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FOURIER MULTIPLIERS ON ANISOTROPIC MIXED-NORM SPACES OF DISTRIBUTIONS

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Abstract

A new general Hörmander type condition involving anisotropies and mixed norms is introduced, and boundedness results for Fourier multipliers on anisotropic Besov and Triebel-Lizorkin spaces of distributions with mixed Lebesgue norms are obtained. As an application, the continuity of such operators is established on mixed Sobolev and Lebesgue spaces too. Some lifting properties and equivalent norms are obtained as well.

1. Introduction

The study of spaces of functions and distributions and operators on such spaces play an essential role in harmonic analysis. Several branches of both pure and applied mathematics make extensive use of such spaces, including in the study of partial differential equations, approximation theory, probability, statistics, and signal processing.

Some of the most general and applicable families of functions spaces in analysis are the Besov and Triebel-Lizorkin spaces. The two families are interesting in their own right, but their importance also stem from the fact that several of the classical function spaces such as Lebesgue, Hardy, BMO, Sobolev, and Hölder spaces can be recovered as special cases. The Besov and Triebel-Lizorkin spaces have been studied for many different reasons and in a variety of settings and circumstances. For further details we refer the reader to [8], [9], [12], [21], [22], [25], [26], [36] and to the references found therein.

The purpose of this article is to study Fourier multipliers in the general setting of anisotropic Triebel-Lizorkin spaces based on mixed-norm Lebesgue spaces. Let us now elaborate further on this particular setting.

Anisotropic phenomena appear naturally in various fields of analysis, both pure and applied. A classical example found in [23] is the case of differential operators with anisotropic symbols. Such operators naturally introduce a need for anisotropic function classes containing functions compatible with the

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particular anisotropy. The notion of anisotropic Besov spaces goes back to Nikol′skiĭ [31] and the notion of anisotropic Triebel-Lizorkin spaces can be found in Triebel's book [35, p. 269]. More recently, anisotropic Besov spaces in a more general setting have been studied by Bownik [8] and the anisotropic Triebel-Lizorkin spaces have been studied by Bownik and Ho [9].

Mixed-norm Lebesgue spaces are useful as a framework for several problems arising in physics demanding different regularity in every direction (e.g. in time and in space). Mixed-norm Besov and Triebel-Lizorkin spaces have been studied during this decade by Johnsen and Sickel as one may see for example [21]. Here we will work on anisotropic mixed-norm Besov and Triebel-Lizorkin spaces.

Fourier multipliers form one of the fundamental and most important classes of operators in harmonic analysis. Their importance is emphasized by their close link to partial differential operators through the Fourier transform, and there has been a continuous interest in the study of boundedness properties of multipliers on L^p and other spaces since the seminal work by Marcinkiewicz [29], Mihlin [30] and Hörmander [18]. Numerous variations and generalizations of the above works have been produced during the past years, see [4], [5], [10], [13], [14], [16], [24], [28], [37], [19] and the references therein.

In this article we prove boundedness of a suitable class of Fourier multipliers on anisotropic Besov and Triebel-Lizorkin spaces with mixed Lebesgue norms. Moreover, we also introduce a new general Hörmander-type class of multipliers naturally adapted to mixed norms and to the general anisotropic setting.

Let us summarize the main contributions in this paper.

(α) We introduce a new and general condition, involving mixed norms and the anisotropy, for Hörmander multipliers, see equation (3.4).

(β) We prove the boundedness of Fourier multipliers on anisotropic mixed-norm Besov and Triebel-Lizorkin space, see Theorem 3.6.

(γ) The continuity of Fourier multipliers on mixed Lebesgue and Sobolev spaces will be obtained, under the new condition as well, see Corollaries 4.1–4.3.

(δ) An equivalent norm characterization for the anisotropic mixed-norm Besov and Triebel-Lizorkin spaces is revisited, see Corollary 4.4.

