equivalent:*

(i) *$\boldsymbol{\gamma}$ is determinate;*

(ii) *if (μ_1, ν_1) and (μ_2, ν_2) are representing pairs for $\boldsymbol{\Gamma}_1, \boldsymbol{\Gamma}_2 \in \mathsf{PDE}(\boldsymbol{\gamma})$, respectively, then $\mu_1 = \mu_2$;*

(iii) *if (μ_1, ν_1) and (μ_2, ν_2) are representing pairs for $\boldsymbol{\Gamma}_1, \boldsymbol{\Gamma}_2 \in \mathsf{PDE}(\boldsymbol{\gamma})$, respectively, then $\mu_1 = \mu_2$ and $\nu_1(\mathbb{T}) = \nu_2(\mathbb{T})$.*

Moreover, if (i) *holds, then every $\boldsymbol{\Gamma} \in \mathsf{PDE}(\boldsymbol{\gamma})$ is quasi-determinate.*

Proof. (i) $\Rightarrow$ (iii). The determinacy of $\boldsymbol{\gamma}$ and Lemma 2 yield

$$\mu_1 + \nu_1(\mathbb{T})\delta_0 = \mu_2 + \nu_2(\mathbb{T})\delta_0.$$

By the definition of a representing pair for an element of $\mathsf{PDE}(\boldsymbol{\gamma})$, both measures μ_1 and μ_2 have no atom at 0. Hence, it follows that $\mu_1 = \mu_2$ and $\nu_1(\mathbb{T}) = \nu_2(\mathbb{T})$.

(iii) $\Rightarrow$ (ii). Obvious.

(ii) $\Rightarrow$ (i). Suppose that λ_1 and λ_2 are representing measures for $\boldsymbol{\gamma}$. Note that the measures λ_1 and λ_2 have the following unique decompositions

$$\lambda_1 = \mu_1 + t_1\delta_0 \quad \text{and} \quad \lambda_2 = \mu_2 + t_2\delta_0, \tag{5}$$

where μ_1 and μ_2 are Borel measures on $\mathbb{C}^*$ and t_1, t_2 are nonnegative real numbers. Then one can verify that $(\mu_1, t_1\delta_1)$ and $(\mu_2, t_2\delta_1)$ are representing pairs for some $\boldsymbol{\Gamma}_1, \boldsymbol{\Gamma}_2 \in \mathsf{PDE}(\boldsymbol{\gamma})$, respectively. It follows from (ii) that $\mu_1 = \mu_2$, and thus

$$t_j \overset{(5)}{=} \lambda_j(\mathbb{C}) - \mu_j(\mathbb{C}^*) = \gamma_{0,0} - \mu_1(\mathbb{C}^*), \quad j = 1, 2,$$

which, by (5) again, implies that $\lambda_1 = \lambda_2$.

To prove the "moreover" part, assume that (i) holds. Let (μ_1, ν_1) and (μ_2, ν_2) be representing pairs for the same positive definite extension of $\boldsymbol{\gamma}$ on $\mathfrak{N}_+$. Then, by (ii), $\mu_1 = \mu_2$, and consequently, by (2) with $m + n = 0$, we have

$$\int_{\mathbb{T}} z^{2m}\, \mathrm{d}\nu_1(z) = \int_{\mathbb{T}} z^{2m}\, \mathrm{d}\nu_2(z), \quad m = 0, \pm 1, \pm 2, \ldots.$$

Applying the measure transport theorem yields

$$\int_{\mathbb{T}} z^{m}\, \mathrm{d}\nu_1 \circ \varphi^{-1}(z) = \int_{\mathbb{T}} z^{m}\, \mathrm{d}\nu_2 \circ \varphi^{-1}(z), \quad m = 0, \pm 1, \pm 2, \ldots.$$

Hence, by the determinacy of Herglotz moment problem (see Section 3), $\nu_1 \circ \varphi^{-1} = \nu_2 \circ \varphi^{-1}$, which completes the proof.

Corollary 4. *Let $\boldsymbol{\gamma}\colon \mathfrak{N} \to \mathbb{C}$ be a determinate complex moment sequence. Then for every nonzero finite Borel measure ν on $\mathbb{T}$, there exists a unique triplet $(\mu, s, \boldsymbol{\Gamma})$ such that μ is a Borel measure on $\mathbb{C}^*$, s is a nonnegative real number, $\boldsymbol{\Gamma} \in \mathsf{PDE}(\boldsymbol{\gamma})$ and $(\mu, s\nu)$ is a representing pair for $\boldsymbol{\Gamma}$.*

Proof. Let ϱ be a representing measure for $\boldsymbol{\gamma}$. Set $s = \varrho(\{0\})/\nu(\mathbb{T})$ and $\mu = \varrho - \varrho(\{0\})\delta_0$. It is easily seen that $(\mu, s\nu)$ is a representing pair for some $\boldsymbol{\Gamma} \in \mathsf{PDE}(\boldsymbol{\gamma})$, cf. (2). The uniqueness is a direct consequence of Theorem 3.

One may illustrate Corollary 4 by considering particular choices, extreme in a sense, of the measure ν: ν being the Lebesgue measure on $\mathbb{T}$ or with ν being the Dirac measure δ_1 on $\mathbb{T}$ at 1.

Proposition 5. *Let $\boldsymbol{\gamma}\colon \mathfrak{N} \to \mathbb{C}$ be a complex moment sequence which has a representing measure μ such that*

$$\mu(\{0\}) > 0. \tag{6}$$

Then the cardinality of $\mathsf{PDE}(\boldsymbol{\gamma})$ is equal to $\mathfrak{c}$.

PROOF. Set $\alpha = \mu(\{0\})$. It is a simple matter to verify that the sequences $\boldsymbol{\Gamma}_1$ and $\boldsymbol{\Gamma}_2$ on $\mathfrak{N}_+$ defined via (2) by the pairs $(\mu - \alpha\delta_0, \alpha\delta_1)$ and $(\mu - \alpha\delta_0, \alpha\delta_{\mathrm{i}})$, respectively, are both in $\mathsf{PDE}(\boldsymbol{\gamma})$; here $\mathrm{i} = \sqrt{-1}$. Then any convex combination of $\boldsymbol{\Gamma}_1$ and $\boldsymbol{\Gamma}_2$ is in $\mathsf{PDE}(\boldsymbol{\gamma})$. Using (2) with $m + n = 0$, where m is odd, we deduce that $\boldsymbol{\Gamma}_1 \neq \boldsymbol{\Gamma}_2$. As a consequence, the cardinality of $\mathsf{PDE}(\boldsymbol{\gamma})$ is at least $\mathfrak{c}$. On the other hand, because $\mathsf{PDE}(\boldsymbol{\gamma}) \subset \mathbb{C}^{\mathfrak{N}_+}$ and the cardinality of $\mathbb{C}^{\mathfrak{N}_+}$ is $\mathfrak{c}$, we conclude that the cardinality of $\mathsf{PDE}(\boldsymbol{\gamma})$ is equal to $\mathfrak{c}$.

It is clear that in the case of a determinate complex moment sequence the zero may or may not be an atom of its representing measure and both instances can occur (e.g., the sequences $(1, 0, 0, \ldots)$ and $(1, 1, 1, \ldots)$ are complex moment sequences with the representing measures δ_0 and δ_1, respectively). One can ask whether the same is true for indeterminate complex moment sequences. This question is answered in the affirmative in Section 3 (see Proposition 9 and the subsequent parts).

If $\boldsymbol{\gamma}: \mathfrak{N} \to \mathbb{C}$ is determinate complex moment sequence and μ is its representing measure such that $\mu(\{0\}) = 0$, then $\mathsf{PDE}(\boldsymbol{\gamma})$ is a single point set, as the following Theorem shows.

THEOREM 6. *Let $\boldsymbol{\gamma}$ be a complex sequence defined on $\mathfrak{N}$. Then the following statements are equivalent:*

(i) *$\boldsymbol{\gamma}$ is a determinate complex moment sequence with a representing measure μ such that $\mu(\{0\}) = 0$;*

(ii) $\mathsf{PDE}(\boldsymbol{\gamma}) = \{\boldsymbol{\Gamma}\}$ *and $\boldsymbol{\Gamma}$ has the property that $\mu_1 = \mu_2$ whenever* $(\mu_1, 0)$ *and* $(\mu_2, 0)$ *are representing pairs for $\boldsymbol{\Gamma}$;*

(iii) $\mathsf{PDE}(\boldsymbol{\gamma}) = \{\boldsymbol{\Gamma}\}$ *and $\boldsymbol{\Gamma}$ is determinate.*

Moreover, if $\boldsymbol{\Gamma}$ is as in (iii)*, then* $(\mu, 0)$ *is its representing pair, where μ is as in* (i).

PROOF. (i) $\Rightarrow$ (iii) Note that $\mathsf{PDE}(\boldsymbol{\gamma}) \neq \varnothing$ because, by Lemma 2, $(\mu, 0)$ is a representing pair for some $\boldsymbol{\Gamma} \in \mathsf{PDE}(\boldsymbol{\gamma})$. Let (μ_1, ν_1) and (μ_2, ν_2) be representing pairs for some extensions $\boldsymbol{\Gamma}_1, \boldsymbol{\Gamma}_2 \in \mathsf{PDE}(\boldsymbol{\gamma})$, respectively. It follows from Theorem 3 that $\mu_1 = \mu_2$. In turn, Lemma 2 yields

$$\mu = \mu_1 + \nu_1(\mathbb{T})\delta_0 = \mu_2 + \nu_2(\mathbb{T})\delta_0,$$

which by our assumption $\mu(\{0\}) = 0$ gives $\nu_1(\mathbb{T}) = \nu_2(\mathbb{T}) = 0$. Thus the pairs (μ_1, ν_1) and (μ_2, ν_2) are equal to $(\mu, 0)$. As a consequence, $\mathsf{PDE}(\boldsymbol{\gamma}) = \{\boldsymbol{\Gamma}\}$, the extension $\boldsymbol{\Gamma}$ is determinate and $(\mu, 0)$ is a representing pair for $\boldsymbol{\Gamma}$.

