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A BICATEGORICAL INTERPRETATION FOR RELATIVE CUNTZ-PIMSNER ALGEBRAS

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Abstract

We interpret the construction of relative Cuntz-Pimsner algebras of correspondences in terms of the correspondence bicategory, as a reflector into a certain sub-bicategory. This generalises a previous characterisation of absolute Cuntz-Pimsner algebras of proper correspondences as colimits in the correspondence bicategory.

1. Introduction

A relative Cuntz-Pimsner algebra is defined by a triple $(A, \mathscr{E}, J)$, where A is a C*-algebra, $\mathscr{E}$ is a C*-correspondence from A to itself, that is, a Hilbert A-module with a nondegenerate left action of A by adjointable operators, $\varphi: A \to \mathbb{B}(\mathscr{E})$, and $J \lhd A$ is an ideal that acts on $\mathscr{E}$ by compact operators, that is, $\varphi(J) \subseteq \mathbb{K}(\mathscr{E})$. The Cuntz-Pimsner covariance condition is only required on J. This variation on a definition by Pimsner [19] was introduced in [17]. Many important C*-algebras may be described as relative Cuntz-Pimsner algebras (see, for instance, [13]).

We view the correspondence $\mathscr{E}$ as a generalised endomorphism of A. If $\mathscr{E}$ comes from an automorphism α of A, then the relative Cuntz-Pimsner algebra for $J = A$ is naturally isomorphic to the crossed product $A \rtimes_\alpha \mathbb{Z}$. So we may view Cuntz-Pimsner algebras as analogues of crossed products for automorphisms. This is made precise in [2] by viewing both crossed products and Cuntz-Pimsner algebras as colimits of diagrams in the bicategory of C*-correspondences. The interpretation of Cuntz-Pimsner algebras in [2] is limited, however, to *proper* correspondences, that is, $\varphi(A) \subseteq \mathbb{K}(\mathscr{E})$, and the "absolute" case $J = A$. This article is concerned with another bicategorical interpretation of the Cuntz-Pimsner algebra construction, which needs no properness and extends to the relative case.

Our results use the equivalence between C*-algebras with a $\mathbb{T}$-action and Fell bundles over $\mathbb{Z}$ (see [1]). The spectral decomposition of a $\mathbb{T}$-action β on

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a C*-algebra B gives a Fell bundle $(B_n)_{n\in\mathbb{N}}$ over the group $\mathbb{Z}$ whose section C*-algebra $C^*((B_n)_{n\in\mathbb{N}})$ is canonically isomorphic to B; namely,

$$B_n := \{ b \in B : \beta_z(b) = z^n \cdot b \}$$

for $n \in \mathbb{Z}$ with the multiplication, involution and norm from B. Conversely, the section C*-algebra of any Fell bundle over $\mathbb{Z}$ carries a canonical gauge action of $\mathbb{T}$. The Fell bundle underlying a Cuntz-Pimsner algebra is *semi-saturated*, that is, $B_n \cdot B_m = B_{n+m}$ if $n, m \geq 0$ (or if $n, m \leq 0$). Here and below, $X \cdot Y$ means the closed linear span of $\{ x \cdot y : x \in X,\ y \in Y \}$. By the results of [1], a semi-saturated Fell bundle is determined by its fibres B_0 and B_1: B_0 is a C*-algebra, B_1 is a Hilbert B_0-bimodule, and the crossed product for the Hilbert B_0-bimodule B_1 is isomorphic to the section C*-algebra of the Fell bundle generated by B_0 and B_1.

Thus we split the construction of Cuntz-Pimsner algebras with their canonical $\mathbb{T}$-action into two steps. The first builds the Hilbert bimodule $\mathcal{O}^1_{J,\mathcal{E}}$ over $\mathcal{O}^0_{J,\mathcal{E}}$, the second takes the crossed product for this Hilbert bimodule. When we include the gauge action, then the second step is reversible using the spectral decomposition. This article interprets the first step in the construction as a reflector to a sub-bicategory. A Hilbert bimodule is a C*-correspondence with an additional left inner product, which is unique if it exists. Thus Hilbert bimodules form a full sub-bicategory in the correspondence bicategory. We describe a bicategory whose objects are the triples $(A, \mathcal{E}, J)$ needed to define a relative Cuntz-Pimsner algebra. Those triples where $\mathcal{E}$ is a Hilbert bimodule and J is Katsura's ideal for $\mathcal{E}$ form a full sub-bicategory. We show that the construction of $(\mathcal{O}^0_{J,\mathcal{E}}, \mathcal{O}^1_{J,\mathcal{E}})$ is a *reflector* onto this sub-bicategory. Roughly speaking, a reflector approximates a given object by an object in the sub-bicategory in the optimal way. More precisely, it is a left (bi)adjoint to the inclusion of the sub-bicategory.

We gradually work up to such bicategorical considerations. Section 2 deals with known properties of relative Cuntz-Pimsner algebras. We also discuss their Fell bundle structure coming from the gauge action, and we show that the Cuntz-Pimsner algebra $\mathcal{O}_{J,\mathcal{E}}$ is the crossed product of its gauge-fixed point algebra $\mathcal{O}^0_{J,\mathcal{E}}$ by the Hilbert $\mathcal{O}^0_{J,\mathcal{E}}$-bimodule $\mathcal{O}^1_{J,\mathcal{E}}$. Section 2 culminates in a result about the functoriality of relative Cuntz-Pimsner algebras, which goes back to an idea of Schweizer [20]. We correct his idea and extend it to the relative case by defining proper covariant correspondences between triples $(A, \mathcal{E}, J)$ so that they induce correspondences between the associated relative Cuntz-Pimsner algebras.

This construction is upgraded in Section 3 to a homomorphism of bicategories (or "functor") from a certain bicategory $\mathfrak{C}^{\mathbb{N}}_{\mathrm{pr}}$ to the $\mathbb{T}$-equivariant correspondence bicategory $\mathfrak{C}^{\mathbb{T}}$. The objects of $\mathfrak{C}^{\mathbb{N}}_{\mathrm{pr}}$ are the triples $(A, \mathscr{E}, J)$ needed to define a relative Cuntz-Pimsner algebra, the arrows are the proper covariant correspondences introduced in Section 2, and the 2-arrows are isomorphisms of covariant correspondences. Whereas Schweizer reduces to ordinary categories by identifying isomorphic correspondences, bicategories are crucial for our purposes, as in [2].

Then we define a sub-bicategory $\mathfrak{C}^{\mathbb{N}}_{\mathrm{pr},*}$ by restricting to Hilbert bimodules instead of correspondences. We prove a crucial statement about covariant correspondences, namely, that proper covariant correspondences $(A, \mathscr{E}, J) \to (B, \mathscr{G}, I_{\mathscr{G}})$ are "equivalent" to proper covariant correspondences $(\mathcal{O}^0_{J,\mathscr{E}}, \mathcal{O}^1_{J,\mathscr{E}}, I_{\mathscr{E}}) \to (B, \mathscr{G}, I_{\mathscr{G}})$ for all $(B, \mathscr{G}, I_{\mathscr{G}})$ in $\mathfrak{C}^{\mathbb{N}}_{\mathrm{pr},*}$, that is, for a Hilbert B-bimodule $\mathscr{G}$ and Katsura's ideal $I_{\mathscr{G}}$.

Section 4 introduces the bicategorical language to understand this fact: it says that a certain arrow

$$\upsilon_{(A,\mathscr{E},J)}\colon (A, \mathscr{E}, J) \longrightarrow (\mathcal{O}^0_{J,\mathscr{E}}, \mathcal{O}^1_{J,\mathscr{E}}, I_{\mathscr{E}})$$

is a universal arrow from $(A, \mathscr{E}, J)$ to $\mathfrak{C}^{\mathbb{N}}_{\mathrm{pr},*}$. The existence of universal arrows implies an adjunction (see [9]). So general bicategory theory upgrades the "equivalence" observed above to our main statement, namely, that the sub-bicategory $\mathfrak{C}^{\mathbb{N}}_{\mathrm{pr},*} \subseteq \mathfrak{C}^{\mathbb{N}}_{\mathrm{pr}}$ is reflective and that the reflector homomorphism $\mathfrak{C}^{\mathbb{N}}_{\mathrm{pr}} \to \mathfrak{C}^{\mathbb{N}}_{\mathrm{pr},*}$ acts on objects by mapping $(A, \mathscr{E}, J)$ to $(\mathcal{O}^0_{J,\mathscr{E}}, \mathcal{O}^1_{J,\mathscr{E}}, I_{\mathcal{O}^1_{J,\mathscr{E}}})$. We describe this reflector in detail and show that its composite with the crossed product homomorphism $\mathfrak{C}^{\mathbb{N}}_{\mathrm{pr},*} \to \mathfrak{C}^{\mathbb{T}}$ is the relative Cuntz-Pimsner algebra homomorphism $\mathfrak{C}^{\mathbb{N}}_{\mathrm{pr}} \to \mathfrak{C}^{\mathbb{T}}$ described in Section 3. The definitions of bicategories, homomorphisms, transformations, and modifications are recalled in the appendix, together with some examples related to the correspondence bicategory.

We thank the referee for several suggestions that helped to improve the presentation of our results. Several results in this article are extended to product systems over quasi-lattice ordered monoids in [21].

2. Preliminaries

In this section, we recall basic results on Cuntz-Pimsner algebras, their gauge action and Fell bundle structure. We correct and generalise an idea by Schweizer on the functoriality of Cuntz-Pimsner algebras for covariant correspondences.

2.1. Correspondences

Let $\mathscr{F}_1, \mathscr{F}_2$ be Hilbert B-modules. Let $\mathbb{B}(\mathscr{F}_1, \mathscr{F}_2)$ be the space of adjointable operators from $\mathscr{F}_1$ to $\mathscr{F}_2$. Let $|\xi\rangle\langle\eta| \in \mathbb{B}(\mathscr{F}_1, \mathscr{F}_2)$ for $\xi \in \mathscr{F}_2$ and $\eta \in \mathscr{F}_1$

be the *generalised rank*-1 *operator* defined by $|\xi\rangle\langle\eta|(\zeta) := \xi\langle\eta|\zeta\rangle_B$. Let $\mathbb{K}(\mathcal{F}_1, \mathcal{F}_2)$ be the closed linear span of $|\xi\rangle\langle\eta|$ for $\xi \in \mathcal{F}_1$ and $\eta \in \mathcal{F}_2$. Elements of $\mathbb{K}(\mathcal{F}_1, \mathcal{F}_2)$ are called *compact operators*. We abbreviate $\mathbb{B}(\mathcal{F}) := \mathbb{B}(\mathcal{F}, \mathcal{F})$ and $\mathbb{K}(\mathcal{F}) := \mathbb{K}(\mathcal{F}, \mathcal{F})$ if $\mathcal{F} = \mathcal{F}_1 = \mathcal{F}_2$.

LEMMA 2.1. *Let $\mathcal{E}_1 \subseteq \mathcal{F}_1$ and $\mathcal{E}_2 \subseteq \mathcal{F}_2$ be Hilbert B-submodules. There is a unique map $\mathbb{K}(\mathcal{E}_1, \mathcal{E}_2) \to \mathbb{K}(\mathcal{F}_1, \mathcal{F}_2)$ that maps $|\xi\rangle\langle\eta| \in \mathbb{K}(\mathcal{E}_1, \mathcal{E}_2)$ to $|\xi\rangle\langle\eta| \in \mathbb{K}(\mathcal{F}_1, \mathcal{F}_2)$ for all $\xi \in \mathcal{E}_2$, $\eta \in \mathcal{E}_1$. This map is injective.*

PROOF. Composing with the inclusion $\mathcal{E}_2 \subseteq \mathcal{F}_2$ maps compact operators $\mathcal{E}_1 \to \mathcal{E}_2$ to bounded linear maps $\mathcal{E}_1 \to \mathcal{F}_2$. Since a rank-1 operator $|\xi\rangle\langle\eta| \in \mathbb{K}(\mathcal{E}_1, \mathcal{E}_2)$ is mapped to a rank-1 operator, we get a well defined, injective map $\mathbb{K}(\mathcal{E}_1, \mathcal{E}_2) \to \mathbb{K}(\mathcal{E}_1, \mathcal{F}_2)$. Taking adjoints maps $\mathbb{K}(\mathcal{E}_1, \mathcal{F}_2)$ bijectively onto $\mathbb{K}(\mathcal{F}_2, \mathcal{E}_1)$. As above, this embeds into $\mathbb{K}(\mathcal{F}_2, \mathcal{F}_1)$. Taking adjoints again gives a map $\mathbb{K}(\mathcal{E}_1, \mathcal{F}_2) \to \mathbb{K}(\mathcal{F}_1, \mathcal{F}_2)$. The composite map $\mathbb{K}(\mathcal{E}_1, \mathcal{E}_2) \to \mathbb{K}(\mathcal{F}_1, \mathcal{F}_2)$ maps $|\xi\rangle\langle\eta| \in \mathbb{K}(\mathcal{E}_1, \mathcal{E}_2)$ to $|\xi\rangle\langle\eta| \in \mathbb{K}(\mathcal{F}_1, \mathcal{F}_2)$ as desired. This determines the map uniquely.

DEFINITION 2.2. Let A and B be C*-algebras. A *correspondence* from A to B is a Hilbert B-module $\mathcal{E}$ with a *nondegenerate* left action of A through a *-homomorphism $\varphi\colon A \to \mathbb{B}(\mathcal{E})$. A correspondence is *proper* if $\varphi(A) \subseteq \mathbb{K}(\mathcal{E})$. It is *faithful* if φ is injective. We write $\mathcal{E}\colon A \rightsquigarrow B$ to say that $\mathcal{E}$ is a correspondence from A to B.

In order for C*-correspondences to form a bicategory, we need to assume the left action to be nondegenerate. Otherwise, the isomorphism $A \otimes_A \mathcal{E} \cong \mathcal{E}$ fails and we no longer have unit arrows.

DEFINITION 2.3. Let A and B be C*-algebras. A *Hilbert A, B-bimodule* is a (right) Hilbert B-module $\mathcal{E}$ with a *left* Hilbert A-module structure $\langle \cdot \mid \cdot \rangle_A$ such that $\langle \xi \mid \eta\rangle_A \zeta = \xi\langle\eta|\zeta\rangle_B$ for all $\xi, \eta, \zeta \in \mathcal{E}$.

If $\mathcal{E}$ is a Hilbert A, B-bimodule, then A acts by adjointable operators on $\mathcal{E}$ and B acts by adjointable operators for the left Hilbert A-module structure, that is, $\langle \xi b \mid \eta\rangle_A = \langle \xi \mid \eta b^*\rangle_A$ for all $\xi, \eta \in \mathcal{E}$ and all $b \in B$. In particular, $\mathcal{E}$ is an A, B-bimodule. The next lemma characterises which correspondences may be enriched to Hilbert bimodules:

LEMMA 2.4 (see [8, Example 1.6]). *A correspondence $\mathcal{E}\colon A \rightsquigarrow B$ carries a Hilbert A, B-bimodule structure if and only if there is an ideal $I \triangleleft A$ such that the left action on $\mathcal{E}$ restricts to a *-isomorphism $I \cong \mathbb{K}(\mathcal{E})$. In this case, the ideal I and the left inner product are unique, and $I = \langle \mathcal{E} \mid \mathcal{E}\rangle_A$.*

DEFINITION 2.5. Let $\mathcal{E}_1, \mathcal{E}_2\colon A \rightsquigarrow B$ be C*-correspondences. A *correspondence isomorphism* $\mathcal{E}_1 \Rightarrow \mathcal{E}_2$ is a unitary A, B-bimodule isomorphism from $\mathcal{E}_1$

to $\mathscr{E}_2$. We write "$\Rightarrow$" because these isomorphisms are the 2-arrows in bicategories that we are going to construct.