Notation

We will denote by $\mathscr{F}(f)(\xi) := \hat{f}(\xi) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi}\, dx$ the Fourier transform of (suitably nice) f, where $x \cdot \xi := x_1\xi_1 + \cdots + x_n\xi_n$ is the standard inner product on $\mathbb{R}^n$. The inverse Fourier transform is then given by $\mathscr{F}^{-1} f(x) := \hat{f}(-x)$. For $\vec{t} = (t_1, \dots, t_n) \in \mathbb{R}^n$ with $t_1, \dots, t_n \neq 0$, we set

$1/\vec{t} := (1/t_1, \ldots, 1/t_n)$. The sets of positive and non-negative integers will be denoted by $\mathbb{N}$ and $\mathbb{N}_0$ respectively. For γ a multi-index $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{N}_0^n$, we denote by $|\gamma| := \gamma_1 + \cdots + \gamma_n$ its length and we set $\partial^\gamma f := \partial_{x_1}^{\gamma_1} \cdots \partial_{x_n}^{\gamma_n} f$. By $\mathscr{S} := \mathscr{S}(\mathbb{R}^n)$ we denote the Schwartz class on $\mathbb{R}^n$ and by $\mathscr{S}'$ its dual; the tempered distributions. For $N \in \mathbb{N}_0$ the space of N-times differential functions on $\mathbb{R}^n$ is denoted by $\mathscr{C}^N$. Finally, any positive constant will be denoted c, or as c_α if it depends on a significant parameter α.

2. Preliminaries

2.1. Mixed-norm Lebesgue spaces

Let $\vec{p} = (p_1, \ldots, p_n)$, with $0 < p_1, \ldots, p_n \leq \infty$, and let $f : \mathbb{R}^n \to \mathbb{C}$ be measurable. We say that f lies in $L^{\vec{p}} = L^{\vec{p}}(\mathbb{R}^n)$ if

$$\|f\|_{\vec{p}} := \left(\int_{\mathbb{R}} \cdots \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x_1, \ldots, x_n)|^{p_1} dx_1 \right)^{p_2/p_1} dx_2 \right)^{p_3/p_2} \cdots dx_n \right)^{1/p_n} < \infty,$$

with the standard modification when $p_k = \infty$, for some $1 \leq k \leq n$. The quasi-norm $\|\cdot\|_{\vec{p}}$, is a norm when $\min(p_1, \ldots, p_n) \geq 1$ and turns $L^{\vec{p}}$ into a Banach space. For further properties of $L^{\vec{p}}$ see for example [1], [2], [3], [15], [27].

Let $R \subset \mathbb{R}^n$. We define $\|f\|_{L^{\vec{p}}(R)} := \|f \chi_R\|_{\vec{p}}$, where χ_R is the characteristic function of R. For example when $R := I_1 \times \cdots \times I_n \subset \mathbb{R}^n$ is a rectangle, we obtain

$$\|f\|_{L^{\vec{p}}(R)} = \left(\int_{I_n} \cdots \left(\int_{I_2} \left(\int_{I_1} |f(x_1, \ldots, x_n)|^{p_1} dx_1 \right)^{p_2/p_1} dx_2 \right)^{p_3/p_2} \cdots dx_n \right)^{1/p_n}.$$

For $\vec{p} \in [1, \infty]^n$ we define the *conjugate* $\vec{p'} := (p_1', \ldots, p_n') \in [1, \infty]^n$ by requiring that $1/p_k + 1/p_k' = 1$ for every $k = 1, \ldots, n$.

The *mixed Hölder-inequality* (see e.g. [3]) is the following estimate: for every $\vec{p} = [1, \infty]^n$, $f \in L^{\vec{p}}$ and $g \in L^{\vec{p'}}$ we have

$$\left| \int_{R^n} f(x) \overline{g}(x)\, dx \right| \leq \|f\|_{\vec{p}} \, \|g\|_{\vec{p'}} . \tag{2.1}$$

We will also need an adapted version of the Hausdorff-Young inequality. The *mixed Hausdorff-Young's Theorem* [3] asserts that if $\vec{t} = (t_1, \ldots, t_n)$ with $1 \leq t_n \leq t_{n-1} \leq \cdots \leq t_1 \leq 2$, then for every $f \in \mathscr{S}$,

$$\|\hat{f}\|_{\vec{t'}} \leq \|f\|_{\vec{t}} . \tag{2.2}$$

2.2. Anisotropic geometry

Let $b, x \in \mathbb{R}^n$ and $\lambda > 0$. We set $\lambda^b x := (\lambda^{b_1} x_1, \dots, \lambda^{b_n} x_n)$. We fix a vector $\vec{a} \in [1, \infty)^n$, and we introduce the anisotropic quasi-homogeneous norm $|\cdot|_{\vec{a}}$ as follows: we set $|0|_{\vec{a}} := 0$, and for $x \neq 0$ we let $|x|_{\vec{a}} := \lambda_0$, where λ_0 is the unique positive number such that $|\lambda_0^{-\vec{a}} x| = 1$. One observes immediately that

$$|\lambda^{\vec{a}} x|_{\vec{a}} = \lambda |x|_{\vec{a}}, \quad \text{for every } \; x \in \mathbb{R}^n, \; \lambda > 0.$$