(iii) $\Rightarrow$ (ii) Obvious.

(ii) $\Rightarrow$ (i) It follows from Theorem 1 that $\boldsymbol{\gamma}$ is a complex moment sequence. According to Proposition 5, if μ is a representing measure for $\boldsymbol{\gamma}$, then $\mu(\{0\}) =$

0, and consequently, by Lemma 2, $(\mu, 0)$ is a representing pair for $\boldsymbol{\Gamma}$. Hence, by the property of $\boldsymbol{\Gamma}$ assumed in (ii), the complex moment sequence $\boldsymbol{\gamma}$ is determinate. This completes the proof.

3. Special classes of complex moment sequences

In the previous section the question of the cardinality of $\mathsf{PDE}(\boldsymbol{\gamma})$ was successfully answered except for the case of an indeterminate complex moment sequence $\boldsymbol{\gamma}$ which has no representing measure with atom at 0. The question arises as to whether such a $\boldsymbol{\gamma}$ may exist. It is well-known that every indeterminate Hamburger moment sequence has a representing measure with an atom at 0 (see [22, Theorem 2.13]). It turns out that in the case of complex moment sequences this does not have to be the case (see Example 12).

We begin by stating a well-known fact on supports of representing measures of a complex moment sequence having at least one representing measure supported in a real algebraic set. In what follows, $\operatorname{supp}\mu$ stands for the closed support of a finite Borel measure on a metric space (see [17, Theorem II.2.1]). In this paper, by "support" we always mean "closed support". Recall that a set $A \subset \mathbb{C}$ is called a *real algebraic set* if $A = \mathscr{Z}_p$ for some $p \in \mathbb{C}[z, \bar{z}]$, where

$$\mathscr{Z}_p \stackrel{\text{def}}{=} \{z \in \mathbb{C}\colon p(z, \bar{z}) = 0\}$$

and $\mathbb{C}[z, \bar{z}]$ stands as usual for the ring of all polynomials in two indeterminates with complex coefficients.

PROPOSITION 7. *If a complex moment sequence $\boldsymbol{\gamma}$ has a representing measure supported in a real algebraic subset A of $\mathbb{C}$, then all the other representing measures for $\boldsymbol{\gamma}$ are also supported in A.*

PROOF. If $A = \mathscr{Z}_p$, where $p \in \mathbb{C}[z, \bar{z}]$, μ_1 and μ_2 are representing measures for $\boldsymbol{\gamma}$ and μ_1 is supported in A, then

$$\int_{\mathbb{C}} |p(z, \bar{z})|^2 \, \mathrm{d}\mu_2(z) = \int_{\mathbb{C}} |p(z, \bar{z})|^2 \, \mathrm{d}\mu_1(z) = \int_A |p(z, \bar{z})|^2 \, \mathrm{d}\mu_1(z) = 0,$$

which implies that $\mu_2(\mathbb{C} \setminus A) = 0$. This completes the proof.

Necessary and sufficient conditions for a complex moment sequence to have a representing measure supported in a given plane algebraic curve were given in [25], [27]. Below we discuss the interplay between a linear Diophantine relation imposed on indices of a complex moment sequence and supports of its representing measures. Given integers k and ℓ such that $k \geqslant 0$, a sequence

$\boldsymbol{\gamma} = \{\gamma_{m,n}\}_{(m,n)\in\mathfrak{N}} \subset \mathbb{C}$ is called (k, ℓ)-*flat* if the following condition holds

$$\gamma_{m,n} = \gamma_{m',n'} \quad \text{whenever } (m, n), (m', n') \in \mathfrak{N} \\ \text{and } km + \ell n = km' + \ell n'. \tag{7}$$

Intuitively speaking, the (k, ℓ)-flatness of $\boldsymbol{\gamma}$ means that the sequence $\boldsymbol{\gamma}$ is constant on each "Diophantine line" $\{(m, n) \in \mathbb{Z}^2: km + \ell n = c\}$ with $c \in \mathbb{Z}$ (as usual $\mathbb{Z}$ stands for the set of all integers).

THEOREM 8. *If $\boldsymbol{\gamma} = \{\gamma_{m,n}\}_{(m,n)\in\mathfrak{N}}$ is a (k, ℓ)-flat complex moment sequence for some (k, ℓ) and μ is its representing measure, then one of the following conditions holds:*

(i) $\operatorname{supp}\mu \subset \{0\} \cup G_r$ *for some integer* $r \geqslant 1$, *where* $G_r = \{z \in \mathbb{C}: z^r = 1\}$*;*

(ii) $\operatorname{supp}\mu \subset \mathbb{T}$*;*

(iii) $\operatorname{supp}\mu \subset \mathbb{R}$.

Moreover, if $k \neq |\ell|$ (resp., $k = -\ell$, $k = \ell$), then the condition (i) (*resp.,* (ii), (iii)) *holds. If additionally $k \neq |\ell|$ and $\ell \leqslant 0$, then $0 \notin \operatorname{supp}\mu$.*

Conversely, if $\boldsymbol{\gamma}$ is a complex moment sequence with a representing measure μ satisfying one of the conditions (i)–(iii)*, then $\boldsymbol{\gamma}$ is (k, ℓ)-flat, where*

$$(k, \ell) = \begin{cases} (1, 1), & \textit{if } \text{(i)} \textit{ holds and } r = 1, \\ (1, r-1), & \textit{if } \text{(i)} \textit{ holds and } r > 1, \\ (1, -1), & \textit{if } \text{(ii)} \textit{ holds,} \\ (1, 1), & \textit{if } \text{(iii)} \textit{ holds.} \end{cases} \tag{8}$$

PROOF. Assume that $\boldsymbol{\gamma}$ is a (k, ℓ)-flat complex moment sequence for some (k, ℓ) and μ is its representing measure. In the case of $k = \ell = 0$ the measure $\gamma_{0,0}\delta_1$ is a (necessarily unique) representing measure for $\boldsymbol{\gamma}$. Hence, its support satisfies the conditions (i)–(iii).

Suppose $\ell > 0$. Note that the pairs $(m, n) = (\ell, \ell)$ and $(m', n') = (0, k+\ell)$ satisfy (7). The same is true for the pairs $(k + \ell, 0)$ and (k, k). Hence, by the (k, ℓ)-flatness of $\boldsymbol{\gamma}$, we have

$$\int_{\mathbb{C}} |z^\ell - \bar{z}^k|^2 \,\mathrm{d}\mu(z) = \gamma_{\ell,\ell} - \gamma_{k+\ell,0} - \gamma_{0,k+\ell} + \gamma_{k,k} = 0,$$

and so

$$\operatorname{supp}\mu \subset \mathscr{Z}_p \quad \text{with } p(z, \bar{z}) = z^\ell - \bar{z}^k. \tag{9}$$

If $k \neq \ell$, then the equality $z^\ell = \bar{z}^k$ implies that $|z| = 1$ for $z \neq 0$ and, consequently, $z^{k+\ell} = 1$. This shows that $\mathscr{Z}_p \setminus \{0\} \subset G_{k+\ell}$. Therefore, μ

satisfies (i). In turn, if $k = \ell$, then $\boldsymbol{\gamma}$ is $(1, 1)$-flat. Applying (9) to $k = \ell = 1$, we see that $\operatorname{supp} \mu \subset \mathscr{Z}_p = \mathbb{R}$, which means that μ obeys (iii).

Now consider the case $\ell < 0$. Note that the pairs $(m, n) = (k - \ell, k - \ell)$ and $(m', n') = (k, -\ell)$ satisfy (7). The same is true for the pairs $(-\ell, k)$ and $(0, 0)$. Then the (k, ℓ)-flatness of $\boldsymbol{\gamma}$ implies that

$$\int_{\mathbb{C}} |z^{-\ell}\bar{z}^k - 1|^2 \, \mathrm{d}\mu(z) = \gamma_{k-\ell,k-\ell} - \gamma_{-\ell,k} - \gamma_{k,-\ell} + \gamma_{0,0} = 0,$$

which means that $\operatorname{supp} \mu \subset \mathscr{Z}_p$ with $p(z, \bar{z}) = z^{-\ell}\bar{z}^k - 1$. As a consequence, $\mathscr{Z}_p \subset \mathbb{T}$, which shows that μ satisfies (ii) (observe that if $z \in \mathscr{Z}_p$ and $k+\ell \neq 0$, then $z \in G_{|k+\ell|}$).

In the remaining case of $\ell = 0$, which is analogous to that of $k = 0$, μ satisfies the condition (ii) (in fact, $\operatorname{supp} \mu \subset G_k$).

To prove the reverse implication, assume that $\boldsymbol{\gamma}$ is a complex moment sequence with a representing measure μ satisfying one the conditions (i)–(iii). First, we consider the case (i), that is $\operatorname{supp} \mu \subset \{0\} \cup G_r$ for some integer $r \geqslant 1$. Then

$$\gamma_{m,n} = c\,\delta_{m+n,0} + \sum_{j=0}^{r-1} a_j \mathrm{e}^{2\pi j(m-n)\mathrm{i}/r}, \quad (m, n) \in \mathfrak{N},$$

where $c = \mu(\{0\})$ and $a_j = \mu(\{\mathrm{e}^{2\pi j\mathrm{i}/r}\})$ for $j = 0, \ldots, r - 1$. If $r = 1$, then $\gamma_{m,n} = c\,\delta_{m+n,0} + a_0$ for $(m, n) \in \mathfrak{N}$, which means that $\boldsymbol{\gamma}$ is $(1, 1)$-flat. This covers the first choice in (8). It is a matter of routine to verify that if $r > 1$, then $\boldsymbol{\gamma}$ is $(1, r - 1)$-flat. This covers the second choice in (8). It is easily seen that if (ii) holds, then

$$\gamma_{m,n} = \begin{cases} \gamma_{m-n,0}, & \text{if } m \geqslant n, \\ \gamma_{0,n-m}, & \text{otherwise}, \end{cases} \quad (m, n) \in \mathfrak{N},$$

which implies that $\gamma_{m,n}$ depends on $m - n$. This covers the third case in (8). Finally, if (iii) holds, then $\gamma_{m,n} = \gamma_{m+n,0}$ for all $(m, n) \in \mathfrak{N}$, which covers the fourth case in (8). This completes the proof.