Let D be another C*-algebra, $\mathscr{F}$ a Hilbert D-module, and $\varphi\colon B \to \mathbb{B}(\mathscr{F})$ a *-homomorphism. The tensor product $\mathscr{E} \otimes_\varphi \mathscr{F}$ is a Hilbert D-module described, for instance, in [16]. For $\xi \in \mathscr{E}$, we define an operator

$$T_\xi\colon \mathscr{F} \longrightarrow \mathscr{E} \otimes_\varphi \mathscr{F}, \qquad \eta \mapsto \xi \otimes \eta.$$

It is adjointable with $T_\xi^*(\zeta \otimes \eta) = \varphi(\langle \xi | \zeta \rangle)\eta$ on elementary tensors (see [19]). Hence

$$T_\xi T_\zeta^* = |\xi\rangle\langle\zeta| \otimes 1, \qquad T_\zeta^* T_\xi = \varphi(\langle \zeta | \xi \rangle),$$

where $|\xi\rangle\langle\zeta| \otimes 1$ is the image of $|\xi\rangle\langle\zeta|$ under the canonical map $\mathbb{B}(\mathscr{E}) \to \mathbb{B}(\mathscr{E} \otimes_\varphi \mathscr{F})$, $T \mapsto T \otimes 1$. Hence the operator T_ξ for $\xi \in \mathscr{E}$ is compact if and only if $\varphi(\langle \xi | \xi \rangle) = T_\xi^* T_\xi$ is compact.

LEMMA 2.6 ([19, Corollary 3.7]). *Let $J := \varphi^{-1}(\mathbb{K}(\mathscr{F})) \lhd A$ and let $T \in \mathbb{K}(\mathscr{E})$. The operator $T \otimes 1$ on $\mathscr{E} \otimes_A \mathscr{F}$ is compact if and only if $T \in \mathbb{K}(\mathscr{E} \cdot J)$ (see Lemma 2.1 for the inclusion $\mathbb{K}(\mathscr{E} \cdot J) \subseteq \mathbb{K}(\mathscr{E})$).*

In particular, if $\varphi(A) \subseteq \mathbb{K}(\mathscr{F})$, then $T \otimes 1 \in \mathbb{K}(\mathscr{E} \otimes_\varphi \mathscr{F})$ for all $T \in \mathbb{K}(\mathscr{E})$.

2.2. C*-*algebras of correspondences*

Let $\mathscr{E}\colon A \rightsquigarrow A$ be a correspondence over A. Let $\varphi\colon A \to \mathbb{B}(\mathscr{E})$ be the left action. Let $\mathscr{E}^{\otimes n}$ be the n-fold tensor product of $\mathscr{E}$ over A. By convention, $\mathscr{E}^{\otimes 0} := A$. Let $\mathscr{E}^+ := \bigoplus_{n=0}^\infty \mathscr{E}^{\otimes n}$ be the *Fock space* of $\mathscr{E}$ (see [19]). Define

$$t_\xi^n\colon \mathscr{E}^{\otimes n} \longrightarrow \mathscr{E}^{\otimes n+1}, \qquad \eta \mapsto \xi \otimes \eta,$$

for $n \geq 0$ and $\xi \in \mathscr{E}$; this is the operator T_ξ above for $\mathscr{F} = \mathscr{E}^{\otimes n}$. The operators t_ξ^n combine to an operator $t_\xi \in \mathbb{B}(\mathscr{E}^+)$, that is, $t_\xi|_{\mathscr{E}^{\otimes n}} = t_\xi^n$. Let $\varphi_\infty\colon A \to \mathbb{B}(\mathscr{E}^+)$ be the obvious representation by block diagonal operators and let $t_\infty\colon \mathscr{E} \to \mathbb{B}(\mathscr{E}^+)$ be the linear map $\xi \mapsto t_\xi$.

DEFINITION 2.7. The *Toeplitz* C*-*algebra $\mathscr{T}_\mathscr{E}$* of $\mathscr{E}$ is the C*-subalgebra of $\mathbb{B}(\mathscr{E}^+)$ generated by $\varphi_\infty(A) + t_\infty(\mathscr{E})$.

Let J be an ideal of A with $\varphi(J) \subseteq \mathbb{K}(\mathscr{E})$. Let P_0 be the projection in $\mathbb{B}(\mathscr{E}^+)$ that is the identity on $A \subseteq \mathscr{E}^+$ and that vanishes on $\mathscr{E}^{\otimes n}$ for $n \geq 1$. Then $J_0 := \varphi_\infty(J)P_0$ is contained in $\mathscr{T}_\mathscr{E}$. The ideal in $\mathscr{T}_\mathscr{E}$ generated by J_0 is equal to $\mathbb{K}(\mathscr{E}^+ J) \subseteq \mathbb{K}(\mathscr{E}^+)$.

DEFINITION 2.8 ([17, Definition 2.18]). The *relative Cuntz-Pimsner algebra $\mathcal{O}_{J,\mathscr{E}}$* of $\mathscr{E}$ with respect to J is $\mathscr{T}_\mathscr{E} / \mathbb{K}(\mathscr{E}^+ J)$.

The following three cases are particularly important. First, if $J = \{0\}$, then $\mathcal{O}_{J,\mathcal{E}}$ is the Toeplitz C*-algebra $\mathcal{T}_{\mathcal{E}}$. Secondly, if $J = \varphi^{-1}(\mathbb{K}(\mathcal{E}))$ and φ is injective, then $\mathcal{O}_{J,\mathcal{E}}$ is the algebra $\tilde{\mathcal{O}}_{\mathcal{E}}$ defined by Pimsner [19]. Third, if J is *Katsura's ideal*

$$I_{\mathcal{E}} := \varphi^{-1}(\mathbb{K}(\mathcal{E})) \cap (\ker \varphi)^{\perp}, \tag{2.9}$$

then $\mathcal{O}_{I_{\mathcal{E}},\mathcal{E}}$ is Katsura's Cuntz-Pimsner algebra as defined in [13].

PROPOSITION 2.10. *Katsura's ideal $I_{\mathcal{E}}$ in* (2.9) *is the largest ideal J in A with $\varphi(J) \subseteq \mathbb{K}(\mathcal{E})$ for which the canonical map $A \to \mathcal{O}_{J,\mathcal{E}}$ is injective.*

PROOF. That $\pi_{I_{\mathcal{E}}}$ is injective is shown in [13, Proposition 4.9] or [17, Proposition 2.21]. The ideal $I_{\mathcal{E}}$ is maximal with this property because any ideal $J \triangleleft A$ with $\varphi(J) \subseteq \mathbb{K}(\mathcal{E})$ and $J \not\subseteq (\ker \varphi)^{\perp}$ must contain $a \in J$ with $\varphi(a) = 0$. Then $\varphi_{\infty}(a) \in \varphi_{\infty}(J) \cdot P_0$ becomes 0 in $\mathcal{O}_{J,\mathcal{E}}$.

DEFINITION 2.11. Let $\mathcal{E}: A \rightsquigarrow A$ be a correspondence and B a C*-algebra. A *representation* of $\mathcal{E}$ in B is a pair (π, t), where $\pi: A \to B$ is a *-homomorphism, $t: \mathcal{E} \to B$ is a linear map, and

(1) $\pi(a)t(\xi) = t(\varphi(a)\xi)$ for all $a \in A$ and $\xi \in \mathcal{E}$;

(2) $t(\xi)^* t(\eta) = \varphi(\langle \xi | \eta \rangle_A)$ for all $\xi, \eta \in \mathcal{E}$.

These conditions imply $t(\xi)\pi(a) = t(\xi a)$ for all $\xi \in \mathcal{E}$ and $a \in A$.

In particular, $(\varphi_{\infty}, t_{\infty})$ is a representation of $\mathcal{E}$ in the Toeplitz C*-algebra $\mathcal{T}_{\mathcal{E}}$. This representation is universal in the following sense:

PROPOSITION 2.12 ([19, Theorem 3.4], [17, Theorem 2.12]). *Any representation (π, t) of $\mathcal{E}$ in a C*-algebra B is of the form $(\tilde{\pi} \circ \varphi_{\infty}, \tilde{\pi} \circ t_{\infty})$ for a unique *-homomorphism $\tilde{\pi}: \mathcal{T}_{\mathcal{E}} \to B$. Conversely, $(\tilde{\pi} \circ \varphi_{\infty}, \tilde{\pi} \circ t_{\infty})$ is a representation of $\mathcal{E}$ for any *-homomorphism $\tilde{\pi}: \mathcal{T}_{\mathcal{E}} \to B$.*

LEMMA 2.13 ([19], [4, Proposition 4.6.3]). *For any representation (π, t) of $\mathcal{E}$, there is a unique *-homomorphism $\pi^1: \mathbb{K}(\mathcal{E}) \to B$ with $\pi^1(|\xi\rangle\langle\eta|) = t_{\xi} t_{\eta}^*$ for all $\xi, \eta \in \mathcal{E}$.*

PROPOSITION 2.14 ([17, Theorem 2.19]). *The representation $\tilde{\pi}$ of $\mathcal{T}_{\mathcal{E}}$ associated to a representation (π, t) of $\mathcal{E}$ factors through the quotient $\mathcal{O}_{J,\mathcal{E}}$ of $\mathcal{T}_{\mathcal{E}}$ if and only if*

$$\pi(a) = \pi^1(\varphi(a)) \qquad \text{for all } a \in J.$$

In this case, we call the representation covariant on J.

Let (π_J, t_J) be the canonical representation of $\mathcal{E}$ in $\mathcal{O}_{J,\mathcal{E}}$. Proposition 2.14 says that (π_J, t_J) is the universal representation of $\mathcal{E}$ that is covariant on J.

PROPOSITION 2.15. *A representation* (π, t) *in* B *is covariant on* J *if and only if* $\pi(J) \subseteq t(\mathscr{E}) \cdot B$.

PROOF. Let $a \in J$. Then $\pi^1(\varphi(a))$ is contained in the closed linear span of $t(\mathscr{E})t(\mathscr{E})^*$ and hence in $t(\mathscr{E}) \cdot B$. So $\pi(a) \in t(\mathscr{E}) \cdot B$ is necessary for $\pi(a) = \pi^1(\varphi(a))$. Conversely, assume $\pi(a) \in t(\mathscr{E}) \cdot B$ for all $a \in J$. We have $\pi(a) \cdot t(\xi) = t(\varphi(a)\xi) = \pi^1(\varphi(a))t(\xi)$ for all $\xi \in \mathscr{E}$ (see [13, Lemma 2.4]). Hence $(\pi(a) - \pi^1(\varphi(a))) \cdot t(\mathscr{E}) \cdot B = 0$. Since $\pi(a^*), \pi^1(\varphi(a^*)) \in t(\mathscr{E}) \cdot B$, we get $(\pi(a) - \pi^1(\varphi(a))) \cdot (\pi(a) - \pi^1(\varphi(a)))^* = 0$. This is equivalent to $\pi(a) = \pi^1(\varphi(a))$.

2.3. Gauge action and Fell bundle structure

Let $\mathscr{E}\colon A \rightsquigarrow A$ be a correspondence and let $J \lhd A$ be an ideal with $\varphi(J) \subseteq \mathbb{K}(\mathscr{E})$. If (π, t) is a representation of $\mathscr{E}$ that is covariant on J, then so is $(\pi, z \cdot t)$ for $z \in \mathbb{T}$. This operation on representations comes from an automorphism of the relative Cuntz-Pimsner algebra $\mathcal{O}_{J,\mathscr{E}}$ by its universal property. These automorphisms define a continuous action γ of $\mathbb{T}$ on $\mathcal{O}_{J,\mathscr{E}}$, called the *gauge action*. Let

$$\mathcal{O}^n_{J,\mathscr{E}} := \{ b \in \mathcal{O}_{J,\mathscr{E}} : \gamma_z(b) = z^n b \text{ for all } z \in \mathbb{T} \}$$

for $n \in \mathbb{Z}$ be the nth spectral subspace. These spectral subspaces form a Fell bundle over $\mathbb{Z}$, that is, $\mathcal{O}^n_{J,\mathscr{E}} \cdot \mathcal{O}^m_{J,\mathscr{E}} \subseteq \mathcal{O}^{n+m}_{J,\mathscr{E}}$ and $(\mathcal{O}^n_{J,\mathscr{E}})^* = \mathcal{O}^{-n}_{J,\mathscr{E}}$ for all $n, m \in \mathbb{Z}$. In particular, for $J = \{0\}$ we get a gauge action on $\mathcal{T}_{\mathscr{E}}$ and corresponding spectral subspaces $\mathcal{T}^n_{\mathscr{E}} \subseteq \mathcal{T}_{\mathscr{E}}$. Explicitly, the gauge action on $\mathcal{T}_{\mathscr{E}}$ comes from the obvious $\mathbb{N}$-grading on $\mathscr{E}^+$: if $x \in \mathcal{T}_{\mathscr{E}}$, then $x \in \mathcal{T}^n_{\mathscr{E}}$ if and only if $x(\mathscr{E}^{\otimes k}) \subseteq \mathscr{E}^{\otimes n+k}$ for all $k \in \mathbb{N}$; this means $x|_{\mathscr{E}^{\otimes k}} = 0$ if $k + n < 0$. And $\mathcal{O}^n_{J,\mathscr{E}}$ is the image of $\mathcal{T}^n_{\mathscr{E}}$ in $\mathcal{O}_{J,\mathscr{E}}$.

LEMMA 2.16. *Let* $n \in \mathbb{Z}$. *The subspace* $\mathcal{O}^n_{J,\mathscr{E}}$ *in* $\mathcal{O}_{J,\mathscr{E}}$ *is the closed linear span of* $t_J(\xi_1)t_J(\xi_2)\cdots t_J(\xi_k) \cdot t_J^*(\eta_\ell)\cdots t_J^*(\eta_2)t_J^*(\eta_1)$ *for* $\xi_i, \eta_j \in \mathscr{E}$, $k - \ell = n$. *If* $n \in \mathbb{N}$, *then*

$$\mathcal{O}^n_{J,\mathscr{E}} \cong \mathscr{E}^{\otimes n} \otimes_A \mathcal{O}^0_{J,\mathscr{E}}$$

as a correspondence $A \rightsquigarrow \mathcal{O}^0_{J,\mathscr{E}}$. *The Fell bundle* $(\mathcal{O}^k_{J,\mathscr{E}})_{k\in\mathbb{Z}}$ *is semi-saturated, that is,* $\mathcal{O}^k_{J,\mathscr{E}} \cdot \mathcal{O}^\ell_{J,\mathscr{E}} = \mathcal{O}^{k+\ell}_{J,\mathscr{E}}$ *if* $k, \ell \geq 0$.

PROOF. Let $b \in \mathcal{O}^n_{J,\mathscr{E}}$ and let $\epsilon > 0$. Then b is ϵ-close to a finite linear combination b_ϵ of monomials $t_J(\xi_1)t_J(\xi_2)\cdots t_J(\xi_k) \cdot t_J^*(\eta_\ell)\cdots t_J^*(\eta_2)t_J^*(\eta_1)$ with $k, \ell \in \mathbb{N}$. Define

$$p_n(x) := \int_{\mathbb{T}} z^{-n}\gamma_z(x)\,\mathrm{d}z, \qquad x \in \mathcal{O}_{J,\mathscr{E}}.$$

This is a contractive projection from $\mathcal{O}_{J,\mathcal{E}}$ onto $\mathcal{O}^n_{J,\mathcal{E}}$. Since $p_n(b) = b$ and $\|p_n\| \le 1$, we have $\|b - p_n(b_\epsilon)\| \le \epsilon$ as well. Inspection shows that p_n maps a monomial $t_J(\xi_1)t_J(\xi_2)\cdots t_J(\xi_k)\cdot t_J^*(\eta_\ell)\cdots t_J^*(\eta_2)t_J^*(\eta_1)$ to itself if $k-\ell = n$ and kills it otherwise. Hence $\mathcal{O}^n_{J,\mathcal{E}}$ is the closed linear span of such monomials with $k - \ell = n$.