From this we notice that $|\cdot|_{\vec{a}}$ is not a norm unless $\vec{a} = (1, \dots, 1)$, where it coincides with the Euclidean norm.

We have a link between the anisotropic and the Euclidean geometry (see [7], [9]): there exist constants $c_1, c_2 > 0$ such that for every $x \in \mathbb{R}^n$,

$$c_1(1 + |x|_{\vec{a}})^{a_m} \leq 1 + |x| \leq c_2(1 + |x|_{\vec{a}})^{a_M}, \tag{2.3}$$

where we write $a_m := \min_{1 \leq j \leq n} a_j$ and $a_M := \max_{1 \leq j \leq n} a_j$.

Finally, we will need the so-called *homogeneous dimension*:

$$\nu := |\vec{a}| = a_1 + \cdots + a_n.$$

2.3. Anisotropic mixed-norm Triebel-Lizorkin and Besov spaces

In this section we define the anisotropic mixed-norm smoothness spaces needed for our analysis, and we discuss some corresponding Fefferman-Stein vector-valued maximal function estimates.

Let $\varphi_0 \in \mathscr{S}$ (the class of Schwartz functions) be such that

$$\operatorname{supp}(\widehat{\varphi_0}) \subseteq 2^{\vec{a}}[-2, 2]^n =: R_0, \tag{2.4}$$

$$|\widehat{\varphi_0}(\xi)| \geq c > 0, \quad \text{if } \; \xi \in 2^{\vec{a}}[-5/3, 5/3]^n, \tag{2.5}$$

and let $\varphi \in \mathscr{S}$ be such that

$$\operatorname{supp}(\hat{\varphi}) \subseteq [-2, 2]^n \setminus (-1/2, 1/2)^n =: \widetilde{R_1}, \tag{2.6}$$

$$|\hat{\varphi}(\xi)| \geq c > 0, \quad \text{if } \; \xi \in [-5/3, 5/3]^n \setminus (-3/5, 3/5)^n. \tag{2.7}$$

Note that it is possible to choose φ and φ_0 satisfying the partition of unity condition

$$\widehat{\varphi_0}(\xi) + \sum_{j=1}^{\infty} \hat{\varphi}(2^{-j\vec{a}}\xi) = 1, \quad \text{for every } \; \xi \in \mathbb{R}^n. \tag{2.8}$$

We define the "rectangular version" of the annulus by

$$R_j := 2^{j\vec{a}} \widetilde{R_1}, \; j \geq 1, \quad \text{and} \quad R_j := \emptyset, \; j < 0. \tag{2.9}$$

Note that the punctured rectangle $\widetilde{R_1}$ can be expressed as the (almost) disjoint union of $k_n := 2^{3n} - 2^n$ closed, dyadic cubes $\{Q_\mu\}_{\mu=1,\dots,k_n}$ of side-length 2^{-2}. Then for every $j \geq 1$ we have that $R_j = \bigcup_{\mu=1}^{k_n} 2^{j\vec{a}} Q_\mu$. For every $\vec{p} \in (0,\infty]^n$, we have a two-sided estimate of the mixed-norm on R_j as

$$\|f\|_{L^{\vec{p}}(R_j)} \asymp \sum_{\mu=1}^{k_n} \|f\|_{L^{\vec{p}}(2^{j\vec{a}} Q_\mu)}\,, \tag{2.10}$$

where the constants in the equivalence depends only on $\vec{p}$ and n.

We write $\varphi_j(x) := 2^{\nu j}\varphi(2^{j\vec{a}}x)$, for $j \in \mathbb{N}$. We then have $\widehat{\varphi_j}(\xi) = \widehat{\varphi}(2^{-j\vec{a}}\xi)$, so by (2.6)

$$\operatorname{supp}(\widehat{\varphi_j}) \subseteq 2^{j\vec{a}} \operatorname{supp}(\widehat{\varphi}) \subseteq R_j, \quad \text{for every } j \in \mathbb{N}. \tag{2.11}$$

Let us now recall the definition of anisotropic mixed-norm Triebel-Lizorkin and Besov spaces (see for example [21]).