We now proceed to complex moment sequences induced by Hamburger moment sequences. We say that a sequence $\boldsymbol{s} = \{s_n\}_{n=0}^{\infty} \subset \mathbb{R}$ is a *Hamburger moment sequence* if there exists a Borel measure τ on $\mathbb{R}$ such that

$$s_n = \int_{\mathbb{R}} x^n \, \mathrm{d}\tau(x), \quad n \geqslant 0; \tag{10}$$

such a measure is called a *representing measure* for $\boldsymbol{s}$. If τ in (10) is unique, then $\boldsymbol{s}$ is called a *determinate* Hamburger moment sequence.

PROPOSITION 9. *Let* $\boldsymbol{s} = \{s_n\}_{n=0}^{\infty}$ *be a sequence of real numbers. Define* $\boldsymbol{\gamma} = \{\gamma_{m,n}\}_{(m,n)\in\mathfrak{N}}$ *by*

$$\gamma_{m,n} = s_{m+n}, \quad (m,n) \in \mathfrak{N}. \tag{11}$$

Then for $\boldsymbol{\gamma}$ *given by* (11) *the following assertions hold:*

(i) $\boldsymbol{s}$ *is a Hamburger moment sequence if and only if* $\boldsymbol{\gamma}$ *is a complex moment sequence, or equivalently, if and only if* $\boldsymbol{\gamma}$ *is a complex moment sequence whose every representing measure is supported in* $\mathbb{R}$*;*

(ii) $\boldsymbol{s}$ *is a determinate Hamburger moment sequence if and only if* $\boldsymbol{\gamma}$ *is a determinate complex moment sequence;*

(iii) *if* $\boldsymbol{s}$ *is an indeterminate Hamburger moment sequence and* $x_0 \in \mathbb{R}$*, then* $\boldsymbol{\gamma}$ *is an indeterminate complex moment sequence which has infinitely many representing measures* μ *such that* $\mu(\{x_0\}) > 0$*; moreover, the cardinality of* $\mathsf{PDE}(\boldsymbol{\gamma})$ *is equal to* $\mathfrak{c}$*.*

PROOF. (i) Suppose that $\boldsymbol{s}$ is a Hamburger moment sequence with a representing measure τ. It is easily seen that the Borel measure μ on $\mathbb{C}$ defined by

$$\mu(\sigma) = \tau(\sigma \cap \mathbb{R}), \quad \sigma \in \mathfrak{B}(\mathbb{C}), \tag{12}$$

is a representing measure for $\boldsymbol{\gamma}$ supported in $\mathbb{R}$. Suppose now that $\boldsymbol{\gamma}$ is a complex moment sequence with a representing measure ϱ. Since $\boldsymbol{\gamma}$ is (1, 1)-flat, Theorem 8 implies that ϱ is supported in $\mathbb{R}$. As a consequence, $\boldsymbol{s}$ is a Hamburger moment sequence with the representing measure τ defined by

$$\tau(\sigma) = \varrho(\sigma), \quad \sigma \in \mathfrak{B}(\mathbb{R}). \tag{13}$$

The above argument concerning the support of ϱ also establishes the second equivalence in (i).

(ii) This is a direct consequence of (12) and (13) and the fact that each representing measure for $\boldsymbol{\gamma}$ is supported in $\mathbb{R}$ as shown in (i).

(iii) Suppose $\boldsymbol{s}$ is an indeterminate Hamburger moment sequence. Then, by [22, Theorem 2.13] (see also [23, Theorem 5]), there exist two representing measures τ_0 and τ_1 of $\boldsymbol{s}$ such that $\tau_0(\{x_0\}) = 0$ and $\tau_1(\{x_0\}) > 0$. Taking convex combinations $\tau_\alpha \overset{\text{def}}{=} \alpha\tau_1 + (1-\alpha)\tau_0$ for $\alpha \in (0, 1)$, we get representing measures for $\boldsymbol{s}$ such that $\tau_\alpha(\{x_0\}) > 0$ for all $\alpha \in (0, 1]$. It is easily seen that the corresponding representing measures μ_α of $\boldsymbol{\gamma}$ given by (12) have the property that $\mu_\alpha(\{x_0\}) > 0$ for $\alpha \in (0, 1]$ and $\mu_\alpha \neq \mu_\beta$ for all $\alpha, \beta \in (0, 1]$ such that $\alpha \neq \beta$. The "moreover" part follows from Proposition 5.

Regarding Proposition 9, one should provide an example of an indeterminate Hamburger moment sequence. One of the possible choices is the famous example $\{\mathrm{e}^{(n+1)^2/4}\}_{n=0}^{\infty}$ due to Stieltjes (see [24]). Below, we present a class of indeterminate Hamburger moment sequences introduced recently in [8].

EXAMPLE 10. Fix a nonzero complex function ω on $\mathbb{R}$ of class $\mathscr{C}^{\infty}$ whose support is compact and define the sequence $\{a_n^{\omega}\}_{n=0}^{\infty}$ by

$$a_n^{\omega} = (-\mathrm{i})^n \int_{\mathbb{R}} \frac{\mathrm{d}^n \omega}{\mathrm{d}x^n}(x)\, \overline{\omega(x)}\, \mathrm{d}x, \quad n = 0, 1, 2, \ldots$$

This is an indeterminate Hamburger moment sequence with striking properties. Namely, one can find continuum explicitly described representing measures for $\{a_n^{\omega}\}_{n=0}^{\infty}$ such that

- the support of each of them is in arithmetic progression,
- the supports of all these measures together partition $\mathbb{R}$,
- all of them are of infinite order,

where the latter means that for any such measure τ, the codimension of polynomials in $L^2(\tau)$ is infinite. All these three conditions hold under the assumption that the Fourier transform of ω does not vanish on $\mathbb{R}$ (such ω always exists!). Hence, for any $x_0 \in \mathbb{R}$, there exists a representing measure τ of $\{a_n^{\omega}\}_{n=0}^{\infty}$ in the above mentioned family such that $\tau(\{x_0\}) > 0$.

Instead of Hamburger moment sequences we may consider Herglotz moment sequences and the complex moment sequences induced by them. We say that a sequence $\boldsymbol{s} = \{s_n\}_{n=-\infty}^{\infty}$ of complex numbers is a *Herglotz (trigonometric) moment sequence* if there exists a Borel measure ϱ on $\mathbb{T}$ such that

$$s_n = \int_{\mathbb{T}} z^n \, \mathrm{d}\varrho(z), \quad n = 0, \pm 1, \pm 2, \ldots; \tag{14}$$

such a measure is called a *representing measure* for $\boldsymbol{s}$. Every Herglotz moment sequence is determinate, that is the measure ϱ in (14) is unique (see [1, Theorem 5.1.2]; see also [4, Theorem 1.7.2]). It is worth mentioning that a Herglotz moment sequence $\boldsymbol{s}$ has the following Hermitian symmetry property:

$$s_n = \overline{s_{-n}}, \quad n = 0, \pm 1, \pm 2, \ldots.$$

This means that such $\boldsymbol{s}$ is uniquely determined by its entries s_n with $n = 0, 1, 2, \ldots$.

PROPOSITION 11. *Let* $\boldsymbol{s} = \{s_n\}_{n=-\infty}^{\infty}$ *be a sequence of complex numbers. Define* $\boldsymbol{\gamma} = \{\gamma_{m,n}\}_{(m,n)\in\mathfrak{R}}$ *by*

$$\gamma_{m,n} = s_{m-n}, \quad (m, n) \in \mathfrak{R}.$$

Then $\boldsymbol{s}$ *is a Herglotz moment sequence if and only if* $\boldsymbol{\gamma}$ *is a complex moment sequence. Moreover, if* $\boldsymbol{s}$ *is a Herglotz moment sequence, then* $\boldsymbol{\gamma}$ *is a determinate complex moment sequence whose unique representing measure is supported in* $\mathbb{T}$, $\mathsf{PDE}(\boldsymbol{\gamma}) = \{\boldsymbol{\Gamma}\}$ *and* $\boldsymbol{\Gamma}$ *is determinate.*

PROOF. If $\boldsymbol{s}$ is a Herglotz moment sequence with a representing measure ϱ, then $\boldsymbol{\gamma}$ is a complex moment sequence with the representing measure μ given by

$$\mu(\sigma) = \varrho(\sigma \cap \mathbb{T}), \quad \sigma \in \mathfrak{B}(\mathbb{C}).$$

Suppose that $\boldsymbol{\gamma}$ is a complex moment sequence with a representing measure ν. Since $\boldsymbol{\gamma}$ is $(1, -1)$-flat, Theorem 8 ensures us that the support of ν is contained in $\mathbb{T}$. Thus the Borel measure ϱ on $\mathbb{T}$ given by

$$\varrho(\sigma) = \nu(\sigma), \quad \sigma \in \mathfrak{B}(\mathbb{T}),$$

is a representing measure for $\boldsymbol{s}$. The “moreover” part follows from the determinacy of Herglotz moment sequences, the fact that $\nu(\{0\}) = 0$ and Theorem 6.

Regarding Proposition 5, we show that it may happen that a complex moment sequence is indeterminate and that none of its representing measures satisfies (6).