The monomials generating $\mathcal{O}^{k+\ell}_{J,\mathcal{E}}$ for $k, \ell \ge 0$ are obviously in $\mathcal{O}^k_{J,\mathcal{E}} \cdot \mathcal{O}^\ell_{J,\mathcal{E}}$. Hence the first statement immediately implies the last one. There is an isometric A, $\mathcal{O}^0_{J,\mathcal{E}}$-bimodule map

$$\mathcal{E}^{\otimes n} \otimes_A \mathcal{O}^0_{J,\mathcal{E}} \longrightarrow \mathcal{O}^n_{J,\mathcal{E}}, \qquad \xi_1 \otimes \cdots \otimes \xi_n \otimes y \mapsto t_J(\xi_1)\cdots t_J(\xi_n)\cdot y.$$

The first statement implies that its image is dense, so it is unitary.

The Fell bundle $(\mathcal{O}^n_{J,\mathcal{E}})_{n\in\mathbb{Z}}$ need not be saturated, that is, $\mathcal{O}^n_{J,\mathcal{E}} \cdot \mathcal{O}^{-n}_{J,\mathcal{E}}$ may differ from $\mathcal{O}^0_{J,\mathcal{E}}$.

The next theorem will split the construction of relative Cuntz-Pimsner algebras into two steps. The first builds the Hilbert $\mathcal{O}^0_{J,\mathcal{E}}$-bimodule $\mathcal{O}^1_{J,\mathcal{E}}$, the second takes the crossed product $\mathcal{O}^0_{J,\mathcal{E}} \rtimes \mathcal{O}^1_{J,\mathcal{E}}$ for this Hilbert bimodule, as defined in [1]:

THEOREM 2.17. *The relative Cuntz-Pimsner algebra is $\mathbb{T}$-equivariantly isomorphic to the crossed product of $\mathcal{O}^0_{J,\mathcal{E}}$ by the Hilbert $\mathcal{O}^0_{J,\mathcal{E}}$-bimodule $\mathcal{O}^1_{J,\mathcal{E}}$ and to the full or reduced section* C*-*algebra of the Fell bundle $(\mathcal{O}^n_{J,\mathcal{E}})_{n\in\mathbb{Z}}$.*

PROOF. The Fell bundle $(\mathcal{O}^n_{J,\mathcal{E}})_{n\in\mathbb{Z}}$ is semi-saturated by Lemma 2.16. Now the results of Abadie-Eilers-Exel [1] imply our claims.

A Hilbert bimodule $\mathcal{G}$ on a C*-algebra B is the same as a Morita-Rieffel equivalence between two ideals in B or, briefly, a partial Morita-Rieffel equivalence on B (this point of view is explained in [5]). The crossed product $B \rtimes \mathcal{G}$ generalises the partial crossed product for a partial automorphism. Many results about crossed products for automorphisms extend to Hilbert bimodule crossed products. In particular, the standard criteria for simplicity and detection and separation of ideals are extended in [15].

PROPOSITION 2.18. *The following conditions are equivalent:*

(1) *the map $\pi_J\colon A \to \mathcal{O}^0_{J,\mathcal{E}}$ is an isomorphism;*

(2) *the map $\varphi\colon J \to \mathbb{K}(\mathcal{E})$ is an isomorphism;*

(3) *the correspondence $\mathcal{E}$ comes from a Hilbert bimodule and $J = I_{\mathcal{E}}$.*

PROOF. If $J = I_{\mathcal{E}}$ is Katsura's ideal, then everything follows from [13, Proposition 5.18]. So it remains to observe that (1) and (2) fail if $J \neq I_{\mathcal{E}}$. Lemma 2.4 shows that $\mathcal{E}$ comes from a Hilbert bimodule if and only if there is an ideal I in A so that $\varphi|_I\colon I \to \mathbb{K}(\mathscr{E})$ is an isomorphism. In this case, I is the largest ideal on which φ restricts to an injective map into $\mathbb{K}(\mathscr{E})$. So $I = I_{\mathscr{E}}$. Thus (2) $\iff$ (3).

If $J \not\subseteq I_{\mathscr{E}}$, then $A \to \mathcal{O}_{J,\mathscr{E}}$ is not injective by Proposition 2.10. So (1) implies $J \subseteq I_{\mathscr{E}}$. If $J \subseteq I_{\mathscr{E}}$ and (1) holds, then the map $A \to \mathcal{O}_{I_{\mathscr{E}},\mathscr{E}}$ is still surjective because $\mathcal{O}_{I_{\mathscr{E}},\mathscr{E}}$ is a quotient of $\mathcal{O}_{J,\mathscr{E}}$, and it is also injective by Proposition 2.10. Hence $\mathcal{O}_{I_{\mathscr{E}},\mathscr{E}} = \mathcal{O}_{J,\mathscr{E}}$. This implies $\mathbb{K}(\mathscr{E}^+ I_{\mathscr{E}}) = \mathbb{K}(\mathscr{E}^+ J)$ and hence $I_{\mathscr{E}} = J$ because of the direct summand A in $\mathscr{E}^+$.

PROPOSITION 2.19. *Let $\mathscr{G}$ be a Hilbert B-bimodule and let $I_{\mathscr{G}}$ be Katsura's ideal for $\mathscr{G}$. Then $\mathcal{O}_{I_{\mathscr{G}},\mathscr{G}} \cong B \rtimes \mathscr{G}$ $\mathbb{T}$-equivariantly.*

PROOF. Theorem 2.17 identifies $\mathcal{O}_{I_{\mathscr{G}},\mathscr{G}} \cong \mathcal{O}^0_{I_{\mathscr{G}},\mathscr{G}} \rtimes \mathcal{O}^1_{I_{\mathscr{G}},\mathscr{G}}$. Proposition 2.18 gives $B \cong \mathcal{O}^0_{I_{\mathscr{G}},\mathscr{G}}$, and the isomorphism $\mathcal{O}^1_{I_{\mathscr{G}},\mathscr{G}} \cong \mathscr{G} \otimes_B \mathcal{O}^0_{I_{\mathscr{G}},\mathscr{G}}$ from Lemma 2.16 implies that $\mathscr{G} \cong \mathcal{O}^1_{I_{\mathscr{G}},\mathscr{G}}$ as a Hilbert B-bimodule.

2.4. Functoriality of relative Cuntz-Pimsner algebras

Schweizer [20] has defined "covariant homomorphisms" and "covariant correspondences" between self-correspondences and has asserted that they induce *-homomorphisms and correspondences between the associated Toeplitz and absolute Cuntz-Pimsner algebras. For the proof of functoriality for covariant correspondences he refers to a preprint that never got published. In fact, there are some technical pitfalls. We correct his statement here, and also add a condition to treat relative Cuntz-Pimsner algebras.

Throughout this subsection, let $\mathscr{E}\colon A \rightsquigarrow A$ and $\mathscr{G}\colon B \rightsquigarrow B$ be correspondences and let $J_A \subseteq \varphi^{-1}(\mathbb{K}(\mathscr{E}))$ and $J_B \subseteq \varphi^{-1}(\mathbb{K}(\mathscr{G}))$ be ideals.

DEFINITION 2.20. A *covariant correspondence* from $(A, \mathscr{E}, J_A)$ to $(B, \mathscr{G}, J_B)$ is a pair $(\mathscr{F}, V)$, where $\mathscr{F}$ is a correspondence $A \rightsquigarrow B$ with $J_A \cdot \mathscr{F} \subseteq \mathscr{F} \cdot J_B$ and V is a correspondence isomorphism $\mathscr{E} \otimes_A \mathscr{F} \Rightarrow \mathscr{F} \otimes_B \mathscr{G}$. A covariant correspondence is *proper* if $\mathscr{F}$ is proper.

PROPOSITION 2.21. *A proper covariant correspondence $(\mathscr{F}, V)$ from $(A, \mathscr{E}, J_A)$ to $(B, \mathscr{G}, J_B)$ induces a proper $\mathbb{T}$-equivariant correspondence $\mathcal{O}_{\mathscr{F},V}\colon \mathcal{O}_{J_A,\mathscr{E}} \rightsquigarrow \mathcal{O}_{J_B,\mathscr{G}}$.*

Here a $\mathbb{T}$*-equivariant correspondence* between two $\mathbb{T}$-C*-algebras A and B is an A, B-correspondence $\mathscr{E}$ with a $\mathbb{T}$-action such that $z \cdot (a \cdot \xi \cdot b) = (z \cdot a) \cdot (z \cdot \xi) \cdot (z \cdot b)$ and $\langle z \cdot \xi | z \cdot \xi_2 \rangle = z \cdot \langle \xi | \xi_2 \rangle$ for all $z \in \mathbb{T}$, $a \in A$, $\xi, \xi_2 \in \mathscr{E}$, $b \in B$.

The construction of $\mathcal{O}_{\mathscr{F},V}$ in the proof below is the basis for all results in the following sections. It depends on some subtle details in the definition of a covariant correspondence. Schweizer [20] claims such a result also for non-proper correspondences, and he allows V to be a non-adjointable isometry. In fact, the proof below will show that a pair $(\mathcal{F}, V)$ where V is only a non-adjointable isometry induces a correspondence between the Toeplitz C*-algebras. It is unclear, however, when this correspondence descends to one between the absolute or relative Cuntz-Pimsner algebras. And we need $\mathcal{E}$ or $\mathcal{F}$ to be proper. The case when $\mathcal{E}$ is proper is quite different, easier, and already treated in [2]. An important new insight in this article is how to construct $\mathcal{O}_{\mathcal{F},V}$ if $\mathcal{F}$ is proper instead of $\mathcal{E}$.

Proof of Proposition 2.21. The canonical *-homomorphism $\pi_{J_B}\colon B \to \mathcal{O}_{J_B,\mathcal{G}}$ allows us to view $\mathcal{O}_{J_B,\mathcal{G}}$ as a proper correspondence $B \rightsquigarrow \mathcal{O}_{J_B,\mathcal{G}}$. Thus $\mathcal{F}_{\mathcal{O}} := \mathcal{F} \otimes_B \mathcal{O}_{J_B,\mathcal{G}}$ becomes a proper correspondence $A \rightsquigarrow \mathcal{O}_{J_B,\mathcal{G}}$, that is, a Hilbert $\mathcal{O}_{J_B,\mathcal{G}}$-module with a representation $\pi\colon A \to \mathbb{K}(\mathcal{F}_{\mathcal{O}})$. The $\mathbb{T}$-action on $\mathcal{O}_{J_B,\mathcal{G}}$ induces a $\mathbb{T}$-action on $\mathcal{F}_{\mathcal{O}}$ because $\pi_{J_B}(B) \subseteq \mathcal{O}^0_{J_B,\mathcal{G}}$. We are going to define a map $t\colon\mathcal{E} \to \mathbb{K}(\mathcal{F}_{\mathcal{O}})$ such that (π, t) is a representation of $(A, \mathcal{E})$ on $\mathcal{F}_{\mathcal{O}}$ that is covariant on J_A. Then Proposition 2.14 yields a representation $\tilde{\pi}\colon \mathcal{O}_{J_A,\mathcal{E}} \to \mathbb{K}(\mathcal{F}_{\mathcal{O}})$. This is the desired correspondence $\mathcal{O}_{J_A,\mathcal{E}} \rightsquigarrow \mathcal{O}_{J_B,\mathcal{G}}$.

There is an isometry $\mu_{\mathcal{G}}\colon \mathcal{G} \otimes_B \mathcal{O}_{J_B,\mathcal{G}} \Rightarrow \mathcal{O}_{J_B,\mathcal{G}}$, $\zeta \otimes y \mapsto t_\infty(\zeta)\cdot y$, of correspondences $B \rightsquigarrow \mathcal{O}_{J_B,\mathcal{G}}$. Usually, it is not unitary. We define an isometry

$$\begin{aligned} V^!\colon \mathcal{E} \otimes_A \mathcal{F}_{\mathcal{O}} &= \mathcal{E} \otimes_A \mathcal{F} \otimes_B \mathcal{O}_{J_B,\mathcal{G}} \\ &\xRightarrow{V\otimes 1} \mathcal{F} \otimes_B \mathcal{G} \otimes_B \mathcal{O}_{J_B,\mathcal{G}} \xRightarrow{1\otimes\mu_{\mathcal{G}}} \mathcal{F} \otimes_B \mathcal{O}_{J_B,\mathcal{G}} = \mathcal{F}_{\mathcal{O}}. \end{aligned}$$

It yields a map t from $\mathcal{E}$ to the space of bounded operators on $\mathcal{F}_{\mathcal{O}}$ by $t(\xi)(\eta) := V^!(\xi \otimes \eta)$. To show that $t(\xi)$ is adjointable, we need that $\mathcal{F}_{\mathcal{O}}$ is a proper correspondence $A \rightsquigarrow \mathcal{O}_{J_B,\mathcal{G}}$: then $T_\xi \in \mathbb{K}(\mathcal{F}_{\mathcal{O}}, \mathcal{E} \otimes_A \mathcal{F}_{\mathcal{O}})$, and composition with $V^!$ maps this into $\mathbb{K}(\mathcal{F}_{\mathcal{O}})$ by Lemma 2.1. So even $t(\xi) \in \mathbb{K}(\mathcal{F}_{\mathcal{O}})$ for all $\xi \in \mathcal{E}$.

We claim that the pair (π, t) is a representation. We have $\pi(a)t(\xi) = t(\varphi(a)\xi)$ because $V^!$ is a left A-module map. And $t(\xi_1)^*t(\xi_2) = \pi(\langle\xi_1|\xi_2\rangle)$ holds because

$$\begin{aligned} \langle t(\xi_1)\eta_1 | t(\xi_2)\eta_2\rangle &= \big\langle V^!(\xi_1 \otimes \eta_1) | V^!(\xi_2 \otimes \eta_2)\big\rangle \\ &= \langle \xi_1 \otimes \eta_1 | \xi_2 \otimes \eta_2\rangle = \langle \eta_1 | \pi(\langle\xi_1|\xi_2\rangle)\eta_2\rangle . \end{aligned}$$

If $J_A = 0$, then we are done at this point, and we have not yet used that V is unitary. So the Toeplitz C*-algebra of a correspondence remains functorial for proper covariant correspondences where V is not unitary.

It remains to prove that π is covariant on J_A. By Proposition 2.15, this is equivalent to $\pi(J_A)(\mathcal{F}_{\mathcal{O}}) \subseteq t(\mathcal{E})(\mathcal{F}_{\mathcal{O}})$. And $J_B \cdot \mathcal{O}_{J_B,\mathcal{G}} \subseteq t_{J_B}(\mathcal{G}) \cdot \mathcal{O}_{J_B,\mathcal{G}}$ holds because the canonical representation of $(B, \mathcal{G})$ on $\mathcal{O}_{J_B,\mathcal{G}}$ is covariant on J_B.

Since $J_A \cdot \mathcal{F} \subseteq \mathcal{F} \cdot J_B$ by assumption,

$$J_A \cdot \mathcal{F}_{\mathcal{O}} \subseteq \mathcal{F} \otimes J_B \cdot \mathcal{O}_{J_B,\mathcal{G}} \subseteq \mathcal{F} \otimes t_{J_B}(\mathcal{G}) \cdot \mathcal{O}_{J_B,\mathcal{G}} = (1 \otimes \mu_{\mathcal{G}})(\mathcal{F} \otimes_B \mathcal{G} \otimes_B \mathcal{O}_{J_B,\mathcal{G}}).$$

Since V is unitary, we may rewrite this further as $V^!(\mathcal{E} \otimes_A \mathcal{F} \otimes_B \mathcal{O}_{J_B,\mathcal{G}}) = t(\mathcal{E}) \cdot \mathcal{F}_{\mathcal{O}}$. This finishes the proof that (π, t) is covariant on J_A. The operators $t(\xi)$ for $\xi \in \mathcal{E}$ are homogeneous of degree 1 for the $\mathbb{T}$-action. Thus $\tilde{\pi}$ is $\mathbb{T}$-equivariant.