For $s \in \mathbb{R}$, $\vec{p} \in (0,\infty)^n$, $q \in (0,\infty]$ and $\vec{a} \in [1,\infty)^n$, the *anisotropic mixed-norm Triebel-Lizorkin space* $F^s_{\vec{p}q}(\vec{a})$ is defined, as the set of all $f \in \mathcal{S}'$ such that

$$\|f\|_{F^s_{\vec{p}q}(\vec{a})} := \left\| \left(\sum_{j=0}^{\infty} (2^{sj} |\varphi_j * f|)^q \right)^{1/q} \right\|_{\vec{p}} < \infty.$$

For $s \in \mathbb{R}$, $\vec{p} \in (0,\infty]^n$, $q \in (0,\infty]$ and $\vec{a} \in [1,\infty)^n$, the *anisotropic mixed-norm Besov space* $B^s_{\vec{p}q}(\vec{a})$ is defined, as the set of all $f \in \mathcal{S}'$ such that

$$\|f\|_{B^s_{\vec{p}q}(\vec{a})} := \left(\sum_{j=0}^{\infty} (2^{sj} \|\varphi_j * f\|_{\vec{p}})^q \right)^{1/q} < \infty,$$

with the ℓ_q-norm replaced by $\sup_j$ if $q = \infty$ for both $F^s_{\vec{p}q}(\vec{a})$ and $B^s_{\vec{p}q}(\vec{a})$. Note that $F^s_{pq}(\vec{a}) = F^s_{\vec{p}q}(\vec{a})$ for $\vec{p} = (p,\dots,p)$ (and the same holds for Besov spaces). Further properties of $F^s_{\vec{p}q}(\vec{a})$ and $B^s_{\vec{p}q}(\vec{a})$ can be found in [20], [17], [32], [33].

REMARK 2.1. One can easily verify that the definition of mixed-norm Triebel-Lizorkin and Besov spaces based on test functions with Fourier transforms having supports in a classical annulus, such as considered in e.g. [21], is equivalent to the above definitions.

2.4. Maximal operators

Maximal operators will be an essential tool in the proof of our main result. Let $1 \le k \le n$. We define for $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$,

$$M_k f(x) := \sup_{I \in I^k_x} \frac{1}{|I|} \int_I |f(x_1, \dots, y_k, \dots, x_n)|\, dy_k,$$

where I^k_x is the set of all intervals I in $\mathbb{R}_{x_k}$ containing x_k.

We will use extensively the following *iterated maximal operator*: for $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, we let

$$\mathcal{M}_{\vec{r}} f(x) := \big(M_n\big(\cdots M_2(M_1|f|^{r_1})^{r_2/r_1} \cdots\big)^{r_n/r_{n-1}}\big)^{1/r_n}(x),$$

for $\vec{r} \in (0,\infty)^n$ and $x \in \mathbb{R}^n$.

We shall need a variation of the *Fefferman-Stein* vector-valued maximal inequality (see [2], [21]): if $\vec{p} = (p_1, \dots, p_n) \in (0,\infty)^n$, $q \in (0,\infty]$ and $\vec{r} = (r_1, \dots, r_n) \in (0,\infty)^n$ with $r_k < \min(p_1, \dots, p_k, q)$ for every $k = 1, \dots, n$, then

$$\bigg\| \Big(\sum_{j \ge 0} \big(\mathcal{M}_{\vec{r}}(f_j)\big)^q \Big)^{1/q} \bigg\|_{\vec{p}} \le c \bigg\| \Big(\sum_{j \ge 0} |f_j|^q \Big)^{1/q} \bigg\|_{\vec{p}}. \tag{2.12}$$