EXAMPLE 12. Consider an indeterminate Hamburger moment sequence $\boldsymbol{s} = \{s_n\}_{n=0}^{\infty}$ (see e.g., Example 10). Let τ be a representing measure for $\boldsymbol{s}$. Define the Borel measure μ on $\mathbb{C}$ by

$$\mu(\sigma) = \tau((\sigma - \mathrm{i}) \cap \mathbb{R}), \quad \sigma \in \mathfrak{B}(\mathbb{C}). \tag{15}$$

Define the complex moment sequence $\boldsymbol{\gamma} = \{\gamma_{m,n}\}_{(m,n)\in\mathfrak{N}}$ by

$$\gamma_{m,n} = \int_{\mathbb{C}} z^m \bar{z}^n \,\mathrm{d}\mu(z), \quad m, n \geqslant 0.$$

It follows from (15) and the indeterminacy of $\boldsymbol{s}$ that $\boldsymbol{\gamma}$ is indeterminate. By (15) again, the measure μ is supported in $\mathbb{R} + \mathrm{i}$. Since $\mathbb{R} + \mathrm{i} = \mathscr{Z}_p$ with $p(z, \bar{z}) = z - \bar{z} - 2\mathrm{i}$, we infer from Proposition 7 that each representing measure for $\boldsymbol{\gamma}$ is supported in $\mathbb{R} + \mathrm{i}$, and consequently none of them satisfies (6).

4. Representing measures whose support is Zariski dense

Our goal in this section is to establish a wide class of complex moment sequences, each of which has the following properties:

1° it is indeterminate;

2° none of its representing measures has an atom at 0;

3° all its representing measures have supports dense in $\mathbb{C}$ with respect to the Zariski topology.

Let us recall that the Zariski topology on $\mathbb{C}$ consists of all the sets of the form $\mathbb{C}\setminus\mathscr{Z}_p$, where $p \in \mathbb{C}[z, \bar{z}]$, and it satisfies the T_1 separation axiom. We refer the reader to [2], [5] for more information on the Zariski topology. The property 3° means that the support of any representing measure for the complex moment sequence under consideration is not contained in a proper real algebraic set $\mathscr{Z}_p$, as opposed to Example 12.

For a technical reason, it is much simpler to state and prove our result in terms of two-dimensional Hamburger moment problem; afterwards we will turn back to complex moment sequences. We say that $\boldsymbol{a} = \{a_{m,n}\}_{m,n=0}^{\infty} \subset \mathbb{R}$ is a *two-dimensional Hamburger moment sequence* if there exists a Borel measure ϱ on $\mathbb{R}^2$, called a *representing measure* for $\boldsymbol{a}$, such that

$$a_{m,n} = \int_{\mathbb{R}^2} x^m y^n \,\mathrm{d}\varrho(x, y), \quad m, n \geqslant 0.$$

If such ϱ is unique, $\boldsymbol{a}$ is called *determinate*.

We begin with recalling a known fact which is indispensable in this section. For the reader's convenience, we include its simple proof (see [18] for more on this topic).

LEMMA 13. *If $\boldsymbol{a} = \{a_{m,n}\}_{m,n=0}^{\infty}$ is a two-dimensional Hamburger moment sequence and ϱ its representing measure, then $\{a_{m,0}\}_{m=0}^{\infty}$ (resp., $\{a_{0,n}\}_{n=0}^{\infty}$) is a Hamburger moment sequence with the representing measure $\varrho \circ \pi_1^{-1}$ (resp., $\varrho \circ \pi_2^{-1}$), and*

$$\operatorname{supp} \varrho \circ \pi_1^{-1} = \overline{\pi_1(\operatorname{supp} \varrho)} \quad (\textit{resp.},\ \operatorname{supp} \varrho \circ \pi_2^{-1} = \overline{\pi_2(\operatorname{supp} \varrho)}\,), \qquad (16)$$

where $\pi_j \colon \mathbb{R}^2 \to \mathbb{R}$, $j = 1, 2$, are mappings given by

$$\pi_1(x, y) = x \text{ and } \pi_2(x, y) = y \quad \text{for } (x, y) \in \mathbb{R}^2. \qquad (17)$$

PROOF. The equalities in (16) follow from [26, Lemma 3.2]. Using the measure transport theorem, we get

$$a_{m,0} = \int_{\mathbb{R}^2} x^m \,\mathrm{d}\varrho(x, y) = \int_{\mathbb{R}} x^m \,\mathrm{d}\varrho \circ \pi_1^{-1}(x), \quad m \geqslant 0.$$

A similar argument applies to $\{a_{0,n}\}_{n=0}^{\infty}$.

The following shows that in some cases supports of representing measures can be localized in non-algebraic subsets of $\mathbb{C}$ (cf. Proposition 7; see also Proposition 25).

PROPOSITION 14. *Let* $\boldsymbol{a} = \{a_{m,n}\}_{m,n=0}^{\infty}$ *be a two-dimensional Hamburger moment sequence and* ϱ *its representing measure. Then the following assertions hold:*

(i) *if the set* $\pi_1(\operatorname{supp}\varrho)$ *(resp.,* $\pi_2(\operatorname{supp}\varrho)$*) is bounded, then* $\{a_{m,0}\}_{m=0}^{\infty}$ *(resp.,* $\{a_{0,n}\}_{n=0}^{\infty}$*) is a determinate Hamburger moment sequence;*

(ii) *if* $\{a_{m,0}\}_{m=0}^{\infty}$ *(resp.,* $\{a_{0,n}\}_{n=0}^{\infty}$*) is a determinate Hamburger moment sequence, then for any representing measure* $\tilde{\varrho}$ *for* $\boldsymbol{a}$*,*

$$\operatorname{supp}\tilde{\varrho} \subset \overline{\pi_1(\operatorname{supp}\varrho)} \times \mathbb{R} \quad (\textit{resp.,}\ \operatorname{supp}\tilde{\varrho} \subset \mathbb{R} \times \overline{\pi_2(\operatorname{supp}\varrho)}\,), \tag{18}$$

$$\overline{\pi_1(\operatorname{supp}\tilde{\varrho})} = \overline{\pi_1(\operatorname{supp}\varrho)} \quad (\textit{resp.,}\ \overline{\pi_2(\operatorname{supp}\tilde{\varrho})} = \overline{\pi_2(\operatorname{supp}\varrho)}\,). \tag{19}$$

PROOF. By symmetry, it suffices to consider the case of $\{a_{m,0}\}_{m=0}^{\infty}$.

(i) If $\pi_1(\operatorname{supp}\varrho)$ is bounded, then by Lemma 13 and (16) the Hamburger moment sequence $\{a_{m,0}\}_{m=0}^{\infty}$ has a compactly supported representing measure and as such is determinate (see [12, p. 50]).

(ii) Suppose $\{a_{m,0}\}_{m=0}^{\infty}$ is a determinate Hamburger moment sequence and $\tilde{\varrho}$ is a representing measure for $\boldsymbol{a}$. By Lemma 13, $\tilde{\varrho}\circ\pi_1^{-1} = \varrho\circ\pi_1^{-1}$. This together with (16) implies the first equality in (19). Set $S = \operatorname{supp}\varrho\circ\pi_1^{-1}$. Then the equality $\tilde{\varrho}\circ\pi_1^{-1} = \varrho\circ\pi_1^{-1}$ yields

$$0 = \tilde{\varrho}\circ\pi_1^{-1}(\mathbb{R}\setminus S) = \tilde{\varrho}(\mathbb{R}^2\setminus(S\times\mathbb{R})).$$

Since $\mathbb{R}^2\setminus(S\times\mathbb{R})$ is an open subset of $\mathbb{R}^2$, we see that $\operatorname{supp}\tilde{\varrho}\subset S\times\mathbb{R}$. Combined with (16), this gives the first inclusion in (18).

We now turn to the main result of this section.

THEOREM 15. *Let* $\boldsymbol{s} = \{s_m\}_{m=0}^{\infty}$ *be a determinate Hamburger moment sequence such that its unique representing measure* μ *has infinite support and* $\mu(\{0\}) = 0$*, and let* $\boldsymbol{t} = \{t_n\}_{n=0}^{\infty}$ *be an indeterminate Hamburger moment sequence. Then the sequence* $\boldsymbol{s}\otimes\boldsymbol{t} = \{(\boldsymbol{s}\otimes\boldsymbol{t})_{m,n}\}_{m,n=0}^{\infty}$ *defined by*

$$(\boldsymbol{s}\otimes\boldsymbol{t})_{m,n} = s_m t_n, \quad m,n \geqslant 0,$$

is a two-dimensional Hamburger moment sequence satisfying the conditions 1°, 2° *and* 3° *of page* 274 *with* $\mathbb{R}^2$ *in place of* $\mathbb{C}$.

PROOF. With no loss of generality, we can assume that $t_0 = 1$. The indeterminacy of $\boldsymbol{t}$ implies that

$$\text{the support of any representing measure for } \boldsymbol{t} \text{ is infinite.} \tag{20}$$

It follows from the Fubini theorem that the mapping $\nu \mapsto \mu \otimes \nu$ acts between the set of all representing measures for $\boldsymbol{t}$ and the set of all representing measures for $\boldsymbol{s} \otimes \boldsymbol{t}$, where $\mu \otimes \nu$ stands for the product measure of μ and ν. Since $\mu(\mathbb{R}) \neq 0$, the mapping is easily seen to be injective. Hence, by the indeterminacy of $\boldsymbol{t}$, the sequence $\boldsymbol{s} \otimes \boldsymbol{t}$ is indeterminate. This shows 1°.

For 2°, take any representing measure ϱ for $\boldsymbol{a} \stackrel{\text{def}}{=} \boldsymbol{s} \otimes \boldsymbol{t}$. Since $a_{m,0} = s_m$ for all integers $m \geqslant 0$ and $\boldsymbol{s}$ is determinate, we infer from Lemma 13 that $\mu = \varrho \circ \pi_1^{-1}$, which implies that

$$0 = \mu(\{0\}) = \varrho(\{0\} \times \mathbb{R}).$$

This yields 2°.