EXAMPLE 2.22. Let $A = B$ and $J = J_A = J_B \neq \{0\}$ and let $\mathcal{E} \subseteq \mathcal{G}$ be an A-invariant Hilbert submodule. Then the identity correspondence $\mathcal{F} = A$ with the inclusion map $\mathcal{E} \otimes_A \mathcal{F} \cong \mathcal{E} \hookrightarrow \mathcal{G} \cong \mathcal{F} \otimes_B \mathcal{G}$ is a covariant correspondence in the notation of Schweizer. There is indeed a canonical *-homomorphism $\mathcal{T}_{\mathcal{E}} \to \mathcal{T}_{\mathcal{G}}$. But it need not descend to the relative Cuntz-Pimsner algebras because $\varphi_{\mathcal{G}}(a) \in \mathbb{K}(\mathcal{G})$ for $a \in J$ need not be the extension of $\varphi_{\mathcal{E}}(a) \in \mathbb{K}(\mathcal{E})$ given by Lemma 2.1. So the Cuntz-Pimsner covariance conditions for $\mathcal{O}_{J,\mathcal{E}}$ and $\mathcal{O}_{J,\mathcal{G}}$ may be incompatible. We ask V to be unitary to avoid this problem.

EXAMPLE 2.23. Turn $\mathcal{O}^0_{J,\mathcal{E}}$, into a proper C*-correspondence $A \rightsquigarrow \mathcal{O}^0_{J,\mathcal{E}}$ with the obvious left action of A. We claim that the proper correspondence $\mathcal{O}^0_{J,\mathcal{E}}\colon A \rightsquigarrow \mathcal{O}^0_{J,\mathcal{E}}$ with the isomorphism from Lemma 2.16 is a proper covariant correspondence from $\mathcal{E}\colon A \rightsquigarrow A$ with the ideal J to $\mathcal{O}^1_{J,\mathcal{E}}\colon \mathcal{O}^0_{J,\mathcal{E}} \rightsquigarrow \mathcal{O}^0_{J,\mathcal{E}}$ with Katsura's ideal $I_{\mathcal{O}^1_{J,\mathcal{E}}}$. It remains to show that $J \cdot \mathcal{O}^0_{J,\mathcal{E}} \subseteq \mathcal{O}^0_{J,\mathcal{E}} \cdot I_{\mathcal{O}^1_{J,\mathcal{E}}} = I_{\mathcal{O}^1_{J,\mathcal{E}}}$. Since $\mathcal{O}^1_{J,\mathcal{E}}$ is a Hilbert bimodule, Katsura's ideal is equal to the range ideal of the left inner product, that is, the closed linear span of xy^* for all $x, y \in \mathcal{O}^1_{J,\mathcal{E}}$. This contains $\mathbb{K}(\mathcal{E})$ for $x, y \in \mathcal{E}$, which in turn contains J by the Cuntz-Pimsner covariance condition on J (see Proposition 2.14). So $J \cdot \mathcal{O}^0_{J,\mathcal{E}} \subseteq I_{\mathcal{O}^1_{J,\mathcal{E}}}$. The relative Cuntz-Pimsner algebra of $(\mathcal{O}^0_{J,\mathcal{E}}, \mathcal{O}^1_{J,\mathcal{E}}, I_{\mathcal{O}^1_{J,\mathcal{E}}})$ is again $\mathcal{O}^0_{J,\mathcal{E}}$ by Proposition 2.18. The correspondence $\mathcal{O}^0_{J,\mathcal{E}} \rightsquigarrow \mathcal{O}^0_{J,\mathcal{E}}$ associated to the covariant correspondence above is just the identity correspondence on $\mathcal{O}^0_{J,\mathcal{E}}$.

REMARK 2.24. If $J_A = 0$ or $J_B = \varphi^{-1}(\mathbb{K}(\mathcal{G}))$, then the condition $J_A \cdot \mathcal{F} \subseteq \mathcal{F} \cdot J_B$ for covariant correspondences $(A, \mathcal{E}, J_A) \to (B, \mathcal{G}, J_B)$ always holds and so may be left out. This is clear if $J_A = 0$. Assume $J_B = \varphi^{-1}(\mathbb{K}(\mathcal{G}))$. Since $\mathcal{F}$ is proper, J_A acts on $\mathcal{E} \otimes_A \mathcal{F} \cong \mathcal{F} \otimes_B \mathcal{G}$ by compact operators by Lemma 2.6. Again by Lemma 2.6, this implies $J_A \subseteq \mathbb{K}(\mathcal{F} \cdot J_B)$. Thus $J_A \cdot \mathcal{F} \subseteq \mathcal{F} \cdot J_B$.

EXAMPLE 2.25. Covariant correspondences are related to the *T-pairs* used by Katsura [14] to describe the ideal structure of relative Cuntz-Pimsner algebras. For this, we specialise to covariant correspondences out of $(A, \mathcal{E}, J)$ where the underlying correspondence comes from a quotient map $A \to A/I$.

That is, $\mathcal{F} = A/I\colon A \rightsquigarrow A/I$ for an ideal $I \triangleleft A$. When is this part of a covariant correspondence from $(A, \mathcal{E}, J)$ to $(A/I, \mathcal{E}', J')$ for some $\mathcal{E}'$, J'? More precisely, $\mathcal{E}'$ is a correspondence $A/I \rightsquigarrow A/I$ and J' is an ideal in A/I that acts on $\mathcal{E}'$ by compact operators.

There are natural isomorphisms $\mathcal{E} \otimes_A \mathcal{F} \cong \mathcal{E}/\mathcal{E}I$ and $\mathcal{F} \otimes_{A/I} \mathcal{E}' \cong \mathcal{E}'$ as correspondences $A \rightsquigarrow A/I$. So the only possible choice for $\mathcal{E}'$ is $\mathcal{E}' := \mathcal{E}/\mathcal{E}I$ with a left A/I-action which gives the canonical A-action when composed with the quotient map $A \to A/I$. Such a correspondence $\mathcal{E}/\mathcal{E}I\colon A/I \rightsquigarrow A/I$ exists if and only if $\mathcal{E}$ is *positively invariant*, that is, $I\mathcal{E} \subseteq \mathcal{E}I$. Assume this to be the case.

The ideal $J' \triangleleft A/I$ is equivalent to its preimage in A, which is an ideal $I' \triangleleft A$ that contains I. For a covariant correspondence, we require $J\mathcal{F} \subseteq \mathcal{F}J'$, which means that $J \subseteq I'$; and for $(A/I, \mathcal{E}', J')$ to define a relative Cuntz-Pimsner algebra, we require the ideal J' or, equivalently, I', to act by compact operators on $\mathcal{E}' := \mathcal{E}/\mathcal{E}I$. There is an isomorphism $\mathcal{E} \otimes_A \mathcal{F} \cong \mathcal{F} \otimes_A \mathcal{E}'$. (It is unique up to an automorphism of $\mathcal{E}/\mathcal{E}I$, that is, a unitary operator on $\mathcal{E}/\mathcal{E}I$ that also commutes with the left action of A or A/I, but this shall not concern us.) So we get a covariant correspondence in this case. It induces a correspondence from $\mathcal{O}_{J,\mathcal{E}}$ to $\mathcal{O}_{J',\mathcal{E}'}$ by Proposition 2.21. Since our covariant correspondence is a covariant homomorphism, the correspondence built in Proposition 2.21 comes from a $\mathbb{T}$-equivariant *-homomorphism, which turns out to be surjective. So a pair of ideals (I, I') as above induces a $\mathbb{T}$-equivariant quotient or, equivalently, a $\mathbb{T}$-invariant ideal in $\mathcal{O}_{J,\mathcal{E}}$.

Sometimes different pairs (I, I') produce the same quotient of $\mathcal{O}_{J,\mathcal{E}}$. If I'/I contains elements that act by 0 on $\mathbb{K}(\mathcal{E}/\mathcal{E}I)$, then the map $A/I \to \mathcal{O}_{J',\mathcal{E}'}$ is not injective by Proposition 2.10. Then we may enlarge I without changing the relative Cuntz-Pimsner algebra. When we add the condition that no non-zero element of I'/I acts by a compact operator on $\mathcal{E}/\mathcal{E} \cdot I$, then we get exactly the *T-pairs* with $J \subseteq I'$ of [14]. The T-pairs (I, I') with $J \subseteq I'$ correspond bijectively to gauge-invariant ideals of $\mathcal{O}_{J,\mathcal{E}}$ by [14, Proposition 11.9].

3. Bicategories of correspondences and Hilbert bimodules

We are going to enrich the relative Cuntz-Pimsner algebra construction to a homomorphism (functor) from a suitable bicategory of covariant correspondences to the $\mathbb{T}$-equivariant correspondence bicategory. Most of the work is already done in Proposition 2.21, which describes how this homomorphism acts on arrows. It remains to define the appropriate bicategories and write down the remaining data of a homomorphism.

The correspondence bicategory of C*-algebras and related bicategories have been discussed in [6], [7], [5], [2]. We recall basic bicategorical definitions in the appendix for the convenience of the reader. Here we go through these notions much more quickly. Let $\mathfrak{C}$ be the correspondence bicategory. It has C*-algebras as objects, C*-correspondences as arrows, and correspondence isomorphisms as 2-arrows. The composition is the tensor product $\otimes_B$ of C*-correspondences.

Given any bicategory $\mathfrak{D}$, there is a bicategory $\mathfrak{C}^{\mathfrak{D}}$ with homomorphisms $\mathfrak{D} \to \mathfrak{C}$ as objects, transformations between these homomorphisms as arrows, and modifications between these transformations as 2-arrows (see the appendix for these notions). There is also a continuous version of this for a locally compact, topological bicategory $\mathfrak{D}$. In particular, we shall use the $\mathbb{T}$-equivariant correspondence bicategory $\mathfrak{C}^{\mathbb{T}}$. Its objects are C*-algebras with a continuous $\mathbb{T}$-action. Its arrows are $\mathbb{T}$-equivariant C*-correspondences, and 2-arrows are $\mathbb{T}$-equivariant isomorphisms of C*-correspondences.

When $\mathfrak{D}$ is the monoid $(\mathbb{N}, +)$, we may simplify the bicategory $\mathfrak{C}^{\mathfrak{D}}$ (see [2, Section 5]). An object in it is equivalent to a C*-algebra A with a self-correspondence $\mathcal{E}\colon A \rightsquigarrow A$. An arrow is equivalent to a covariant correspondence (without the condition $J_A\mathcal{F} \subseteq \mathcal{F} J_B$), and a 2-arrow is equivalent to an isomorphism between two covariant correspondences. The bicategory $\mathfrak{C}^{\mathbb{N}}_{\mathrm{pr}}$ that we need is a variant of $\mathfrak{C}^{\mathbb{N}}$ where we add the ideal J and allow only proper covariant correspondences as arrows.

DEFINITION 3.1. The bicategory $\mathfrak{C}^{\mathbb{N}}_{\mathrm{pr}}$ has the following data (see Definition A.1):

- Objects are triples $(A, \mathcal{E}, J)$, where A is a C*-algebra, $\mathcal{E}\colon A \rightsquigarrow A$ is a C*-correspondence, and $J \subseteq \varphi^{-1}(\mathbb{K}(\mathcal{E}))$ is an ideal.
- Arrows $(A, \mathcal{E}, J) \to (A_1, \mathcal{E}_1, J_1)$ are proper covariant correspondences $(\mathcal{F}, u)$ from $(A, \mathcal{E}, J)$ to $(A_1, \mathcal{E}_1, J_1)$, that is, $\mathcal{F}$ is a proper correspondence $A \rightsquigarrow A_1$ with $J\mathcal{F} \subseteq \mathcal{F} J_1$ and u is a correspondence isomorphism $\mathcal{E} \otimes_A \mathcal{F} \Rightarrow \mathcal{F} \otimes_{A_1} \mathcal{E}_1$.
- 2-Arrows $(\mathcal{F}_0, u_0) \Rightarrow (\mathcal{F}_1, u_1)$ are isomorphisms of covariant correspondences, that is, correspondence isomorphisms $w\colon \mathcal{F}_0 \Rightarrow \mathcal{F}_1$ for which the following diagram commutes:

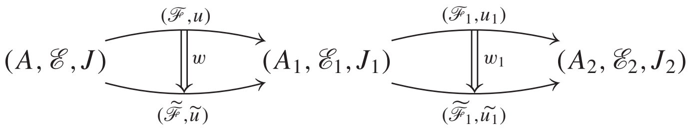
$$\begin{array}{ccc} \mathcal{E} \otimes_A \mathcal{F}_0 & \xRightarrow{u_0} & \mathcal{F}_0 \otimes_{A_1} \mathcal{E}_1 \\ {\scriptstyle 1_{\mathcal{E}}\otimes w}\Downarrow & & \Downarrow{\scriptstyle w\otimes 1_{\mathcal{E}_1}} \\ \mathcal{E} \otimes_A \mathcal{F}_1 & \xRightarrow{u_1} & \mathcal{F}_1 \otimes_{A_1} \mathcal{E}_1 \end{array}$$

- The vertical product of 2-arrows

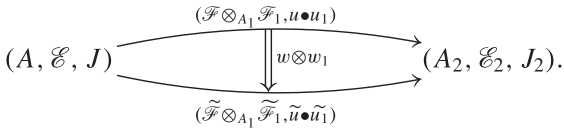
$$w_0\colon (\mathcal{F}_0, u_0) \Rightarrow (\mathcal{F}_1, u_1), \qquad w_1\colon (\mathcal{F}_1, u_1) \Rightarrow (\mathcal{F}_2, u_2)$$

is the usual product $w_1 \cdot w_0 \colon \mathcal{F}_0 \to \mathcal{F}_2$. This is indeed a 2-arrow from $(\mathcal{F}_0, u_0)$ to $(\mathcal{F}_2, u_2)$. And the vertical product is associative and unital. Thus the arrows $(A, \mathcal{E}, J) \to (A_1, \mathcal{E}_1, J_1)$ and the 2-arrows between them form a category $\mathfrak{C}^{\mathbb{N}}_{\mathrm{pr}}((A, \mathcal{E}, J), (A_1, \mathcal{E}_1, J_1))$.

- Let
$$(\mathcal{F}, u) \colon (A, \mathcal{E}, J) \to (A_1, \mathcal{E}_1, J_1),$$
$$(\mathcal{F}_1, u_1) \colon (A_1, \mathcal{E}_1, J_1) \to (A_2, \mathcal{E}_2, J_2)$$
be arrows. Their product is $(\mathcal{F}_1, u_1) \circ (\mathcal{F}, u) := (\mathcal{F} \otimes_{A_1} \mathcal{F}_1, u \bullet u_1)$, where $u \bullet u_1$ is the composite correspondence isomorphism
$$\mathcal{E} \otimes_A \mathcal{F} \otimes_{A_1} \mathcal{F}_1 \xrightarrow{u \otimes 1_{\mathcal{F}_1}} \mathcal{F} \otimes_{A_1} \mathcal{E}_1 \otimes_{A_1} \mathcal{F}_1 \xrightarrow{1_{\mathcal{F}} \otimes u_1} \mathcal{F} \otimes_{A_1} \mathcal{F}_1 \otimes_{A_2} \mathcal{E}_2.$$
- The horizontal product for a diagram of arrows and 2-arrows



is the 2-arrow



This horizontal product and the product of arrows combine to composition bifunctors
$$\mathfrak{C}^{\mathbb{N}}_{\mathrm{pr}}((A, \mathcal{E}, J), (A_1, \mathcal{E}_1, J_1)) \times \mathfrak{C}^{\mathbb{N}}_{\mathrm{pr}}((A_1, \mathcal{E}_1, J_1), (A_2, \mathcal{E}_2, J_2)) \longrightarrow \mathfrak{C}^{\mathbb{N}}_{\mathrm{pr}}((A, \mathcal{E}, J), (A_2, \mathcal{E}_2, J_2)).$$
- The unit arrow on the object $(A, \mathcal{E}, J)$ is the proper covariant correspondence $(A, \iota_{\mathcal{E}})$, where A is the identity correspondence, that is, A with the obvious A-bimodule structure and the inner product $\langle x | y \rangle := x^* y$, and $\iota_{\mathcal{E}}$ is the canonical isomorphism
$$\mathcal{E} \otimes_A A \cong \mathcal{E} \cong A \otimes_A \mathcal{E}$$
built from the right and left actions of A on $\mathcal{E}$.
- The associators and unitors are the same as in the correspondence bicategory. Thus they inherit the coherence conditions needed for a bicategory.