By [21, Proposition 3.11], we obtain the following compound result: for every $\vec{r} \in (0,\infty)^n$ there exists a constant $c > 0$, such that for every $\vec{b} = (b_1, \dots, b_n) \in (0,\infty)^n$ and every f with $\mathrm{supp}(\hat{f}) \subset [-b_1, b_1] \times \cdots \times [-b_n, b_n]$,

$$\sup_{z \in \mathbb{R}^n} \frac{|f(x-z)|}{(1+|b_1 z_1|)^{1/r_1} \cdots (1+|b_n z_n|)^{1/r_n}} \le c \mathcal{M}_{\vec{r}} f(x), \quad x \in \mathbb{R}^n. \tag{2.13}$$

3. Fourier multipliers

3.1. Anisotropic Fourier multipliers

One of the most classical problems in harmonic analysis is the boundedness of Fourier multipliers between suitable function (or distribution) smoothness spaces. A bounded function $m = m(\xi)$ on $\mathbb{R}^n$ is called a multiplier. The corresponding Fourier multiplier operator is given by

$$T_m f(x) := \int_{\mathbb{R}^n} m(\xi) \hat{f}(\xi) e^{ix\cdot\xi}\, d\xi, \quad \text{for every } x \in \mathbb{R}^n,\ f \in \mathcal{S}. \tag{3.1}$$

The question of boundedness of T_m is extremely well studied in the Euclidean setting as well as on manifolds, groups, symmetric spaces, and in many other settings. See for example [16], [18], [24], [28], [30] and the references therein.

Fourier multipliers on Triebel-Lizorkin spaces have been studied by Triebel in [34]. For anisotropic Besov and Triebel-Lizorkin spaces we refer to the articles [4], [5] of Bényi and Bownik.

Yang and Yuan in [36] introduced some general scales; Triebel-Lizorkin-type spaces. For Fourier multipliers on such spaces, see D. Yang et al. [37].

Perhaps the most well-known multiplier conditions (with respect to anisotropic geometry) are the following Mihlin and Hörmander conditions, which we will state as L^∞ and L^2 conditions for compatibility with the new condition that we are going to introduce below.

Let $\alpha \in \mathbb{R}$, $N \in \mathbb{N}$ and $m \in \mathscr{C}^N$, we say that m satisfies the

(1) L^∞-*condition*, when

$$\sup_{|\gamma|\le N}\sup_{\xi\in\mathbb{R}^n}\left|(1+|\xi|_{\vec{a}})^{-\alpha+\vec{a}\cdot\gamma}\partial^\gamma m(\xi)\right| < \infty, \tag{3.2}$$

(2) L^2-*condition*, when

$$\sup_{|\gamma|\le N}\sup_{j\ge 0}\left\{2^{-j\alpha}2^{j\vec{a}\cdot\gamma}2^{-j\nu/2}\,\|\partial^\gamma m\|_{L^2(R_j)}\right\} < \infty, \tag{3.3}$$

where R_j are as in (2.4) and (2.9).

REMARK 3.1. (1) The conditions (3.2) and (3.3) are the anisotropic analogues of the classical inhomogeneous ones, see [4], [5], [18], [30], with the extra parameter $\alpha \in \mathbb{R}$, which allows us to interplay between different smoothness levels as in [10], [37].

(2) Under the isotropic versions of the above conditions, Antonić and Ivec in the recent paper [1], proved the boundedness of Fourier multipliers on mixed Lebesgue and Sobolev spaces.

(3) It is not hard to see that $\|1\|_{L^2(R_j)} = c2^{j\nu/2}$, for every $j \in \mathbb{N}_0$, so the L^2-condition is sharper than the L^∞-condition.

(4) Multipliers on anisotropic homogeneous mixed-norm spaces are considered by the present authors in [11].

3.2. A new class of multipliers

Here we introduce a new class of multipliers replacing the conditions (3.2) and (3.3) above by a mixed-norm $L^{\vec{t}}$-condition of Hörmander type. Before of this we give the following definition.

DEFINITION 3.2. A vector $\vec{t} = (t_1, \dots, t_n) \in [1, 2]^n$ with $1 \le t_n \le t_{n-1} \le \cdots \le t_1 \le 2$ will be called *admissible*. We also denote by $t := 1/t_1 + \cdots + 1/t_n$.

We proceed now to define multiplier classes with respect to both the anisotropy and the mixed-norms.