It remains to prove 3°. Take any representing measure ν for $\boldsymbol{t}$. It is a well-known and easy to prove fact that

$$\operatorname{supp} \mu \otimes \nu = \operatorname{supp} \mu \times \operatorname{supp} \nu. \tag{21}$$

Suppose that a polynomial $p \in \mathbb{C}[x, y]$ vanishes on $\operatorname{supp} \mu \otimes \nu$. We will show that $p = 0$. Indeed, if $x \in \operatorname{supp} \mu$, then by (20) and (21), the polynomial $y \mapsto p(x, y)$ vanishes on an infinite subset of $\mathbb{R}$, and consequently $p(x, y) = 0$ for all $x \in \operatorname{supp} \mu$ and $y \in \mathbb{R}$. Hence, for every $y \in \mathbb{R}$, the polynomial $x \mapsto p(x, y)$ vanishes on an infinite subset of $\mathbb{R}$, and thus $p(x, y) = 0$ for all $x \in \mathbb{R}$. As a consequence, $p = 0$. Suppose, contrary to 3°, that there exists a representing measure ϱ for $\boldsymbol{s} \otimes \boldsymbol{t}$ such the Zariski closure of $\operatorname{supp} \varrho$ is a proper subset of $\mathbb{R}^2$. This means that there exists a nonzero polynomial $q \in \mathbb{C}[x, y]$ such that the measure ϱ is supported in the real algebraic set $S = \{(x, y) \in \mathbb{R}^2 \colon q(x, y) = 0\}$. Since Proposition 7 remains valid for the two-dimensional Hamburger moment problem, we infer that any representing measure for $\boldsymbol{s} \otimes \boldsymbol{t}$ must be supported in S. In particular, this should hold for $\mu \otimes \nu$, which is a contradiction. This yields 3° and completes the proof.

Now, we turn back to the complex case. Recall that there is a one-to-one correspondence between the set of all two-dimensional Hamburger moment sequences $\boldsymbol{a} = \{a_{m,n}\}_{m,n=0}^{\infty}$ and the set of all complex moment sequences $\boldsymbol{\gamma} = \{\gamma_{m,n}\}_{m,n=0}^{\infty}$ given by

$$\gamma_{m,n} = \int_{\mathbb{R}^2} (x+\mathrm{i}y)^m (x-\mathrm{i}y)^n \,\mathrm{d}\varrho(x, y) = \sum_{k,\ell \geqslant 0} \alpha_{k,\ell}^{m,n} a_{k,\ell}, \quad (m, n) \in \mathfrak{N}, \tag{22}$$

where ϱ is a representing measure for $\boldsymbol{a}$ and $\{\alpha_{k,\ell}^{m,n}\}_{k,\ell=0}^{\infty}$, $(m, n) \in \mathfrak{N}$, are systems of complex numbers (each with finitely many nonzero entries) uniquely determined by the equations

$$(x+\mathrm{i}y)^m(x-\mathrm{i}y)^n = \sum_{k,\ell\geqslant 0} \alpha_{k,\ell}^{m,n} x^k y^\ell, \quad x, y \in \mathbb{R}.$$

Moreover, the sets of all representing measures for $\boldsymbol{a}$ and the corresponding $\boldsymbol{\gamma}$ coincide. We refer the reader to [10, Appendix A] for more details. As a consequence, if $\boldsymbol{a} = \boldsymbol{s} \otimes \boldsymbol{t}$, where $\boldsymbol{s}$ and $\boldsymbol{t}$ are as in Theorem 15, then the corresponding $\boldsymbol{\gamma}$ satisfies the conditions 1°, 2° and 3° on page 274.

5. Representing measures on real algebraic sets

In contrast to the previous section, we will now focus on the complex moment sequences having representing measures on real algebraic sets that are different from $\mathbb{C}$. The most satisfactory result establishes a one-to-one correspondence between representing measures for a complex moment sequence and its positive definite extensions on $\mathfrak{N}_+$ in the case when the mapping $\mathbb{C}^* \ni z \mapsto \frac{z}{\bar{z}} \in \mathbb{T}$ restricted to the algebraic set in question is injective (see Theorem 22(v)). In Example 24 we gather some classes of real algebraic sets meeting this requirement. After examining the Witch of Agnesi (one of these examples) we conclude that there is no complex moment counterpart of the partitioning property of the family of N-extremal measures as in one-dimensional Hamburger moment problem (see Proposition 26 and the discussion preceding it).

It is a well-known fact that a mapping $f: X \to Y$ between nonempty sets X and Y is injective if and only if the mapping $2^Y \ni \sigma \longmapsto f^{-1}(\sigma) \in 2^X$ is surjective. Following this, we say that a Borel mapping $f: X \to Y$ between topological Hausdorff spaces X and Y (i.e., a mapping such that $f^{-1}(\sigma) \in \mathfrak{B}(X)$ for all $\sigma \in \mathfrak{B}(Y)$) is *Borel injective* if the related inverse image mapping $\mathfrak{B}(Y) \ni \sigma \longmapsto f^{-1}(\sigma) \in \mathfrak{B}(X)$ is surjective. Below, we show that the notions of injectivity and Borel injectivity coincide for continuous mappings f defined on σ-compact topological Hausdorff spaces.

PROPOSITION 16. *Let $f: X \to Y$ be a mapping between topological Hausdorff spaces X and Y. Then the following assertions hold:*

(i) *if f is Borel injective, then f is injective;*

(ii) *if f is continuous and X is σ-compact, then f is Borel injective if and only if it is injective.*

PROOF. (i) Suppose that, contrary to our claim, $f(x_1) = f(x_2)$ for some distinct points x_1 and x_2 of X. Then the singleton $\{x_1\}$ is closed and so there exists $\sigma' \in \mathfrak{B}(Y)$ such that $\{x_1\} = f^{-1}(\sigma')$. However, $x_2 \in f^{-1}(\sigma')$, which is a contradiction.

(ii) Suppose f is continuous and X is σ-compact. Clearly, f is a Borel mapping. In view of (i), it suffices to prove the "if" part. Assume that f is injective. First, we show that

$$\text{if } F \text{ is a closed subset of } X\text{, then } f(F) \in \mathfrak{B}(Y). \tag{23}$$

Indeed, by σ-compactness of X, there exists a sequence $\{K_n\}_{n=1}^{\infty}$ of compact subsets of X such that $X = \bigcup_{n=1}^{\infty} K_n$. Since each $K_n \cap F$ is compact and consequently, by the continuity of f, each $f(K_n \cap F)$ is compact, we see that

$$f(F) = f\left(\bigcup_{n=1}^{\infty} K_n \cap F\right) = \bigcup_{n=1}^{\infty} f(K_n \cap F) \in \mathfrak{B}(Y),$$

which completes the proof of (23). Set

$$\mathscr{A}_f = \{\sigma \in \mathfrak{B}(X) \colon f(\sigma) \in \mathfrak{B}(Y)\}.$$

It follows from (23) that $X \in \mathscr{A}_f$. Since f is injective and $f(X) \in \mathfrak{B}(Y)$, we deduce that $X \setminus \sigma \in \mathscr{A}_f$ whenever $\sigma \in \mathscr{A}_f$. Clearly, $\bigcup_{n=1}^{\infty} \sigma_n \in \mathscr{A}_f$ whenever $\{\sigma_n\}_{n=1}^{\infty} \subset \mathscr{A}_f$. This means that $\mathscr{A}_f$ is a σ-subalgebra of $\mathfrak{B}(X)$, which, by (23), contains all the open subsets of X. Hence, $\mathscr{A}_f = \mathfrak{B}(X)$, that is $f(\sigma) \in \mathfrak{B}(Y)$ for every $\sigma \in \mathfrak{B}(X)$. To prove the Borel injectivity of f, take $\sigma \in \mathfrak{B}(X)$. Since $\mathscr{A}_f = \mathfrak{B}(X)$, we see that $\sigma' \stackrel{\text{def}}{=} f(\sigma) \in \mathfrak{B}(Y)$. By the injectivity of f, we deduce that $\sigma = f^{-1}(\sigma')$, which completes the proof of Borel injectivity of f.

REMARK 17. Note that in general injective (or even bijective) continuous mappings between topological Hausdorff spaces may not be Borel injective. Indeed, the mapping $f \colon X \to Y$, where X is the real line equipped with the discrete topology and Y is the real line equipped with the Euclidean topology, defined by $f(x) = x$ for $x \in X$, is bijective and continuous, but not Borel injective because $\mathfrak{B}(Y) \subsetneq 2^X$ (see [19, Remarks 2.21]).

In this section we will focus on restrictions of the continuous mapping

$$\psi \colon \mathbb{C}^* \to \mathbb{T}, \quad \psi(z) = \frac{z}{\bar{z}} = \left(\frac{z}{|z|}\right)^2, \quad z \in \mathbb{C}^*. \tag{24}$$

For a given Borel measure μ on $\mathbb{C}$ and a nonempty Borel subset Z of $\mathbb{C}^*$, we denote by $\mu \circ (\psi|_Z)^{-1}$ the transport of the Borel measure $\mu|_{\mathfrak{B}(Z)}$ via $\psi|_Z \colon Z \to \mathbb{T}$ given by

$$(\mu \circ (\psi|_Z)^{-1})(\sigma) \stackrel{\text{def}}{=} \mu((\psi|_Z)^{-1}(\sigma)), \quad \sigma \in \mathfrak{B}(\mathbb{T}), \tag{25}$$

In the context of restrictions of ψ, Proposition 16 can be specified as follows.

COROLLARY 18. *If Z is a σ-compact subset of $\mathbb{C}$ such that $0 \notin Z$, then the mapping $\psi|_Z$ is Borel injective if and only if it is injective.*

Proposition 16 enables us to formulate a geometric criterion for Borel injectivity of restrictions of ψ.