THEOREM 3.2. *There is a homomorphism* $\mathfrak{C}^{\mathbb{N}}_{\mathrm{pr}} \to \mathfrak{C}^{\mathbb{T}}$ *that maps each object* $(A, \mathcal{E}, J)$ *to its relative Cuntz-Pimsner algebra and is the construction of Proposition 2.21 on arrows.*

PROOF. The construction in Proposition 2.21 is "natural" and thus functorial for isomorphisms of covariant correspondences, and it maps the identity covariant correspondence to the identity $\mathbb{T}$-equivariant correspondence on the relative Cuntz-Pimsner algebras. Let $(\mathcal{F}, u)\colon (A, \mathcal{E}, J) \to (A_1, \mathcal{E}_1, J_1)$ and $(\mathcal{F}_1, u_1)\colon (A_1, \mathcal{E}_1, J_1) \to (A_2, \mathcal{E}_2, J_2)$ be covariant correspondences and let $\mathcal{O}_{\mathcal{F},u}$ and $\mathcal{O}_{\mathcal{F}_1,u_1}$ be the associated correspondences of relative Cuntz-Pimsner algebras. By definition, $\mathcal{O}_{\mathcal{F},u} \otimes_{\mathcal{O}_{J_1,\mathcal{E}_1}} \mathcal{O}_{\mathcal{F}_1,u_1}$ and $\mathcal{O}_{\mathcal{F}\otimes_{A_1}\mathcal{F}_1,u\bullet u_1}$ are equal to $(\mathcal{F} \otimes_{A_1} \mathcal{O}_{J_1,\mathcal{F}_1}) \otimes_{\mathcal{O}_{J_1,\mathcal{F}_1}} (\mathcal{F}_1 \otimes_{A_2} \mathcal{O}_{J_2,\mathcal{F}_2})$ and $(\mathcal{F} \otimes_{A_1} \mathcal{F}_1) \otimes_{A_2} \mathcal{O}_{J_2,\mathcal{F}_2}$ as $\mathbb{T}$-equivariant correspondences $A \rightsquigarrow \mathcal{O}_{J_2,\mathcal{F}_2}$. Associators and unit transformations give a canonical $\mathbb{T}$-equivariant isomorphism between these correspondences. This isomorphism also intertwines the representations of $\mathcal{E}$. Hence it is an isomorphism of correspondences $\mathcal{O}_{J,\mathcal{F}} \rightsquigarrow \mathcal{O}_{J_2,\mathcal{F}_2}$. These canonical isomorphisms satisfy the coherence conditions for a homomorphism of bicategories in Definition A.5.

The relative Cuntz-Pimsner algebra $\mathcal{O}_{J,\mathcal{E}}$ is the crossed product $\mathcal{O}^0_{J,\mathcal{E}} \rtimes \mathcal{O}^1_{J,\mathcal{E}}$ by Theorem 2.17. So $\mathcal{O}_{J,\mathcal{E}}$ with the gauge $\mathbb{T}$-action and the Hilbert $\mathcal{O}^0_{J,\mathcal{E}}$-bimodule $\mathcal{O}^1_{J,\mathcal{E}}$ contain the same amount of information. We now study the construction that sends $(A, \mathcal{E}, J)$ to the Hilbert $\mathcal{O}^0_{J,\mathcal{E}}$-bimodule $\mathcal{O}^1_{J,\mathcal{E}}$. The appropriate bicategory of Hilbert bimodules is a sub-bicategory of $\mathfrak{C}^{\mathbb{N}}_{\mathrm{pr}}$:

DEFINITION 3.3. Let $\mathfrak{C}^{\mathbb{N}}_{\mathrm{pr},*} \subseteq \mathfrak{C}^{\mathbb{N}}_{\mathrm{pr}}$ be the full sub-bicategory whose objects are triples $(B, \mathcal{G}, I_{\mathcal{G}})$, where $\mathcal{G}$ is a Hilbert B-bimodule and $I_{\mathcal{G}}$ is Katsura's ideal for $\mathcal{G}$, which is also equal to the range ideal $\langle \mathcal{G} \mid \mathcal{G} \rangle$ of the left inner product on $\mathcal{G}$. The arrows and 2-arrows among objects of $\mathfrak{C}^{\mathbb{N}}_{\mathrm{pr},*}$ are the same as in $\mathfrak{C}^{\mathbb{N}}_{\mathrm{pr}}$, including the condition $I_{\mathcal{E}}\mathcal{F} \subseteq \mathcal{F} I_{\mathcal{G}}$ for covariant correspondences.

When we restrict the relative Cuntz-Pimsner algebra construction $\mathfrak{C}^{\mathbb{N}}_{\mathrm{pr}} \to \mathfrak{C}^{\mathbb{T}}$ to $\mathfrak{C}^{\mathbb{N}}_{\mathrm{pr},*}$, we get the (partial) crossed product construction for Hilbert bimodules by Proposition 2.19. Thus Theorem 3.2 also completes the crossed product for Hilbert bimodules to a functor $\mathfrak{C}^{\mathbb{N}}_{\mathrm{pr},*} \to \mathfrak{C}^{\mathbb{T}}$.

The map that sends $(A, \mathcal{E}, J)$ to $(\mathcal{O}^0_{J,\mathcal{E}}, \mathcal{O}^1_{J,\mathcal{E}}, I_{\mathcal{O}^1_{J,\mathcal{E}}})$ is part of a functor $\mathfrak{C}^{\mathbb{N}}_{\mathrm{pr}} \to \mathfrak{C}^{\mathbb{N}}_{\mathrm{pr},*}$ which, when composed with the crossed product functor $\mathfrak{C}^{\mathbb{N}}_{\mathrm{pr},*} \to \mathfrak{C}^{\mathbb{T}}$, gives the relative Cuntz-Pimsner algebra functor of Theorem 3.2. We do not prove this now because it follows from our main result. The key step is the following universal property of $(\mathcal{O}^0_{J,\mathcal{E}}, \mathcal{O}^1_{J,\mathcal{E}}, I_{\mathcal{O}^1_{J,\mathcal{E}}})$:

PROPOSITION 3.4. *Let* $(A, \mathcal{E}, J)$ *and* $(B, \mathcal{G}, I_{\mathcal{G}})$ *be objects of* $\mathfrak{C}^{\mathbb{N}}_{\mathrm{pr}}$ *and* $\mathfrak{C}^{\mathbb{N}}_{\mathrm{pr},*}$, *respectively. Let*

$$\upsilon_{(A,\mathscr{E},J)}\colon (A,\mathscr{E},J) \longrightarrow (\mathcal{O}^0_{J,\mathscr{E}}, \mathcal{O}^1_{J,\mathscr{E}}, I_{\mathcal{O}^1_{J,\mathscr{E}}})$$

be the covariant correspondence from Example 2.23. Composition with $\upsilon_{(A,\mathscr{E},J)}$ induces a groupoid equivalence

$$\mathfrak{C}^{\mathbb{N}}_{\mathrm{pr}}\big((A,\mathscr{E},J),(B,\mathscr{G},I_{\mathscr{G}})\big) \simeq \mathfrak{C}^{\mathbb{N}}_{\mathrm{pr},*}\big((\mathcal{O}^0_{J,\mathscr{E}}, \mathcal{O}^1_{J,\mathscr{E}}, I_{\mathcal{O}^1_{J,\mathscr{E}}}),(B,\mathscr{G},I_{\mathscr{G}})\big).$$

Recall that $\mathfrak{C}^{\mathbb{N}}_{\mathrm{pr}}((A,\mathscr{E},J),(A_1,\mathscr{E}_1,J_1))$ for objects $(A,\mathscr{E},J)$ and $(A_1,\mathscr{E}_1,J_1)$ of $\mathfrak{C}^{\mathbb{N}}_{\mathrm{pr}}$ denotes the groupoid with arrows $(A,\mathscr{E},J)\to(A_1,\mathscr{E}_1,J_1)$ as objects and 2-arrows among them as arrows.

Proof. We begin with an auxiliary construction. Proposition 2.19 identifies $\mathcal{O}_{I_{\mathscr{G}},\mathscr{G}} \cong B \rtimes \mathscr{G}$ as $\mathbb{Z}$-graded C*-algebras. In particular, $\mathcal{O}^0_{I_{\mathscr{G}},\mathscr{G}} \cong B$ and $\mathcal{O}^1_{I_{\mathscr{G}},\mathscr{G}} \cong \mathscr{G}$, $\mathcal{O}^{-1}_{I_{\mathscr{G}},\mathscr{G}} \cong \mathscr{G}^*$ as Hilbert B-bimodules. Let $(\mathscr{F},u)$ be a proper covariant correspondence $(A,\mathscr{E},J)\to(B,\mathscr{G},I_{\mathscr{G}})$. It induces a proper, $\mathbb{T}$-equivariant correspondence $\mathcal{O}_{\mathscr{F},V} = \bigoplus_{n\in\mathbb{Z}} \mathcal{O}^n_{\mathscr{F},V}$ from $\mathcal{O}_{J,\mathscr{E}}$ to $\mathcal{O}_{I_{\mathscr{G}},\mathscr{G}}$ by Proposition 2.21. By construction, $\mathcal{O}^n_{\mathscr{F},V} = \mathscr{F}\otimes_B \mathcal{O}^n_{I_{\mathscr{G}},\mathscr{G}}$. Thus $\mathcal{O}^0_{\mathscr{F},V} = \mathscr{F}\otimes_B \mathcal{O}^0_{I_{\mathscr{G}},\mathscr{G}} \cong \mathscr{F}\otimes_B B \cong \mathscr{F}$ and $\mathcal{O}^1_{\mathscr{F},V} = \mathscr{F}\otimes_B \mathcal{O}^1_{I_{\mathscr{G}},\mathscr{G}} \cong \mathscr{F}\otimes_B\mathscr{G}$. The left action on $\mathcal{O}_{\mathscr{F},V}$ is a nondegenerate, $\mathbb{T}$-equivariant *-homomorphism $\mathcal{O}_{J,\mathscr{E}} \to \mathbb{K}(\mathcal{O}_{\mathscr{F},V})$. So $\mathcal{O}^0_{J,\mathscr{E}}$ acts on $\mathcal{O}_{\mathscr{F},V}$ by grading-preserving operators. Restricting to the degree-0 part, we get a nondegenerate *-homomorphism $\mathcal{O}^0_{J,\mathscr{E}} \to \mathbb{K}(\mathcal{O}^0_{\mathscr{F},V}) \cong \mathbb{K}(\mathscr{F})$. Let $\mathscr{F}^\#$ be $\mathscr{F}$ viewed as a correspondence $\mathcal{O}^0_{J,\mathscr{E}} \rightsquigarrow B$ in this way.

We now construct an isomorphism of correspondences

$$u^\#\colon \mathcal{O}^1_{J,\mathscr{E}} \otimes_{\mathcal{O}^0_{J,\mathscr{E}}} \mathscr{F}^\# \Rightarrow \mathscr{F}^\#\otimes_B\mathscr{G}.$$

We need two descriptions of $u^\#$. The first shows that it is unitary, the second that it intertwines the left actions of $\mathcal{O}^0_{J,\mathscr{E}}$. The first formula for $u^\#$ uses Lemma 2.16, which gives unitary Hilbert B-module maps

$$\mathcal{O}^1_{J,\mathscr{E}} \otimes_{\mathcal{O}^0_{J,\mathscr{E}}} \mathscr{F}^\# \cong \mathscr{E}\otimes_A \mathcal{O}^0_{J,\mathscr{E}} \otimes_{\mathcal{O}^0_{J,\mathscr{E}}} \mathscr{F}^\# \cong \mathscr{E}\otimes_A\mathscr{F}.$$

Composing with $u\colon \mathscr{E}\otimes_A\mathscr{F} \Rightarrow \mathscr{F}\otimes_B\mathscr{G}$ gives the desired unitary $u^\#$. The second formula for $u^\#$ restricts the left action of $\mathcal{O}_{J,\mathscr{E}}$ on $\mathcal{O}_{\mathscr{F},V}$ to a multiplication map

$$\mathcal{O}^1_{J,\mathscr{E}} \otimes_{\mathcal{O}^0_{J,\mathscr{E}}} \mathscr{F}^\# = \mathcal{O}^1_{J,\mathscr{E}} \otimes_{\mathcal{O}^0_{J,\mathscr{E}}} \mathcal{O}^0_{\mathscr{F},V} \longrightarrow \mathcal{O}^1_{\mathscr{F},V} \cong \mathscr{F}^\#\otimes_B\mathscr{G}. \tag{3.5}$$

This is manifestly $\mathcal{O}^0_{J,\mathscr{E}}$-linear because the isomorphism $\mathscr{F}^\#\otimes_B \mathcal{O}^n_{I_{\mathscr{G}},\mathscr{G}} \cong \mathcal{O}^n_{\mathscr{F},V}$ is by right multiplication and so intertwines the left actions of $\mathcal{O}^0_{J,\mathscr{E}}$. The map in (3.5) maps $t_J(\xi)\otimes\eta \mapsto u(\xi\otimes\eta)$ for all $\xi\in\mathcal{E}$, $\eta\in\mathcal{F}$. This determines it by Lemma 2.16. So both constructions give the same map $u^{\#}$.

We claim that $I_{\mathcal{O}^1_{J,\mathcal{E}}}\cdot\mathcal{F}^{\#}\subseteq\mathcal{F}^{\#}\cdot I_{\mathcal{G}}$ holds, so that the pair $(\mathcal{F}^{\#},u^{\#})$ is a proper covariant correspondence from $(\mathcal{O}^0_{J,\mathcal{E}},\mathcal{O}^1_{J,\mathcal{E}},I_{\mathcal{O}^1_{J,\mathcal{E}}})$ to $(B,\mathcal{G},I_{\mathcal{G}})$. The ideal $I_{\mathcal{O}^1_{J,\mathcal{E}}}$ is equal to the range of the left inner product on $\mathcal{O}^1_{J,\mathcal{E}}$. Using the Fell bundle structure, we may rewrite this as $\mathcal{O}^1_{J,\mathcal{E}}\cdot\mathcal{O}^{-1}_{J,\mathcal{E}}$. Thus

$$I_{\mathcal{O}^1_{J,\mathcal{E}}}\cdot\mathcal{O}^0_{\mathcal{F},V}=\mathcal{O}^1_{J,\mathcal{E}}\cdot\mathcal{O}^{-1}_{J,\mathcal{E}}\cdot\mathcal{O}^0_{\mathcal{F},V}\subseteq\mathcal{O}^1_{J,\mathcal{E}}\cdot\mathcal{O}^{-1}_{\mathcal{F},V}=\mathcal{E}\cdot\mathcal{O}^0_{J,\mathcal{E}}\cdot\mathcal{O}^{-1}_{\mathcal{F},V}=\mathcal{E}\cdot\mathcal{O}^{-1}_{\mathcal{F},V}.$$

The product $\mathcal{E}\cdot\mathcal{O}^{-1}_{\mathcal{F},V}$ uses the representation of $\mathcal{E}$ on $\mathcal{O}_{\mathcal{F},V}$ built in the proof of Proposition 2.21. So $\mathcal{E}\cdot\mathcal{O}^{-1}_{\mathcal{F},V}$ is the image of the map

$$\mathcal{E}\otimes_A\mathcal{F}\otimes_B\mathcal{G}^*\cong\mathcal{F}\otimes_B\mathcal{G}\otimes_B\mathcal{G}^*=\mathcal{F}\cdot I_{\mathcal{G}}.$$

So $I_{\mathcal{O}^1_{J,\mathcal{E}}}\cdot\mathcal{O}^0_{\mathcal{F},V}\subseteq\mathcal{F}\cdot I_{\mathcal{G}}$ as claimed. We have turned a proper covariant correspondence $(\mathcal{F},u)$ from $(A,\mathcal{E},J)$ to $(B,\mathcal{G},I_{\mathcal{G}})$ into a proper covariant correspondence $(\mathcal{F}^{\#},u^{\#})$ from $(\mathcal{O}^0_{J,\mathcal{E}},\mathcal{O}^1_{J,\mathcal{E}},I_{\mathcal{O}^1_{J,\mathcal{E}}})$ to $(B,\mathcal{G},I_{\mathcal{G}})$.