DEFINITION 3.3. Let $\vec{a} \in [1,\infty)^n$. Given $\alpha \in \mathbb{R}$, an admissible $\vec{t}$ and $N \in \mathbb{N}$, we say that the multiplier $m \in \mathscr{C}^N(\mathbb{R}^n)$ satisfies the $L^{\vec{t}}$*-condition*, or that it belongs to the class $\mathscr{A}(\alpha, \vec{t}, N)$, if

$$A_{\alpha,\vec{t},N}(m) := \sup_{|\gamma|\leq N} \sup_{j\geq 0} \left\{ 2^{-j\alpha} 2^{j\vec{a}\cdot\gamma} 2^{-j\vec{a}\cdot 1/\vec{t}} \, \|\partial^\gamma m\|_{L^{\vec{t}}(R_j)} \right\} < \infty, \tag{3.4}$$

where the R_j's are as in (2.4) and (2.9).

REMARK 3.4. We have the followings remarks pertaining to Definition 3.3.

(1) When $\vec{t} = (2,\dots,2)$, the $L^{\vec{t}}$-condition (3.4) coincides with the Hörmander L^2-condition (3.3).

(2) We observe that $\|1\|_{L^{\vec{t}}(R_j)} = c 2^{j\vec{a}\cdot 1/\vec{t}}$, for every $j \in \mathbb{N}_0$, and $\vec{t}$ admissible. Then from (2.3), we conclude that the $L^{\vec{t}}$-condition is sharper than the L^∞-condition.

(3) For every $\vec{t}, \vec{r} \in [1,2]^n$, with $\vec{t} \leq \vec{r}$ (i.e., $t_j \leq r_j$, $j = 1,\dots,n$), we have

$$\begin{aligned} 2^{-j\nu} \|f\|_{L^1(R_j)} &\leq 2^{-j\vec{a}\cdot 1/\vec{t}} \|f\|_{L^{\vec{t}}(R_j)} \\ &\leq 2^{-j\vec{a}\cdot 1/\vec{r}} \|f\|_{L^{\vec{r}}(R_j)} \leq 2^{-j\nu/2} \|f\|_{L^2(R_j)} . \end{aligned}$$

3.3. The main result

We now proceed to state our main result. However, we need first to fix some notation.

DEFINITION 3.5. For every vector $\vec{p} = (p_1,\dots,p_n) \in (0,\infty)^n$ and every $q \in (0,\infty]$, we set $\mu_j := \min(p_1,\dots,p_j,q)$, $j = 1,\dots,n$ for the case of Triebel-Lizorkin and $\mu_j := \min(p_1,\dots,p_j)$, $j = 1,\dots,n$ for the case of Besov spaces. In every case we denote by $\mu := 1/\mu_1 + \cdots + 1/\mu_n$.

Our main multiplier result is the following.

THEOREM 3.6. *Let $\alpha, s \in \mathbb{R}$, $\vec{p} = (p_1,\dots,p_n) \in (0,\infty)^n$, $q \in (0,\infty]$, $\vec{a} \in [1,\infty)^n$. Let $\vec{t}$ be an admissible vector and let $N \in \mathbb{N}$ with $N > \mu + t$, where t, μ as in Definitions 3.2 and 3.5.*

If m is a multiplier in the class $\mathscr{A}(\alpha, \vec{t}, N)$, then the Fourier multiplier T_m is bounded from $F^{s+\alpha}_{\vec{p}q}(\vec{a})$ to $F^{s}_{\vec{p}q}(\vec{a})$ and from $B^{s+\alpha}_{\vec{p}q}(\vec{a})$ to $B^{s}_{\vec{p}q}(\vec{a})$.

PROOF. We shall treat only the case of Triebel-Lizorkin spaces. The Besov space case is similar and we leave it for the reader.

Let (φ_0, φ) be a pair satisfying (2.4)–(2.8). From (2.8), and bearing in mind (2.11), we can verify that there exists $M \in \mathbb{N}$ such that, for every $j \geq 0$

$$\widehat{\varphi_j} = \sum_{k=j-M}^{j+M} \widehat{\varphi_k}\widehat{\varphi_j},$$

with the convention that $\widehat{\varphi_k} \equiv 0$ if $k < 0$.