PROPOSITION 19. *Let Z be a nonempty subset of $\mathbb{C}^*$. Then the mapping $\psi|_Z$ is injective if and only if the intersection of Z and any straight line passing through the origin contains at most one point.*

PROOF. Suppose the intersection of Z and any straight line passing through the origin contains at most one point. Take $z_1, z_2 \in Z$ such that $\psi(z_1) = \psi(z_2)$. Note that there exist $t_1, t_2 \in \mathbb{R}$, such that $|t_1 - t_2| \leqslant \pi$, $z_1 = |z_1|\mathrm{e}^{\mathrm{i}t_1}$ and $z_2 = |z_2|\mathrm{e}^{\mathrm{i}t_2}$. Since $Z \subset \mathbb{C}^*$, we see that $\mathrm{e}^{2\mathrm{i}t_1} = \mathrm{e}^{2\mathrm{i}t_2}$, which gives $t_1 = t_2$ or $|t_1 - t_2| = \pi$. In both cases z_1 and z_2 are points of a straight line passing through the origin, so, by our assumption, $z_1 = z_2$. This proves the injectivity of $\psi|_Z$. The converse implication follows easily from the fact that ψ is constant on any straight line $\{r\mathrm{e}^{\mathrm{i}t} : r \in \mathbb{R}\}$ intersected with $\mathbb{C}^*$, where $t \in [0, \pi)$. This completes the proof.

COROLLARY 20. *Let $Z \subset \mathbb{C}^*$ be a nonempty set. Suppose Δ is a subset of $\mathbb{R}$ such that $|t_1 - t_2| < \pi$ for all $t_1, t_2 \in \Delta$, and $r\colon \Delta \to (0, \infty)$ is a function for which the mapping $\phi\colon \Delta \ni t \longmapsto r(t)\mathrm{e}^{\mathrm{i}t} \in Z$ is surjective. Then ϕ and $\psi|_Z$ are injective.*

Before formulating the main result of this section, we state a crucial lemma which is in the spirit of quasi-determinacy (cf. Theorem 3).

LEMMA 21. *If $\boldsymbol{\gamma}\colon \Re \to \mathbb{C}$ is a complex moment sequence and (μ_1, ν_1) and (μ_2, ν_2) are representing pairs for some $\boldsymbol{\Gamma} \in \mathsf{PDE}(\boldsymbol{\gamma})$, then*

$$\mu_1 \circ \psi^{-1} + \nu_1 \circ \varphi^{-1} = \mu_2 \circ \psi^{-1} + \nu_2 \circ \varphi^{-1}, \tag{26}$$

where φ and ψ are given by (3) *and* (24)*, respectively, whereas $\nu_j \circ \varphi^{-1}$ and $\mu_j \circ \psi^{-1}$ are Borel measures on $\mathbb{T}$ given by* (4) *and* (25)*, respectively.*

PROOF. It follows from the measure transport theorem that

$$\begin{aligned}\int_{\mathbb{C}^*} z^m \bar{z}^{-m}\,\mathrm{d}\mu_j(z) &\overset{(24)}{=} \int_{\mathbb{C}^*} \psi(z)^m\,\mathrm{d}\mu_j(z)\\ &= \int_{\mathbb{T}} z^m\,\mathrm{d}(\mu_j \circ \psi^{-1})(z), \quad m = 0, \pm 1, \pm 2, \ldots,\ j = 1, 2,\end{aligned}$$

and similarly

$$\int_{\mathbb{T}} z^m \bar{z}^{-m} \,\mathrm{d}\nu_j(z) = \int_{\mathbb{T}} z^m \,\mathrm{d}(\nu_j \circ \varphi^{-1})(z), \quad m = 0, \pm 1, \pm 2, \ldots, \; j = 1, 2,$$

which implies that

$$\Gamma_{m,-m} \overset{(2)}{=} \int_{\mathbb{T}} z^m \,\mathrm{d}(\mu_j \circ \psi^{-1} + \nu_j \circ \varphi^{-1})(z), \quad m = 0, \pm 1, \pm 2, \ldots, \; j = 1, 2.$$

Hence, by the determinacy of the Herglotz moment problem (see Section 3), the condition (26) holds. This completes the proof.

In what follows:

- $\psi_p \overset{\text{def}}{=} \psi|_{\mathscr{Z}_p}$ whenever $p \in \mathbb{C}[z, \bar{z}]$ is such that $\mathscr{Z}_p \neq \varnothing$ and $0 \notin \mathscr{Z}_p$;
- $\mathscr{M}(\boldsymbol{\gamma})$ stands for the set of all representing measures for a complex moment sequence $\boldsymbol{\gamma}$ on $\mathfrak{R}$.

We are now in a position to prove the main result of this section.

THEOREM 22. *Let $p \in \mathbb{C}[z, \bar{z}]$ be a polynomial such that $\mathscr{Z}_p \neq \varnothing$ and $0 \notin \mathscr{Z}_p$. Assume that $\boldsymbol{\gamma}\colon \mathfrak{R} \to \mathbb{C}$ is a complex moment sequence which has a representing measure supported in $\mathscr{Z}_p$. Then the following assertions hold:*

(i) *if $\mu \in \mathscr{M}(\boldsymbol{\gamma})$, then* $\operatorname{supp}\mu \subset \mathscr{Z}_p$, $\boldsymbol{\Gamma}(\mu) = \{\Gamma_{m,n}(\mu)\}_{(m,n)\in\mathfrak{R}_+} \in \mathsf{PDE}(\boldsymbol{\gamma})$ *and $(\mu, 0)$ is a representing pair for $\boldsymbol{\Gamma}(\mu)$, where*

$$\Gamma_{m,n}(\mu) = \int_{\mathbb{C}^*} z^m \bar{z}^n \,\mathrm{d}\mu(z), \quad (m, n) \in \mathfrak{R}_+;$$

(ii) *if (μ, ν) is a representing pair for some $\boldsymbol{\Gamma} \in \mathsf{PDE}(\boldsymbol{\gamma})$, then $\mu \in \mathscr{M}(\boldsymbol{\gamma})$ and $\nu = 0$;*

(iii) *if $(\mu_1, 0)$ and $(\mu_2, 0)$ are representing pairs for some $\boldsymbol{\Gamma} \in \mathsf{PDE}(\boldsymbol{\gamma})$, then*

$$\mu_1 \circ \psi_p^{-1} = \mu_2 \circ \psi_p^{-1}; \tag{27}$$

(iv) *if Z is a nonempty closed subset of $\mathscr{Z}_p$ such that $\psi|_Z$ is injective, then the mapping $\mathscr{M}_Z(\boldsymbol{\gamma}) \ni \mu \longmapsto \boldsymbol{\Gamma}(\mu) \in \mathsf{PDE}(\boldsymbol{\gamma})$ is injective, where $\mathscr{M}_Z(\boldsymbol{\gamma}) = \{\mu \in \mathscr{M}(\boldsymbol{\gamma})\colon \operatorname{supp}\mu \subset Z\}$;*

(v) *if ψ_p is injective, then*
   - *the mapping $\mathscr{M}(\boldsymbol{\gamma}) \ni \mu \longmapsto \boldsymbol{\Gamma}(\mu) \in \mathsf{PDE}(\boldsymbol{\gamma})$ is bijective,*
   - *every $\boldsymbol{\Gamma} \in \mathsf{PDE}(\boldsymbol{\gamma})$ is determinate,*
   - $\mathsf{PDE}(\boldsymbol{\gamma})$ *is of cardinality continuum whenever $\boldsymbol{\gamma}$ is indeterminate.*

PROOF. (i) Suppose $\mu \in \mathscr{M}(\boldsymbol{\gamma})$. By Proposition 7, μ is supported in $\mathscr{Z}_p$. Since, by our assumption $0 \notin \mathscr{Z}_p$, we deduce that $\boldsymbol{\Gamma}(\mu) \in \mathsf{PDE}(\boldsymbol{\gamma})$ and $(\mu, 0)$ is a representing pair for $\boldsymbol{\Gamma}(\mu)$ (see Lemma 2).

(ii) Assume that (μ, ν) is a representing pair for some $\boldsymbol{\Gamma} \in \mathsf{PDE}(\boldsymbol{\gamma})$. It follows from Lemma 2 that $\mu + \nu(\mathbb{T})\delta_0 \in \mathscr{M}(\boldsymbol{\gamma})$, and consequently by (i) and the assumption that $0 \notin \mathscr{Z}_p$, we have

$$\nu(\mathbb{T}) = (\mu + \nu(\mathbb{T})\delta_0)(\{0\}) = 0.$$

This implies that $\nu = 0$ and thus, by (2), $\mu \in \mathscr{M}(\boldsymbol{\gamma})$.

(iii) Assume that $(\mu_1, 0)$ and $(\mu_2, 0)$ are representing pairs for some $\boldsymbol{\Gamma} \in \mathsf{PDE}(\boldsymbol{\gamma})$. Then, by Lemma 21, $\mu_1 \circ \psi^{-1} = \mu_2 \circ \psi^{-1}$. By (i) and (ii), the measures μ_1 and μ_2 are supported in $\mathscr{Z}_p$. Therefore, $\mu_j \circ \psi^{-1} = \mu_j \circ \psi_p^{-1}$ for $j = 1, 2$, which yields (27).