Conversely, take a proper covariant correspondence

$$(\mathcal{F},u)\colon(\mathcal{O}^0_{J,\mathcal{E}},\mathcal{O}^1_{J,\mathcal{E}},I_{\mathcal{O}^1_{J,\mathcal{E}}})\longrightarrow(B,\mathcal{G},I_{\mathcal{G}}).$$

Composing it with $\upsilon_{(A,\mathcal{E},J)}$ gives a proper covariant correspondence from $(A,\mathcal{E},J)$ to $(B,\mathcal{G},I_{\mathcal{G}})$. We now simplify this product of covariant correspondences. The underlying correspondence $A\to\mathcal{O}^0_{J,\mathcal{E}}$ in $\upsilon_{(A,\mathcal{E},J)}$ is $\mathcal{O}^0_{J,\mathcal{E}}$, and the isomorphism $\mathcal{E}\otimes_A\mathcal{O}^0_{J,\mathcal{E}}\cong\mathcal{O}^0_{J,\mathcal{E}}\otimes_{\mathcal{O}^0_{J,\mathcal{E}}}\mathcal{O}^1_{J,\mathcal{E}}=\mathcal{O}^1_{J,\mathcal{E}}$ is the one from Lemma 2.16. We identify the tensor product $\mathcal{O}^0_{J,\mathcal{E}}\otimes_{\mathcal{O}^0_{J,\mathcal{E}}}\mathcal{F}$ with $\mathcal{F}$ by the canonical map. Thus the product of $(\mathcal{F},u)$ with $\upsilon_{(A,\mathcal{E},J)}$ is canonically isomorphic to a covariant correspondence $(\mathcal{F}^{\flat},u^{\flat})$ with underlying correspondence $\mathcal{F}^{\flat}=\mathcal{F}\colon A\rightsquigarrow B$ with the left A-action through $\pi_J\colon A\to\mathcal{O}^0_{J,\mathcal{E}}$. The isomorphism $u^{\flat}\colon\mathcal{E}\otimes_A\mathcal{F}^{\flat}\Rightarrow\mathcal{F}^{\flat}\otimes_B\mathcal{G}$ is the composite of the given isomorphism $u\colon\mathcal{O}^1_{J,\mathcal{E}}\otimes_{\mathcal{O}^0_{J,\mathcal{E}}}\mathcal{F}\Rightarrow\mathcal{F}\otimes_B\mathcal{G}$ with the isomorphism $\mathcal{E}\otimes_A\mathcal{O}^0_{J,\mathcal{E}}\cong\mathcal{O}^1_{J,\mathcal{E}}$ from Lemma 2.16.

Now let $(\mathcal{F},u)$ be a proper covariant correspondence from $(A,\mathcal{E},J)$ to $(B,\mathcal{G},I_{\mathcal{G}})$. We claim that

$$(\mathcal{F}^{\#\flat},u^{\#\flat})=(\mathcal{F},u). \tag{3.6}$$

By construction, the underlying Hilbert B-module of $\mathcal{F}^{\#\flat}$ is $\mathcal{F}$. We even have $\mathcal{F}^{\#\flat}=\mathcal{F}$ as correspondences $A\rightsquigarrow B$, that is, the left $\mathcal{O}^0_{J,\mathcal{E}}$-action on $\mathcal{F}^{\#}$ composed with $\pi_J\colon A\to\mathcal{O}^0_{J,\mathcal{E}}$ is the original action of A. The isomorphism

$\mathcal{E} \otimes_A \mathcal{O}^0_{J,\mathcal{E}} \cong \mathcal{O}^1_{J,\mathcal{E}}$ is used both to get $u^\#$ from u and to get $u^{\#\flat}$ from $u^\#$. Unravelling this shows that $u^{\#\flat} = u$.

Next we claim that the map that sends a proper covariant correspondence

$$(\mathcal{F}, u)\colon (\mathcal{O}^0_{J,\mathcal{E}}, \mathcal{O}^1_{J,\mathcal{E}}, I_{\mathcal{O}^1_{J,\mathcal{E}}}) \longrightarrow (B, \mathcal{G}, I_{\mathcal{G}})$$

to $(\mathcal{F}^\flat, u^\flat)$ is injective. This claim and (3.6) imply $(\mathcal{F}^{\flat\#}, u^{\flat\#}) = (\mathcal{F}, u)$, that is, our two operations are inverse to each other. To prove injectivity, we use Proposition 2.21 to build a correspondence $\mathcal{O}_{\mathcal{F},u}\colon \mathcal{O}_{J,\mathcal{E}} \rightsquigarrow \mathcal{O}_{I_{\mathcal{G}},\mathcal{G}}$ from $(\mathcal{F}, u)$. This correspondence determines $(\mathcal{F}, u)$: we can get back $\mathcal{F}$ as its degree-0 part because $\mathcal{O}_{I_{\mathcal{G}},\mathcal{G}} = B \rtimes \mathcal{G}$, and because u and the left $\mathcal{O}^0_{J,\mathcal{E}}$-module structure on $\mathcal{F}$ are both contained in the left $\mathcal{O}_{J,\mathcal{E}}$-module structure on $\mathcal{O}_{\mathcal{F},u}$. An $\mathcal{O}_{J,\mathcal{E}}$-module structure on $\mathcal{O}_{I_{\mathcal{G}},\mathcal{G}}$ is already determined by a representation of $(A, \mathcal{E})$. Since $\mathcal{O}^n_{I_{\mathcal{G}},\mathcal{G}} = \mathcal{O}^0_{I_{\mathcal{G}},\mathcal{G}} \cdot \mathcal{O}^n_{I_{\mathcal{G}},\mathcal{G}}$, this representation is determined by its restriction to $\mathcal{O}^0_{I_{\mathcal{G}},\mathcal{G}} \cong B$. And $(\mathcal{F}^\flat, u^\flat)$ determines the representation of $(A, \mathcal{E})$ on B. Thus $(\mathcal{F}^\flat, u^\flat)$ determines $(\mathcal{F}, u)$.

The constructions of $(\mathcal{F}^\#, u^\#)$ and $(\mathcal{F}^\flat, u^\flat)$ are clearly natural for isomorphisms of covariant correspondences. So they form an isomorphism of categories

$$\mathfrak{C}^{\mathbb{N}}_{\text{pr}}\big((A, \mathcal{E}, J), (B, \mathcal{G}, I_{\mathcal{G}})\big) \cong \mathfrak{C}^{\mathbb{N}}_{\text{pr},*}\big((\mathcal{O}^0_{J,\mathcal{E}}, \mathcal{O}^1_{J,\mathcal{E}}, I_{\mathcal{O}^1_{J,\mathcal{E}}}), (B, \mathcal{G})\big).$$

One piece in this isomorphism is naturally equivalent to the functor that composes with $\upsilon_{(A,\mathcal{E},J)}$. Hence this functor is an equivalence of categories, as asserted.

4. The reflector from correspondences to Hilbert bimodules

We now strengthen Proposition 3.4 using some general results on adjunctions of homomorphisms between bicategories. We first recall the related and better known results about ordinary categories and functors.

Let $\mathcal{C}$ and $\mathcal{B}$ be categories. Let $R\colon \mathcal{C} \to \mathcal{B}$ be a functor and $b \in \operatorname{ob}\mathcal{B}$. An object $c \in \operatorname{ob}\mathcal{C}$ with an arrow $\upsilon\colon b \to R(c)$ is called a *universal arrow* from b to R if, for each $x \in \operatorname{ob}\mathcal{C}$ and each $f \in \mathcal{B}(b, R(x))$, there is a unique $g \in \mathcal{C}(c, x)$ with $R(g) \circ \upsilon = f$. Equivalently, the maps

$$\mathcal{C}(c, x) \longrightarrow \mathcal{B}(b, R(x)), \qquad g \mapsto R(g) \circ \upsilon, \tag{4.1}$$

are bijective for all $x \in \operatorname{ob}\mathcal{C}$. The functor R has a left adjoint $L\colon \mathcal{B} \to \mathcal{C}$ if and only if such universal arrows exist for all $x \in \operatorname{ob}\mathcal{C}$. The left adjoint functor $L\colon \mathcal{B} \to \mathcal{C}$ is uniquely determined up to natural isomorphism. It maps $b \mapsto c$ on objects, and the isomorphisms (4.1) become natural in both b and x when we replace c by $L(b)$. An adjunction between L and R may also be expressed through its unit and counit, that is, natural transformations $L \circ R \Rightarrow \mathrm{id}_{\mathscr{C}}$ and $\mathrm{id}_{\mathscr{B}} \Rightarrow R \circ L$ such that the induced transformations $L \Rightarrow L \circ R \circ L \Rightarrow L$ and $R \Rightarrow R \circ L \circ R \Rightarrow R$ are unit transformations.

A subcategory $\mathscr{C} \subseteq \mathscr{B}$ is called *reflective* if the inclusion functor $R: \mathscr{C} \to \mathscr{B}$ has a left adjoint $L: \mathscr{B} \to \mathscr{C}$. The functor L is called a *reflector*. The case we care about is a bicategorical version of a full subcategory. If $\mathscr{C} \subseteq \mathscr{B}$ is a full subcategory, then we may choose $L \circ R$ to be the identity functor on $\mathscr{C}$ and the counit $L \circ R \Rightarrow \mathrm{id}_{\mathscr{C}}$ to be the unit natural transformation.

Fiore [9] carries the story of adjoint functors over to homomorphisms between 2-categories (which he calls "pseudo functors"), that is, bicategories where the associators and unitors are identity 2-arrows. The bicategories we need are not 2-categories. But any bicategory is equivalent to a 2-category by MacLane's Coherence Theorem. Hence Fiore's definitions and results apply in bicategories as well. We shorten notation by speaking of "universal" arrows and "adjunctions" instead of "biuniversal" arrows and "biadjunctions." A 2-category is also a category with some extra structure. So leaving out the prefix "bi" may cause confusion in that setting. But it will always be clear whether we mean the categorical or bicategorical notions.

Definition 4.2 ([9, Definition 9.4]). Let $\mathscr{B}$ and $\mathscr{C}$ be bicategories, $R: \mathscr{C} \to \mathscr{B}$ a homomorphism, and $b \in \operatorname{ob}\mathscr{B}$. Let $c \in \operatorname{ob}\mathscr{C}$ and let $g: b \to R(c)$ be an arrow in $\mathscr{B}$. The pair (c, g) is a *universal arrow* from b to R if, for every $x \in \operatorname{ob}\mathscr{C}$, the following functor is an equivalence of categories:

$$g^*: \mathscr{C}(c, x) \longrightarrow \mathscr{B}(b, R(x)), \qquad f \mapsto R(f) \cdot g, \quad w \mapsto R(w) \bullet 1_g.$$

Universal arrows are called *left biliftings* by Street [22].

We can now reformulate Proposition 3.4:

Proposition 4.3. *Let $(A, \mathscr{E}, J) \in \operatorname{ob}\mathfrak{C}^{\mathbb{N}}_{\mathrm{pr}}$. The covariant correspondence $\upsilon_{(A,\mathscr{E},J)}$ from $(A, \mathscr{E}, J)$ to $(\mathcal{O}^0_{J,\mathscr{E}}, \mathcal{O}^1_{J,\mathscr{E}}, I_{\mathscr{E}})$ is a universal arrow from $(A, \mathscr{E}, J)$ to the inclusion homomorphism $\mathfrak{C}^{\mathbb{N}}_{\mathrm{pr},*} \to \mathfrak{C}^{\mathbb{N}}_{\mathrm{pr}}$.*

There are two alternative definitions of adjunctions, based on equivalences between morphism categories or on units and counits. These are spelled out, respectively, by Fiore in [9, Definition 9.8] and by Gurski in [12, Definition 2.1]. We shall use Fiore's definition.

Definition 4.4 ([9, Definition 9.8]). Let $\mathscr{B}$ and $\mathscr{C}$ be bicategories. An *adjunction* between them consists of

- two homomorphisms $L: \mathscr{B} \to \mathscr{C}$, $R: \mathscr{C} \to \mathscr{B}$;

- equivalences of categories

$$\varphi_{b,c}\colon \mathscr{C}(L(b), c) \simeq \mathscr{B}(b, R(c))$$

for all $b \in \operatorname{ob}\mathscr{B}$, $c \in \operatorname{ob}\mathscr{C}$;
- natural equivalences of functors

$$\begin{array}{ccccc}
\mathscr{C}(L(b_1), c_1) & \xrightarrow{f^*} & \mathscr{C}(L(b_2), c_1) & \xrightarrow{g_*} & \mathscr{C}(L(b_2), c_2) \\
\Big\downarrow{\scriptstyle \varphi_{b_1,c_1}} & & \Swarrow & & \Big\downarrow{\scriptstyle \varphi_{b_2,c_2}} \\
\mathscr{B}(b_1, R(c_1)) & \xrightarrow[f^*]{} & \mathscr{B}(b_2, R(c_1)) & \xrightarrow[g_*]{} & \mathscr{B}(b_2, R(c_2))
\end{array}$$

for all arrows $f\colon b_2 \to b_1$, $g\colon c_1 \to c_2$ in $\mathscr{B}$ and $\mathscr{C}$.

These are subject to a coherence condition. In brief, the functors $\varphi_{b,c}$ and the natural equivalences form a transformation between the two homomorphisms

$$\mathscr{B}^{\mathrm{op}} \times \mathscr{C} \rightrightarrows \mathbf{Cat}, \qquad (b, c) \mapsto \mathscr{C}(L(b), c),\ \mathscr{B}(b, R(c)).$$

Here **Cat** is the bicategory of categories (see Example A.2).

THEOREM 4.5 ([9, Theorem 9.17]). *Let $\mathscr{B}$ and $\mathscr{C}$ be bicategories and let $R\colon \mathscr{C} \to \mathscr{B}$ be a homomorphism. It is part of an adjunction if and only if there are universal arrows from c to R for each object $c \in \operatorname{ob}\mathscr{C}$.*

More precisely, let $c_b \in \operatorname{ob}\mathscr{C}$ and $\upsilon_b\colon b \to R(c_b)$ for $b \in \operatorname{ob}\mathscr{C}$ be universal arrows from b to R. Then there is an adjoint homomorphism $L\colon \mathscr{B} \to \mathscr{C}$ that maps $b \mapsto c_b$ on objects. In particular, this assignment is part of a homomorphism of bicategories.

THEOREM 4.6 ([9, Theorem 9.20]). *Two left adjoints $L, L'\colon \mathscr{B} \rightrightarrows \mathscr{C}$ of $R\colon \mathscr{C} \to \mathscr{B}$ are equivalent, that is, there are transformations $L \Rightarrow L'$ and $L' \Rightarrow L$ that are inverse to each other up to invertible modifications.*

Using these general theorems, we may strengthen Proposition 3.4 (in the form of Proposition 4.3) to an adjunction theorem:

COROLLARY 4.7. *The sub-bicategory $\mathfrak{C}^{\mathbb{N}}_{\mathrm{pr},*} \subseteq \mathfrak{C}^{\mathbb{N}}_{\mathrm{pr}}$ is reflective, that is, the inclusion homomorphism $R\colon \mathfrak{C}^{\mathbb{N}}_{\mathrm{pr},*} \to \mathfrak{C}^{\mathbb{N}}_{\mathrm{pr}}$ has a left adjoint (reflector) $L\colon \mathfrak{C}^{\mathbb{N}}_{\mathrm{pr}} \to \mathfrak{C}^{\mathbb{N}}_{\mathrm{pr},*}$. On objects, this adjoint homomorphism maps*

$$(A, \mathscr{E}, J) \mapsto (\mathcal{O}^0_{J,\mathscr{E}}, \mathcal{O}^1_{J,\mathscr{E}}, I_{\mathcal{O}^1_{J,\mathscr{E}}}).$$

The homomorphism L is determined uniquely up to equivalence by Theorem 4.6. So we have characterised the construction of relative Cuntz-Pimsner algebras in bicategorical terms, as the reflector for the full sub-bicategory $\mathfrak{C}^{\mathbb{N}}_{\mathrm{pr},*} \subseteq \mathfrak{C}^{\mathbb{N}}_{\mathrm{pr}}$. By Corollary 4.7, the relative Cuntz-Pimsner algebra construction is part of a homomorphism $L\colon \mathfrak{C}^{\mathbb{N}}_{\mathrm{pr}} \to \mathfrak{C}^{\mathbb{N}}_{\mathrm{pr},*}$. For instance, this implies the following:

COROLLARY 4.8. *The relative Cuntz-Pimsner algebras $\mathcal{O}_{J,\mathcal{E}}$ and $\mathcal{O}_{J_1,\mathcal{E}_1}$ are Morita equivalent if there is a Morita equivalence $\mathcal{F}$ between $\mathcal{E}$ and $\mathcal{E}_1$ as in* [18, Definition 2.1] *with $J \cdot \mathcal{F} = \mathcal{F} \cdot J_1$.*

The proof of Theorem 4.5 also describes the adjoint functor. We now describe the reflector $L\colon \mathfrak{C}^{\mathbb{N}}_{\mathrm{pr}} \to \mathfrak{C}^{\mathbb{N}}_{\mathrm{pr},*}$ explicitly, thereby explaining part of the proof of Theorem 4.5. Much of the work in this proof is needed to check that various diagrams of 2-arrows commute. We do not repeat these computations here.