(iv) Assume that Z is a nonempty closed subset of $\mathscr{Z}_p$ such that $\psi|_Z$ is injective. Since Z is σ-compact and $0 \notin Z$, we infer from Corollary 18 that the mapping $\psi|_Z$ is Borel injective. First note that by (i), $\boldsymbol{\Gamma}(\mu) \in \mathsf{PDE}(\boldsymbol{\gamma})$ for every $\mu \in \mathscr{M}(\boldsymbol{\gamma})$. If $\mu_1, \mu_2 \in \mathscr{M}_Z(\boldsymbol{\gamma})$ are such that $\boldsymbol{\Gamma}(\mu_1) = \boldsymbol{\Gamma}(\mu_2)$, then by (i) and (iii), $\mu_1 \circ \psi_p^{-1} = \mu_2 \circ \psi_p^{-1}$. Since the measures μ_1 and μ_2 are supported in Z, we deduce that $\mu_j \circ \psi_p^{-1} = \mu_j \circ (\psi|_Z)^{-1}$ for $j = 1, 2$. Hence, $\mu_1 \circ (\psi|_Z)^{-1} = \mu_2 \circ (\psi|_Z)^{-1}$, which by the Borel injectivity of $\psi|_Z$ leads to $\mu_1 = \mu_2$. As a consequence, the mapping $\mathscr{M}_Z(\boldsymbol{\gamma}) \ni \mu \longmapsto \boldsymbol{\Gamma}(\mu) \in \mathsf{PDE}(\boldsymbol{\gamma})$ is injective.

(v) Assume that ψ_p is injective. It follows from (i) and (iv) that the mapping $\mathscr{M}(\boldsymbol{\gamma}) \ni \mu \longmapsto \boldsymbol{\Gamma}(\mu) \in \mathsf{PDE}(\boldsymbol{\gamma})$ is injective. On the other hand, by (ii), the second term in (2) must be zero whenever $\boldsymbol{\Gamma} \in \mathsf{PDE}(\boldsymbol{\gamma})$ and (μ, ν) is a representing pair for $\boldsymbol{\Gamma}$. This yields the surjectivity of the mapping $\mathscr{M}(\boldsymbol{\gamma}) \ni \mu \longmapsto \boldsymbol{\Gamma}(\mu) \in \mathsf{PDE}(\boldsymbol{\gamma})$. Therefore, it is a bijection. This together with (ii) implies that every $\boldsymbol{\Gamma} \in \mathsf{PDE}(\boldsymbol{\gamma})$ is determinate. To prove the last assertion in (v) assume that $\boldsymbol{\gamma}$ is indeterminate. Since the set $\mathscr{M}(\boldsymbol{\gamma})$ is a convex subset of the set of all Borel measures on $\mathbb{C}$ and $\mathscr{M}(\boldsymbol{\gamma})$ is not a one point set, we see that the cardinality of $\mathscr{M}(\boldsymbol{\gamma})$ is at least continuum. Combined with the injectivity of $\mathscr{M}(\boldsymbol{\gamma}) \ni \mu \longmapsto \boldsymbol{\Gamma}(\mu) \in \mathsf{PDE}(\boldsymbol{\gamma})$, this implies that the set $\mathsf{PDE}(\boldsymbol{\gamma})$ is of cardinality at least $\mathfrak{c}$. Since $\mathsf{PDE}(\boldsymbol{\gamma}) \subset \mathbb{C}^{\mathfrak{N}_+}$ and the cardinality of $\mathbb{C}^{\mathfrak{N}_+}$ is $\mathfrak{c}$, the proof is complete.

COROLLARY 23. *Let $p \in \mathbb{C}[z, \bar{z}]$ be a polynomial such that $\mathscr{Z}_p \neq \varnothing$, $0 \notin \mathscr{Z}_p$ and ψ_p is injective. Suppose $\boldsymbol{\gamma}\colon \mathfrak{N} \to \mathbb{C}$ is a complex moment sequence having a representing measure supported in $\mathscr{Z}_p$. If $\mathsf{PDE}(\boldsymbol{\gamma}) = \{\boldsymbol{\Gamma}\}$, then $\boldsymbol{\gamma}$ is determinate.*

Regarding assertion (v) of Theorem 22, it is advisable to know for which polynomials p, the mapping ψ_p is injective. It is easily seen that the injectivity property of ψ_p fails to hold for most plane algebraic curves, including circles, ellipses, hyperbolas, parabolas, lemniscates, etc. However, we may indicate several polynomials p for which ψ_p is injective. For convenience, in Example 24 below we use the two real variable description of real algebraic sets.

EXAMPLE 24. The mapping ψ_p is injective in any of the following cases:

(1) $p(x, y) = ax + by - c$, where $a, b, c \in \mathbb{R}$, $a^2 + b^2 > 0$ and $c \neq 0$ (a straight line which does not contain the origin);

(2) $p(x, y) = (y - y_0)^\ell - ax^{2k}$, where k is a nonnegative integer, $\ell > 2k$ is an odd integer, $a > 0$ and $y_0 > 0$ (for $k = 1$ and $\ell = 3$ this is a shifted *Neil's semicubical parabola*, cf. [6, p. 93], [16, p. 5]);

(3) $p(x, y) = y(x^2 + a) - b$, where $a, b > 0$ (a generalized *Witch of Agnesi*, cf. [6, p. 94]);

(4) $p(x, y) = ((x - x_0)^2 + y^2)(x - x_0) - 2ay^2$, where $a > 0$ and $x_0 > 0$ (a shifted *cissoid of Diocles*, cf. [6, p. 95], [16, p. 5]);

(5) $p(x, y) = y^\ell x^{2k} - a$, where k is a nonnegative integer, ℓ is an odd positive integer and $a \in \mathbb{R} \setminus \{0\}$.

The injectivity of ψ_p will be deduced from Proposition 19 by verifying that the intersection of $\mathscr{Z}_p$ and any straight line passing through the origin contains at most one point.

The case (1) is obvious. Let p be as in (2). Then $\mathscr{Z}_p$ is located in the upper half-plane. The case of the line $x = 0$ is plain. Since the set $\mathscr{Z}_p \cap \{z \in \mathbb{C} \colon \mathrm{Re}(z) \geqslant 0\}$ is the graph of a strictly increasing concave function on the interval $[0, \infty)$ whose value at 0 is positive, it intersects the line $y = cx$ in exactly one point whenever $c > 0$. The case $c < 0$ follows by the symmetry of $\mathscr{Z}_p$ with respect to the reflection across the line $x = 0$.

Suppose p is as in (3). Then $\mathscr{Z}_p$ is contained in the upper half-plane. The case of the line $x = 0$ is trivial. Since the set $\mathscr{Z}_p \cap \{z \in \mathbb{C} \colon \mathrm{Re}(z) \geqslant 0\}$ is the graph of a strictly decreasing positive function on the interval $[0, \infty)$, it intersects the line $y = cx$ in exactly one point whenever $c > 0$. As above, the case $c < 0$ follows by the symmetry of $\mathscr{Z}_p$ with respect to the reflection across the line $x = 0$.

Let p be as in (4). This time $\mathscr{Z}_p$ is a subset of the right half-plane. Again, the case of the line $y = 0$ is obvious. Since the set $\mathscr{Z}_p \cap \{z \in \mathbb{C} \colon \mathrm{Im}(z) \geqslant 0\}$ is the graph of a strictly increasing convex function on the interval $[x_0, x_0 + 2a)$ that vanishes at x_0, it intersects the line $y = cx$ in exactly one point whenever $c > 0$. The case $c < 0$ follows by the symmetry of $\mathscr{Z}_p$ with respect to the reflection across the line $y = 0$.

Finally, the case when p is as in (5) is straightforward.

Regarding Theorem 22(iv), we note that though in general the mapping ψ given by (24) is not injective on plane algebraic curves, it becomes such on appropriately chosen parts of them, e.g., one branch of a hyperbola, an arc of a parabola, etc. In turn, Proposition 7 which helps to localize the supports of representing measures of a complex moment sequence on a real algebraic set can be enforced with the help of Proposition 14 as follows (cf. Remark 28).

PROPOSITION 25. *If $\boldsymbol{\gamma} = \{\gamma_{m,n}\}_{(m,n)\in\mathfrak{N}}$ is a complex moment sequence which has a representing measure μ supported in a real algebraic set Z such that the set $\pi_1(\operatorname{supp}\mu)$ (resp., $\pi_2(\operatorname{supp}\mu)$) is bounded, then*

$$\operatorname{supp}\tilde{\mu} \subset Z \cap \big(\overline{\pi_1(\operatorname{supp}\mu)} \times \mathbb{R}\big) \quad \big(\textit{resp.,}\ \operatorname{supp}\tilde{\mu} \subset Z \cap \big(\mathbb{R} \times \overline{\pi_2(\operatorname{supp}\mu)}\,\big)\big)$$

for any representing measure $\tilde{\mu}$ for $\boldsymbol{\gamma}$, where π_1 and π_2 are as in (17).

Proposition 25 can be applied e.g. to the Witch of Agnesi (see Example 24(3)). We will show that for such a plane algebraic curve there is no analogue of an N-extremal measure in the following sense. Recall that a representing measure μ of an indeterminate Hamburger moment sequence is said to be *N-extremal* if complex polynomials are dense in $L^2(\mu)$. The supports of N-extremal measures of an indeterminate Hamburger moment sequence have remarkable properties, namely they are infinite, have no accumulation points in $\mathbb{R}$ and form a partition of $\mathbb{R}$ (see [22, Theorem 2.13]; see also [23]). As shown in Proposition 26 below, this is no longer true for supports of representing measures of a complex moment sequence provided at least one of them is contained in the Witch of Agnesi and has no accumulation point therein. An analogue of Proposition 26 can also be formulated and proved for a shifted cissoid of Diocles (see Example 24(4)). We leave the details to the reader.

Below, for brevity, we write $x_n \nearrow \infty$ (resp., $x_n \searrow 0$) if $\{x_n\}_{n=1}^{\infty}$ is a strictly increasing (resp., strictly decreasing) sequence in $\mathbb{R}$ which converges to ∞ (resp., 0).