The homomorphism L maps $(A, \mathcal{E}, J) \mapsto (\mathcal{O}^0_{J,\mathcal{E}}, \mathcal{O}^1_{J,\mathcal{E}}, I_{\mathcal{O}^1_{J,\mathcal{E}}})$ on objects. Let $(A, \mathcal{E}, J)$ and $(A_1, \mathcal{E}_1, J_1)$ be objects of $\mathfrak{C}^{\mathbb{N}}_{\mathrm{pr}}$ and let $(\mathcal{F}, u)\colon (A, \mathcal{E}, J) \to (A_1, \mathcal{E}_1, J_1)$ be proper covariant correspondences. We use the notation of the proof of Proposition 3.4 and write $\bar{\iota}_{\mathcal{E}_1}$ for the canonical isomorphism $\mathcal{E}_1 \otimes_{A_1} \mathcal{O}^0_{J_1,\mathcal{E}_1} \cong \mathcal{O}^1_{J_1,\mathcal{E}_1} \otimes_{\mathcal{O}^0_{J_1,\mathcal{E}_1}} \mathcal{O}^0_{J_1,\mathcal{E}_1}$ from Lemma 2.16, which is the covariance part of $\upsilon_{(A_1,\mathcal{E}_1,J_1)}$. Let

$$L(\mathcal{F}, u)\colon (\mathcal{O}^0_{J,\mathcal{E}}, \mathcal{O}^1_{J,\mathcal{E}}, I_{\mathcal{O}^1_{J,\mathcal{E}}}) \longrightarrow (\mathcal{O}^0_{J_1,\mathcal{E}_1}, \mathcal{O}^1_{J_1,\mathcal{E}_1}, I_{\mathcal{O}^1_{J_1,\mathcal{E}_1}}),$$
$$L(\mathcal{F}, u) := \big((\mathcal{F} \otimes_{A_1} \mathcal{O}^0_{J_1,\mathcal{E}_1})^{\#}, (u \bullet \bar{\iota}_{\mathcal{E}_1})^{\#}\big).$$

In other words, we first compose $(\mathcal{F}, u)$ with $\upsilon_{(A_1,\mathcal{E}_1,J_1)}$ to get a covariant correspondence $(\mathcal{F} \otimes_{A_1} \mathcal{O}^0_{J_1,\mathcal{E}_1}, u \bullet \bar{\iota}_{\mathcal{E}_1})$ from $(A, \mathcal{E}, J)$ to $(\mathcal{O}^0_{J_1,\mathcal{E}_1}, \mathcal{O}^1_{J_1,\mathcal{E}_1}, I_{\mathcal{O}^1_{J_1,\mathcal{E}_1}})$ and then apply the equivalence in Proposition 3.4.

The construction on covariant correspondences above is clearly "natural", that is, functorial for isomorphisms. Explicitly, L maps an isomorphism of covariant correspondences $w\colon (\mathcal{F}, u) \Rightarrow (\mathcal{F}', u')$ to

$$L(w) := (w \otimes 1_{\mathcal{O}^0_{J_1,\mathcal{E}_1}})^{\#}\colon L(\mathcal{F}, u) \Rightarrow L(\mathcal{F}', u').$$

To make L a homomorphism, we also need compatibility data for units and composition of arrows. The construction of L above maps the identity covariant correspondence on $(A, \mathcal{E}, J)$ to $\upsilon^{\#}_{(A,\mathcal{E},J)}\colon (\mathcal{O}^0_{J,\mathcal{E}}, \mathcal{O}^1_{J,\mathcal{E}}, I_{\mathcal{E}}) \to (\mathcal{O}^0_{J,\mathcal{E}}, \mathcal{O}^1_{J,\mathcal{E}}, I_{\mathcal{E}})$. This is canonically isomorphic to the identity covariant correspondence on $(\mathcal{O}^0_{J,\mathcal{E}}, \mathcal{O}^1_{J,\mathcal{E}}, I_{\mathcal{E}})$ because the equivalence in Proposition 3.4 is by composition with $\upsilon_{(A,\mathcal{E},J)}$. This is the unit part in our homomorphism L.

Let $(\mathcal{F}, u)\colon (A, \mathcal{E}, J) \to (A_1, \mathcal{E}_1, J_1)$ and $(\mathcal{F}_1, u_1)\colon (A_1, \mathcal{E}_1, J_1) \to (A_2, \mathcal{E}_2, J_2)$ be proper covariant correspondences. Then the homomorphism L

contains isomorphisms of covariant correspondences

$$\lambda\big((\mathscr{F}, u), (\mathscr{F}_1, u_1)\big)\colon L(\mathscr{F}, u) \circ L(\mathscr{F}_1, u_1) \Rightarrow L\big((\mathscr{F}, u) \circ (\mathscr{F}_1, u_1)\big), \quad (4.9)$$

which are natural for isomorphisms of covariant correspondences and satisfy some coherence conditions when we compose three covariant correspondences or compose with identity covariant correspondences. We take λ to be the isomorphism

$$(\mathscr{F}_0 \otimes_{A_1} \mathcal{O}^0_{J_1,\mathscr{E}_1}) \otimes_{\mathcal{O}^0_{J_1,\mathscr{E}_1}} (\mathscr{F}_1 \otimes_{A_2} \mathcal{O}^0_{J_2,\mathscr{E}_2}) \cong (\mathscr{F}_0 \otimes_{A_1} \mathscr{F}_1) \otimes_{A_2} \mathcal{O}^0_{J_2,\mathscr{E}_2}$$

given by the left action of $\mathcal{O}^0_{J_1,\mathscr{E}_1}$ on $(\mathscr{F}_1 \otimes_{A_2} \mathcal{O}^0_{J_2,\mathscr{E}_2})$ that is constructed in the proof of Proposition 2.21.

The proof of Theorem 4.5 builds λ using only the universality of the arrows $\upsilon_{(A,\mathscr{E},J)}$. By the equivalence of categories in Proposition 3.4, whiskering (horizontal composition) with $\upsilon_{(A,\mathscr{E},J)}$ maps isomorphisms as in (4.9) bijectively to isomorphisms

$$\upsilon_{(A,\mathscr{E},J)} \circ L(\mathscr{F}, u) \circ L(\mathscr{F}_1, u_1) \Rightarrow \upsilon_{(A,\mathscr{E},J)} \circ L\big((\mathscr{F}, u) \circ (\mathscr{F}_1, u_1)\big). \quad (4.10)$$

The construction of L implies $\upsilon_{(A,\mathscr{E},J)} \circ L(\mathscr{F}, u) \circ L(\mathscr{F}_1, u_1) \cong (\mathscr{F}, u) \circ \upsilon_{(A_1,\mathscr{E}_1,J_1)} \circ L(\mathscr{F}_1, u_1) \cong (\mathscr{F}, u) \circ (\mathscr{F}_1, u_1) \circ \upsilon_{(A_2,\mathscr{E}_2,J_2)}$ and $\upsilon_{(A,\mathscr{E},J)} \circ L\big((\mathscr{F}, u) \circ (\mathscr{F}_1, u_1)\big) \cong \big((\mathscr{F}, u) \circ (\mathscr{F}_1, u_1)\big) \circ \upsilon_{(A_2,\mathscr{E}_2,J_2)}$, where we disregard associators. Hence there is a canonical isomorphism of covariant correspondences as in (4.10). This Ansatz produces the same isomorphisms λ as above. We have now described the data of the homomorphism L. Fiore's arguments in [9] show that it is indeed a homomorphism.

PROPOSITION 4.11. *The composite of L and the crossed product homomorphism $\mathfrak{C}^{\mathbb{N}}_{\mathrm{pr},*} \to \mathfrak{C}^{\mathbb{T}}$ is naturally isomorphic to the homomorphism $\mathfrak{C}^{\mathbb{N}}_{\mathrm{pr}} \to \mathfrak{C}^{\mathbb{T}}$ of Theorem 3.2.*

PROOF. Our homomorphisms agree on objects by Proposition 2.18. The proof of Proposition 3.4 constructed the covariant correspondence $(\mathscr{F}^{\#}, u^{\#})$ by taking the degree-0 part in the correspondence constructed in the proof of Proposition 2.21. Thus we may build a natural isomorphism between the functors in question out of the nondegenerate left action of $\mathcal{O}^0_{J_1,\mathscr{E}_1}$ on $\mathcal{O}_{J_1,\mathscr{E}_1}$.

So the reflector L lifts the Cuntz-Pimsner algebra homomorphism $\mathfrak{C}^{\mathbb{N}}_{\mathrm{pr}} \to \mathfrak{C}^{\mathbb{T}}$ to a homomorphism with values in $\mathfrak{C}^{\mathbb{N}}_{\mathrm{pr},*}$. Such a lifting should exist because a Hilbert bimodule and its crossed product with the $\mathbb{T}$-action determine each other.

An adjunction also contains "natural" equivalences of categories

$$\varphi_{b,c}\colon \mathscr{C}(L(b), c) \simeq \mathscr{B}(b, R(c)),$$

where naturality is further data (see Definition 4.4). In the case at hand, these equivalences are exactly the equivalences of categories

$$\upsilon^*_{(A,\mathscr{E},J)}\colon \mathfrak{C}^{\mathbb{N}}_{\mathrm{pr}}\big((A,\mathscr{E},J),(B,\mathscr{G},I_{\mathscr{G}})\big) \simeq \mathfrak{C}^{\mathbb{N}}_{\mathrm{pr},*}\big((\mathcal{O}^0_{J,\mathscr{E}},\mathcal{O}^1_{J,\mathscr{E}},I_{\mathcal{O}^1_{J,\mathscr{E}}}),(B,\mathscr{G},I_{\mathscr{G}})\big)$$

in Proposition 3.4. Their naturality boils down to the canonical isomorphisms of correspondences $\upsilon_{(A,\mathscr{E},J)} \circ L(\mathscr{F},u) \cong (\mathscr{F},u) \circ \upsilon_{(A,\mathscr{E},J)}$, which we have already used above to describe the multiplicativity data λ in the homomorphism L.

Finally, we relate our adjunction to the colimit description of Cuntz-Pimsner algebras in [2]. Let $\mathscr{C}$ and $\mathscr{D}$ be categories. Let $\mathscr{C}^{\mathscr{D}}$ be the category of functors $\mathscr{D} \to \mathscr{C}$, which are also called diagrams of shape $\mathscr{D}$ in $\mathscr{C}$. Identify $\mathscr{C}$ with the subcategory of "constant" diagrams in $\mathscr{C}^{\mathscr{D}}$. This subcategory is reflective if and only if all $\mathscr{D}$-shaped diagrams in $\mathscr{C}$ have a colimit, and the reflector maps a diagram to its colimit.

This remains true for the bicategorical colimits in [2]: by definition, the colimit of a diagram is a universal arrow to a constant diagram. In our context, a constant diagram in $\mathfrak{C}^{\mathbb{N}}_{\mathrm{pr}}$ is an object of the form (B, B, B) that is, the Hilbert B-bimodule is the identity bimodule and $J = B$ as always for objects of $\mathfrak{C}^{\mathbb{N}}_{\mathrm{pr},*}$. Since the condition $J \cdot \mathscr{F} \subseteq \mathscr{F} \cdot B$ always holds, the ideal J plays no role, compare Remark 2.24.

A proper covariant correspondence $(A,\mathscr{E},J) \to (B,B,B)$ is equivalent to a proper correspondence $\mathscr{F}\colon A \rightsquigarrow B$ with an isomorphism $\mathscr{E} \otimes_A \mathscr{F} \Rightarrow \mathscr{F}$ because $\mathscr{F} \otimes_B B \cong \mathscr{F}$. As shown in [2], such a pair is equivalent to a representation (φ, t) of the correspondence $\mathscr{E}$ on $\mathscr{F}$ that is nondegenerate in the sense that $t(\mathscr{E}) \cdot \mathscr{F} = \mathscr{F}$. The properness of $\mathscr{F}$ means that $\varphi(A) \subseteq \mathbb{K}(\mathscr{F})$, which implies $t(\mathscr{E}) \subseteq \mathbb{K}(\mathscr{F})$.

It is shown in [2] that all diagrams of *proper* correspondences of any shape have a colimit. This is probably false for diagrams of non-proper correspondences, such as the correspondence $\ell^2(\mathbb{N})\colon \mathbb{C} \rightsquigarrow \mathbb{C}$ that defines the Cuntz algebra $\mathcal{O}_\infty$. The way around this problem that we found here is to enlarge the sub-bicategory of constant diagrams, allowing diagrams of Hilbert bimodules. In addition, we added an ideal J to have enough data to build *relative* Cuntz-Pimsner algebras.

Since the sub-bicategory $\mathfrak{C} \subseteq \mathfrak{C}^{\mathbb{N}}_{\mathrm{pr}}$ of constant diagrams is contained in $\mathfrak{C}^{\mathbb{N}}_{\mathrm{pr},*}$, we may relate universal arrows to objects in $\mathfrak{C}$ and $\mathfrak{C}^{\mathbb{N}}_{\mathrm{pr},*}$ as follows. Let $(A,\mathscr{E},J)$ be an object of $\mathfrak{C}^{\mathbb{N}}_{\mathrm{pr}}$. Then $\upsilon_{(A,\mathscr{E},J)}\colon (A,\mathscr{E},J) \to (\mathcal{O}^0_{J,\mathscr{E}},\mathcal{O}^1_{J,\mathscr{E}},I_{\mathcal{O}^1_{J,\mathscr{E}}})$

is a universal arrow to an object of $\mathfrak{C}^{\mathbb{N}}_{\mathrm{pr},*}$ by Proposition 4.3. The universality of $\upsilon_{(A,\mathcal{E},J)}$ implies that a universal arrow from $(A, \mathcal{E}, J)$ to a constant diagram factors through $\upsilon_{(A,\mathcal{E},J)}$, and that an arrow from $(\mathcal{O}^0_{J,\mathcal{E}}, \mathcal{O}^1_{J,\mathcal{E}}, I_{\mathcal{O}^1_{J,\mathcal{E}}})$ to a constant diagram is universal if and only if its composite with $\upsilon_{(A,\mathcal{E},J)}$ is universal. In other words, the diagram $(A, \mathcal{E}, J)$ has a colimit if and only if $(\mathcal{O}^0_{J,\mathcal{E}}, \mathcal{O}^1_{J,\mathcal{E}}, I_{\mathcal{O}^1_{J,\mathcal{E}}})$ has one, and then the two colimits are the same. We are dealing with the same colimits as in [2] because the ideal J in $(A, \mathcal{E}, J)$ plays no role for arrows to constant diagrams.

Appendix A. Bicategories

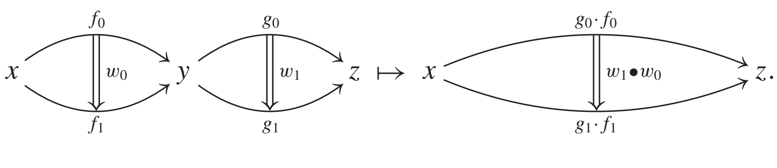
We recall some basic definitions from bicategory theory, following [3], [11]. We also give a few examples with Sections 3 and 4 in mind.

DEFINITION A.1. A *bicategory* $\mathcal{B}$ consists of the following data:

- a set of objects $\mathrm{ob}\,\mathcal{B}$;
- a category $\mathcal{B}(x, y)$ for each pair of objects (x, y); objects of $\mathcal{B}(x, y)$ are called *arrows* (or *morphisms*) from x to y, and arrows in $\mathcal{B}(x, y)$ are called 2-*arrows* (or 2-*morphisms*); the category structure on $\mathcal{B}(x, y)$ gives us a *unit* 2-*arrow* 1_f on each arrow $f: x \to y$, and a *vertical composition* of 2-arrows: $w_0: f_0 \Rightarrow f_1$ and $w_1: f_1 \Rightarrow f_2$ compose to a 2-arrow $w_1 \cdot w_0: f_0 \Rightarrow f_2$;
- composition functors

$$\circ: \mathcal{B}(y, z) \times \mathcal{B}(x, y) \longrightarrow \mathcal{B}(x, z)$$

  for each triple of objects (x, y, z); this contains a *horizontal composition* of 2-arrows as displayed below:



- a unit arrow $1_x \in \mathcal{B}(x, x)$ for each x;
- natural invertible 2-arrows (*unitors*) $r_f: f \cdot 1_x \Rightarrow f$ and $\ell_f: 1_y \cdot f \Rightarrow f$ for all $f \in \mathcal{B}(x, y)$;

- natural isomorphisms

$$\begin{array}{ccc}
\mathscr{B}(x,y)\times\mathscr{B}(y,z)\times\mathscr{B}(z,w) & \xrightarrow{(\circ,1)} & \mathscr{B}(x,z)\times\mathscr{B}(z,w)\\
{\scriptstyle (1,\circ)}\downarrow & \overset{a}{\nearrow\!\!\!\!\nearrow} & \downarrow{\scriptstyle \circ}\\
\mathscr{B}(x,y)\times\mathscr{B}(y,w) & \xrightarrow{\circ} & \mathscr{B}(x,w)
\end{array}$$

that is, natural invertible 2-arrows, called *associators*,

$$a(f_1, f_2, f_3)\colon (f_3 \cdot f_2)\cdot f_1 \underset{\cong}{\Rightarrow} f_3 \cdot (f_2 \cdot f_1),$$

where $f_1\colon x \to y$, $f_2\colon y \to z$ and $f_3\colon z \to w$.

This data must make the following diagrams commute:

$$\begin{array}{ccccc}
((f_4\cdot f_3)\cdot f_2)\cdot f_1 & \Longrightarrow & (f_4\cdot f_3)\cdot(f_2\cdot f_1) & \Longrightarrow & f_4\cdot(f_3\cdot(f_2\cdot f_1))\\
\Downarrow & & & & \Uparrow\\
(f_4\cdot(f_3\cdot f_2))\cdot f_1 & & \Longrightarrow & & f_4\cdot((f_3\cdot f_2)\cdot f_1),
\end{array}$$

$$\begin{array}{ccc}
(f_2\cdot 1_y)\cdot f_1 & \Longrightarrow & f_2\cdot(1_y\cdot f_1)\\
 & \searrow\!\!\!\!\searrow & \Downarrow\\
 & & f_2\cdot f_1,
\end{array}$$

where f_1, f_2, f_3, and f_4 are composable arrows, and the 2-arrows are associators and unitors and horizontal products of them with unit 2-arrows.

We write "$\cdot$" or nothing for vertical products and "$\bullet$" for horizontal products.

EXAMPLE A.2. Categories form a bicategory **Cat** with functors as arrows and natural transformations as 2-arrows. Here the composition of morphisms is strictly associative and unital, that is, **Cat** is even a 2-category.

EXAMPLE A.3. A category $\mathscr{C}$ may be regarded as a bicategory in which the categories $\mathscr{C}(x, y)$ have only identity arrows.

EXAMPLE A.4. The correspondence bicategory $\mathfrak{C}$ is defined in [7] as the bicategory with C*-algebras as objects, correspondences as arrows, and correspondence isomorphisms as 2-arrows. The unit arrow 1_A on a C*-algebra A is A viewed as a Hilbert A-bimodule in the canonical way. The A, B-bimodule structure on $\mathscr{F}$ provides the unitors $A\otimes_A\mathscr{F} \Rightarrow \mathscr{F}$ and $\mathscr{F}\otimes_B B \Rightarrow \mathscr{F}$ for a correspondence $\mathscr{F}\colon A \rightsquigarrow B$. The associators $(\mathscr{E}\otimes_A\mathscr{F})\otimes_B\mathscr{G} \Rightarrow \mathscr{E}\otimes_A(\mathscr{F}\otimes_B\mathscr{G})$ are the obvious isomorphisms.

DEFINITION A.5. Let $\mathscr{B}, \mathscr{C}$ be bicategories. A *homomorphism* $F\colon \mathscr{B} \to \mathscr{C}$ consists of

- a map $F^0\colon \operatorname{ob}\mathscr{B} \to \operatorname{ob}\mathscr{C}$ between the object sets;
- functors $F_{x,y}\colon \mathscr{B}(x, y) \to \mathscr{C}(F^0(x), F^0(y))$ for all $x, y \in \operatorname{ob}\mathscr{B}$;
- natural transformations

$$\begin{array}{ccc} \mathscr{B}(y,z) \times \mathscr{B}(x,y) & \xrightarrow{\quad\circ\quad} & \mathscr{B}(x,z) \\ {\scriptstyle (F_{y,z},F_{x,y})}\big\downarrow & \overset{\varphi_{xyz}}{\nearrow\!\!\!\nearrow} & \big\downarrow{\scriptstyle F_{x,z}} \\ \mathscr{C}(F(y),F(z)) \times \mathscr{C}(F(x),F(y)) & \xrightarrow{\quad\circ\quad} & \mathscr{C}(F(x),F(z)) \end{array}$$

  for all triples x, y, z of objects of $\mathscr{B}$; explicitly, these are natural 2-arrows $\varphi(f_1, f_2)\colon F_{y,z}(f_2) \cdot F_{x,y}(f_1) \Rightarrow F_{x,z}(f_2 \cdot f_1)$;
- 2-arrows $\varphi_x\colon 1_{F(x)} \Rightarrow F_{x,x}(1_x)$ for all objects x of $\mathscr{B}$.

This data must make the following diagrams commute:

$$\begin{array}{ccc} (F_{z,w}(f_3) \cdot F_{y,z}(f_2)) \cdot F_{x,y}(f_1) & \overset{a'}{\Longrightarrow} & F_{z,w}(f_3) \cdot (F_{y,z}(f_2) \cdot F_{x,y}(f_1)) \\ {\scriptstyle \varphi(f_2,f_3)\bullet 1_{F_{x,y}(f_1)}}\Big\Downarrow & & \Big\Downarrow{\scriptstyle 1_{F_{z,w}(f_3)}\bullet\varphi(f_1,f_2)} \\ F_{y,w}(f_3 \cdot f_2) \cdot F_{x,y}(f_1) & & F_{z,w}(f_3) \cdot F_{x,z}(f_2 \cdot f_1) \\ {\scriptstyle \varphi(f_1,f_3\cdot f_2)}\Big\Downarrow & & \Big\Downarrow{\scriptstyle \varphi(f_2\cdot f_1,f_3)} \\ F_{x,w}((f_3 \cdot f_2) \cdot f_1) & \xRightarrow{F_{x,w}(a)} & F_{x,w}(f_3 \cdot (f_2 \cdot f_1)); \end{array} \tag{A.6}$$

$$\begin{array}{ccc} F_{x,y}(f_1) \cdot F_{x,x}(1_x) & \xRightarrow{\varphi(1_x,f_1)} & F_{x,y}(f_1 \cdot 1_x) \\ {\scriptstyle 1_{F_{x,y}(f_1)}\bullet\varphi_x}\Big\Uparrow & & \Big\Downarrow{\scriptstyle F_{x,y}(r_{f_1})} \\ F_{x,y}(f_1) \cdot 1_{F(x)} & \xRightarrow{r'_{F_{x,y}(f_1)}} & F_{x,y}(f_1); \end{array} \tag{A.7}$$

$$\begin{array}{ccc} F_{y,y}(1_y) \cdot F_{x,y}(f_1) & \xRightarrow{\varphi(f_1,1_y)} & F_{x,y}(1_y \cdot f_1) \\ {\scriptstyle \varphi_y\bullet 1_{F_{x,y}(f_1)}}\Big\Uparrow & & \Big\Downarrow{\scriptstyle F_{x,y}(\ell_{f_1})} \\ 1_{F(y)} \cdot F_{x,y}(f_1) & \xRightarrow{\ell'_{F_{x,y}(f_1)}} & F_{x,y}(f_1). \end{array} \tag{A.8}$$

EXAMPLE A.9. A monoid P may be viewed as a category with one object and P as its set of arrows. It may be viewed as a bicategory as well as in

Example A.3. A homomorphism from P to $\mathfrak{C}$ is equivalent to an essential product system $(A, (\mathscr{E}_p)_{p\in P^{\mathrm{op}}}, \mu)$ over P^{op} as defined by Fowler [10]. The condition (A.6) says that the multiplication maps $\mu_{p,q}\colon \mathscr{E}_p \otimes_A \mathscr{E}_q \overset{\simeq}{\to} \mathscr{E}_{qp}$ are associative. The conditions (A.7) and (A.8) mean that $\mu_{1,p}(a \otimes \xi) = \varphi_p(a)\xi$ and $\mu_{p,1}(\xi \otimes a) = \xi a$ for $a \in A$, $\xi \in \mathscr{E}_p$.

A morphism $f\colon x \to y$ in a bicategory $\mathscr{B}$ induces functors

$$f_*\colon \mathscr{B}(c, x) \longrightarrow \mathscr{B}(c, y), \qquad f^*\colon \mathscr{B}(y, c) \longrightarrow \mathscr{B}(x, c)$$

for $c \in \operatorname{ob}\mathscr{B}$ by composing arrows with f and composing 2-arrows horizontally with 1_f on one side (this is also called *whiskering* with f).

Definition A.10. Let $F, G\colon \mathscr{B} \rightrightarrows \mathscr{C}$ be homomorphisms. A *transformation* $\alpha\colon F \Rightarrow G$ consists of

- morphisms $\alpha_x\colon F(x) \to G(x)$ for all $x \in \operatorname{ob}\mathscr{B}$;
- natural transformations

$$\begin{array}{ccc} \mathscr{B}(x, y) & \xrightarrow{\quad F_{x,y}\quad} & \mathscr{C}(F(x), F(y)) \\ {\scriptstyle G_{x,y}}\big\downarrow & \overset{\alpha_{x,y}}{\swarrow\!\!\!\swarrow} & \big\downarrow{\scriptstyle \alpha_{y*}} \\ \mathscr{C}(G(x), G(y)) & \xrightarrow{\quad {\alpha_x}^*\quad} & \mathscr{C}(F(x), G(y)), \end{array}$$

that is, 2-arrows $\alpha_{x,y}(f)\colon \alpha_y F_{x,y}(f) \Rightarrow G_{x,y}(f)\alpha_x$ for all $x, y \in \operatorname{ob}\mathscr{B}$.

This data must make the following diagrams commute:

$$\begin{array}{ccccc} \alpha_z(F_{y,z}(g)F_{x,y}(f)) & \xRightarrow{1\bullet\varphi_F(f,g)} & \alpha_z F_{x,z}(gf) & \xRightarrow{\quad\alpha_{x,z}(gf)\quad} & G_{x,z}(gf)\alpha_x \\ \Updownarrow & & & & \Uparrow{\scriptstyle \varphi_G(f,g)\bullet 1} \\ (\alpha_z F_{y,z}(g))F_{x,y}(f) & & & & (G_{y,z}(g)G_{x,y}(f))\alpha_x \\ {\scriptstyle \alpha_{y,z}(g)\bullet 1}\Downarrow & & & & \Updownarrow \\ (G_{y,z}(g)\alpha_y)F_{x,y}(f) & \Longleftrightarrow & G_{y,z}(g)(\alpha_y F_{x,y}(f)) & \xRightarrow{1\bullet\alpha_{x,y}(f)} & G_{y,z}(g)(G_{x,y}(f)\alpha_x); \end{array}$$

$$\begin{array}{ccccc} \alpha_x F_{x,x}(1_x) & \xLeftarrow{\quad 1_{\alpha_x}\bullet\varphi^F_x\quad} & \alpha_x 1_{F(x)} & \xRightarrow{\quad r\quad} & \alpha_x \\ {\scriptstyle \alpha_{x,x}(1_x)}\Downarrow & & & & \Downarrow{\scriptstyle \ell^{-1}} \\ G_{x,x}(1_x)\alpha_x & & \xLeftarrow{\qquad\qquad \varphi^G_x\bullet 1_{\alpha_x}\qquad\qquad} & & 1_{G(x)}\alpha_x. \end{array}$$

Example A.11. Let G be a group. A transformation between homomorphisms $G \to \mathfrak{C}$ consists of a correspondence $\mathscr{F}\colon A \rightsquigarrow B$ and isomorphisms

$\alpha_s\colon \mathcal{E}_s \otimes_A \mathcal{F} \simeq \mathcal{F} \otimes_B \mathcal{G}_s$ so that the following diagrams commute for all $s, t \in G$:

$$
\begin{array}{ccccc}
(\mathcal{E}_s \otimes_A \mathcal{E}_t) \otimes_A \mathcal{F} & \xRightarrow{w^1_{s,t}\otimes 1} & \mathcal{E}_{st} \otimes_A \mathcal{F} & \xRightarrow{\alpha_{st}} & \mathcal{F} \otimes_B \mathcal{G}_{st} \\
\Updownarrow & & & & \Uparrow {\scriptstyle 1\otimes w^2_{s,t}} \\
\mathcal{E}_s \otimes_A (\mathcal{E}_t \otimes_A \mathcal{F}) & & & & \mathcal{F} \otimes_B (\mathcal{G}_s \otimes_B \mathcal{G}_t) \\
{\scriptstyle 1\otimes\alpha_t} \Downarrow & & & & \Updownarrow \\
\mathcal{E}_s \otimes_A (\mathcal{F} \otimes_B \mathcal{G}_t) & \Longleftrightarrow & (\mathcal{E}_s \otimes_A \mathcal{F}) \otimes_B \mathcal{G}_t & \xRightarrow{\alpha_s\otimes 1} & (\mathcal{F} \otimes_B \mathcal{G}_s) \otimes_B \mathcal{G}_t .
\end{array}
$$

This is called a *correspondence* of Fell bundles (see [7, Proposition 3.23]).

DEFINITION A.12. Let $\alpha, \beta\colon F \Rightarrow G$ be transformations between homomorphisms. A *modification* $\Delta\colon \alpha \Rrightarrow \beta$ is a family of 2-arrows $\Delta_x\colon \alpha_x \Rightarrow \beta_x$ such that for every 2-arrow $w\colon f_1 \Rightarrow f_2$ for arrows $f_1, f_2\colon x \to y$, the following diagram commutes:

$$
\begin{array}{ccc}
\alpha_y F_{x,y}(f_1) & \xRightarrow{\Delta_y \bullet F_{x,y}(w)} & \beta_y F_{x,y}(f_2) \\
{\scriptstyle \alpha_{x,y}(f_1)} \Downarrow & & \Downarrow {\scriptstyle \beta_{x,y}(f_2)} \\
G_{x,y}(f_1)\alpha_x & \xRightarrow{G_{x,y}(w)\bullet \Delta_x} & G_{x,y}(f_2)\beta_x .
\end{array}
$$

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