PROPOSITION 26. *Let $\boldsymbol{\gamma} = \{\gamma_{m,n}\}_{(m,n)\in\mathfrak{N}}$ be a complex moment sequence with a representing measure μ supported in $\mathscr{Z}_p$, where p is as in* Example 24(3). *Assume that* $\operatorname{supp}\mu$ *is infinite and has no accumulation points in $\mathscr{Z}_p$. Then there exists a sequence $\{z_n\}_{n=1}^{\infty} \subset \mathscr{Z}_p$ such that*

(i) $0 \leqslant \operatorname{Re}(z_n) \nearrow \infty$,

(ii) *for any representing measure $\tilde{\mu}$ of $\boldsymbol{\gamma}$,*

$$\tilde{\mu}(\{z_n, -\bar{z}_n\}) = \mu(\{z_n, -\bar{z}_n\}) > 0, \quad n \geqslant 1, \tag{28}$$

$$\operatorname{supp}\tilde{\mu} \subset \{z_n\colon n \geqslant 1\} \cup \{-\bar{z}_n\colon n \geqslant 1\}. \tag{29}$$

PROOF. Since $\operatorname{supp}\mu$ is infinite and has no accumulation points in $\mathscr{Z}_p$, one can show that there exists a sequence $\{z_n\}_{n=1}^{\infty} \subset \mathscr{Z}_p$ such that

$$0 \leqslant \operatorname{Re}(z_n) \nearrow \infty, \tag{30}$$

$$\mu(\{z_n, -\bar{z}_n\}) > 0, \quad n \geqslant 1, \tag{31}$$

$$\operatorname{supp}\mu \subset \{z_n : n \geqslant 1\} \cup \{-\bar{z}_n : n \geqslant 1\}. \tag{32}$$

In fact, the set $\{z_n : n \geqslant 1\} \cup \{-\bar{z}_n : n \geqslant 1\}$ is the smallest subset of $\mathscr{Z}_p$ that contains $\operatorname{supp}\mu$ and is symmetric with respect to the reflection across the line $x = 0$. It follows from (31) and (32) that

$$\pi_2(\operatorname{supp}\mu) = \{\operatorname{Im}(z_n) : n \geqslant 1\} \subset \left[0, \frac{b}{a}\right].$$

Since, by (30), $\operatorname{Im}(z_n) \searrow 0$, we see that $\overline{\pi_2(\operatorname{supp}\mu)} = \{0\} \cup \{\operatorname{Im}(z_n) : n \geqslant 1\}$. This implies that

$$\mathscr{Z}_p \cap \left(\mathbb{R} \times \overline{\pi_2(\operatorname{supp}\mu)}\right) = \{z_n : n \geqslant 1\} \cup \{-\bar{z}_n : n \geqslant 1\}. \tag{33}$$

Let $\tilde{\mu}$ be any representing measure for $\boldsymbol{\gamma}$. In view of (33) and Proposition 25, the measure $\tilde{\mu}$ satisfies (29). It follows from Lemma 13 and Proposition 14 that $\tilde{\mu} \circ \pi_2^{-1} = \mu \circ \pi_2^{-1}$. Since μ and $\tilde{\mu}$ are supported in $\mathscr{Z}_p$ and

$$\mathscr{Z}_p \cap \pi_2^{-1}(\{\operatorname{Im}(z_n)\}) = \{z_n, -\bar{z}_n\}, \quad n \geqslant 1,$$

we conclude that (28) holds. This completes the proof.

REMARK 27. We conclude this section by examining Borel injectivity of ψ_p after transformation by polynomial automorphism. For simplicity of presentation we treat $\mathbb{C}$ as $\mathbb{R}^2$. Let us consider a polynomial automorphism $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$\Phi(x, y) = (x, y + f(x)), \quad x, y \in \mathbb{R},$$

where $f \in \mathbb{R}[x]$ (see [11] for fundamentals of the theory of polynomial automorphisms). Clearly, the inverse of Φ is given by

$$\Phi^{-1}(x, y) = (x, y - f(x)), \quad x, y \in \mathbb{R}.$$

Though polynomial automorphisms preserve many properties of moment sequences (see e.g., [28, Sec. 21] or [7, Proposition 46]), they fail to preserve injectivity of ψ_p. Indeed, if $p(x, y) = y - 1$ for $x, y \in \mathbb{R}$, then ψ_p is injective (see Example 24(1)). Note that $\Phi(\mathscr{Z}_p) = \mathscr{Z}_{p \circ \Phi^{-1}}$ and $p \circ \Phi^{-1}(x, y) = y - f(x) - 1$ for $x, y \in \mathbb{R}$. Let $f(x) = x^2$ for $x \in \mathbb{R}$. Then $\Phi(\mathscr{Z}_p)$ is the parabola $y = x^2 + 1$, which means that $\psi_{p \circ \Phi^{-1}}$ is not injective.

6. An open problem

The following question, partially answered in Theorem 6 and Corollary 23, needs to be solved in full generality.

QUESTION. Assume that $\boldsymbol{\gamma}\colon \mathfrak{N} \to \mathbb{C}$ is a complex moment sequence such that $\mathsf{PDE}(\boldsymbol{\gamma}) = \{\boldsymbol{\Gamma}\}$. Does it follow that $\boldsymbol{\gamma}$ is determinate?

In view of this question it is legitimate to make sure that none of the examples given in this paper solves it in the negative.

REMARK 28. The sequence $\boldsymbol{\gamma}$ from Example 12 is indeterminate and, by Theorem 22(v) and Example 24(1), the set $\mathsf{PDE}(\boldsymbol{\gamma})$ is infinite.

We will show that the same conclusion holds for the indeterminate complex moment sequence $\boldsymbol{\gamma} = \{\gamma_{m,n}\}_{(m,n)\in\mathfrak{N}}$ coming from the two-dimensional Hamburger moment sequence $\boldsymbol{s} \otimes \boldsymbol{t}$ appearing in Theorem 15 if

$$d \stackrel{\text{def}}{=} \sup \operatorname{supp} \mu \in (0, \infty) \quad \text{and} \quad \operatorname{supp} \mu \subset [0, d], \tag{34}$$

where μ is as in Theorem 15. The following argument can be applied to the case when $\operatorname{supp} \mu$ is any compact set, but the assumption (34) allows us to avoid some technical details. In our settings, the complex moment sequence $\boldsymbol{\gamma}$ is defined by (cf. (22))

$$\gamma_{m,n} = \int_{\mathbb{R}^2} (x + \mathrm{i}y)^m (x - \mathrm{i}y)^n \, \mathrm{d}\varrho(x, y), \quad m, n \geqslant 0,$$

where ϱ is any representing measure for $\boldsymbol{s} \otimes \boldsymbol{t}$. The definition of $\boldsymbol{\gamma}$ is independent of the choice of ϱ, and, after identifying $\mathbb{C}$ with $\mathbb{R}^2$, representing measures of $\boldsymbol{s} \otimes \boldsymbol{t}$ and $\boldsymbol{\gamma}$ coincide (see [10, Appendix A]). We will indicate two representing measures ϱ_1 and ϱ_2 for $\boldsymbol{\gamma}$ (equivalently for $\boldsymbol{s} \otimes \boldsymbol{t}$) such that

$$\varrho_1 \circ \psi^{-1} \neq \varrho_2 \circ \psi^{-1}. \tag{35}$$

For this, we first observe that if $\Delta \subset (0, \pi)$ is an open interval and

$$E_\Delta \stackrel{\text{def}}{=} \{\pm r\mathrm{e}^{\mathrm{i}t} \colon r > 0, t \in \Delta\},$$

then

$$E_\Delta = \psi^{-1}(\{\mathrm{e}^{2\mathrm{i}t} \colon t \in \Delta\}). \tag{36}$$

Next, we notice that it is possible to find two representing measures ν_1 and ν_2 for $\boldsymbol{t}$ for which there exist $b_1, b_2 \in \mathbb{R}$ such that

$$b_2 > b_1 > 0, \quad b_1 \in \operatorname{supp} \nu_1 \quad \text{and} \quad \nu_2((0, b_2)) = 0. \tag{37}$$

Indeed, this is always true for any two distinct N-extremal measures of $\boldsymbol{t}$ having atoms in $(0, \infty)$ (up to rearrangement), because supports of N-extremal measures of $\boldsymbol{t}$ form the partition of $\mathbb{R}$ and each of them has no accumulation points in $\mathbb{R}$ (see [22, Theorem 2.13]). Set $\varrho_1 = \mu \otimes \nu_1$ and $\varrho_2 = \mu \otimes \nu_2$. Then $\varrho_1$and ϱ_2 are representing measures for $\boldsymbol{\gamma}$ (see the proof of Proposition 15). Moreover, by (34), (37) and $\mu(\{0\}) = 0$, we have

$$\varrho_2(\Omega) = 0, \tag{38}$$

where

$$\Omega = \big((-\infty, 0] \times \mathbb{R}\big) \cup \big([0, d] \times (0, b_2)\big) \cup \big((d, \infty) \times \mathbb{R}\big),$$

and

$$(d, b_1) \in \operatorname{supp} \mu \times \operatorname{supp} \nu_1 = \operatorname{supp} \varrho_1. \tag{39}$$

Plainly, we can choose a set E_Δ, where $\Delta \subset (0, \pi)$ is an open interval, so that $E_\Delta \subset \Omega$ and $(d, b_1) \in E_\Delta$. This combined with (38), (39) and the fact that E_Δ is an open neighbourhood of (d, b_1) implies that $\varrho_2(E_\Delta) = 0$ and $\varrho_1(E_\Delta) > 0$. Hence, by (36), we get (35). Since, by Theorem 15, none of representing measures of $\boldsymbol{\gamma}$ has an atom at 0, we infer from Lemma 2(ii) that $(\varrho_1, 0)$ and $(\varrho_2, 0)$ are representing pairs for some extensions in $\mathsf{PDE}(\boldsymbol{\gamma})$. It follows from Lemma 21 and (35) that they cannot be representing pairs for the same $\boldsymbol{\Gamma} \in \mathsf{PDE}(\boldsymbol{\gamma})$. This implies that the set $\mathsf{PDE}(\boldsymbol{\gamma})$ is infinite (see the proof of Proposition 5).

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