Let $f \in F^{s+\alpha}_{\vec{p}q}(\vec{a})$. We have for every $\xi \in \mathbb{R}^n$, using equation (3.1),

$$\begin{aligned} \widehat{\varphi_j}(\xi)\widehat{T_m f}(\xi) &= \widehat{\varphi_j}(\xi)m(\xi)\hat{f}(\xi) \\ &= \sum_{k=j-M}^{j+M} m(\xi)\widehat{\varphi_k}(\xi)\widehat{\varphi_j}(\xi)\hat{f}(\xi). \end{aligned} \tag{3.5}$$

We set

$$m_{(j)}(\xi) := 2^{-j\alpha}m(\xi)\sum_{k=j-M}^{j+M}\widehat{\varphi_k}(\xi), \quad \xi \in \mathbb{R}^n,$$

and

$$g_{(j)}(\xi) := m_{(j)}(2^{j\vec{a}}\xi), \quad \xi \in \mathbb{R}^n.$$

In the light of the above, and by the inverse Fourier transform $\mathscr{F}^{-1}$, (3.5) implies

$$\big(\varphi_j * T_m f\big)(x) = 2^{j\alpha}\big(\mathscr{F}^{-1}(m_{(j)}) * (\varphi_j * f)\big)(x), \quad x \in \mathbb{R}^n. \tag{3.6}$$

From the selection of N, we can find $\varepsilon > 0$ such that $N = \mu + t + 2n\varepsilon$. We set

$$N_k := \frac{1}{\mu_k} + \frac{1}{t_k} + 2\varepsilon, \quad \text{for every } k = 1, \dots, n,$$

and hence $N_1 + \cdots + N_n = N$. We also introduce the vector $\vec{r} := (r_1, \dots, r_n)$ where $r_k := (1/\mu_k + \varepsilon)^{-1}$ for every $k = 1, \dots, n$ and thus

$$\frac{1}{r_k} = N_k - \left(\varepsilon + \frac{1}{t_k}\right), \quad \text{for every } k = 1, \dots, n.$$

By (3.6), we obtain

$$\begin{aligned} \big|\big(\varphi_j * T_m f\big)(x)\big| &\leq 2^{j\alpha}\int_{\mathbb{R}^n}\big|\mathscr{F}^{-1}(m_{(j)})(y)\big|\,|\varphi_j * f(x-y)|\,dy \\ &\leq 2^{j\alpha}\sup_{z\in\mathbb{R}^n}\frac{|\varphi_j * f(x-z)|}{\prod_{k=1}^n(1+|2^{ja_k}z_k|)^{1/r_k}} \times I, \quad x \in \mathbb{R}^n, \end{aligned} \tag{3.7}$$

where we have put

$$I := \int_{\mathbb{R}^n} \left|\mathscr{F}^{-1}(m_{(j)})(y)\right| \prod_{k=1}^{n} (1 + |2^{ja_k} y_k|)^{1/r_k}\, dy.$$

Since $\mathrm{supp}(\widehat{\varphi_j * f}) \subset 2^{j\vec{a}} R_1 \subset 2^{j\vec{a}}[-2,2]^n$, we use the maximal inequality (2.13) to obtain that

$$\sup_{z\in\mathbb{R}^n} \frac{|\varphi_j * f(x-z)|}{\prod_{k=1}^{n}(1+|2^{ja_k} z_k|)^{1/r_k}} \le c\mathscr{M}_{\vec{r}}(\varphi_j * f)(x), \quad x \in \mathbb{R}^n. \tag{3.8}$$

Estimation of I. By the definition of $g_{(j)}$, we have $\mathscr{F}^{-1}(m_{(j)})(y) = 2^{j\nu}\mathscr{F}^{-1}(g_{(j)})(2^{j\vec{a}}y)$, so

$$\begin{aligned} I &= \int_{\mathbb{R}^n} 2^{j\nu}\left|\mathscr{F}^{-1}(g_{(j)})(2^{j\vec{a}}y)\right| \prod_{k=1}^{n}(1+|2^{ja_k}y_k|)^{1/r_k}\, dy \\ &= \int_{\mathbb{R}^n} \left|\mathscr{F}^{-1}(g_{(j)})(x)\right| \prod_{k=1}^{n}(1+|x_k|)^{1/r_k}\, dx, \end{aligned}$$

where for the last equality we changed to the variable $x := 2^{j\vec{a}}y$.

We now apply the mixed Hölder-inequality (2.1) for the admissible $\vec{t} \in [1,2]^n$ and obtain

$$I \le \left\| \mathscr{F}^{-1}(g_{(j)})(x) \prod_{k=1}^{n}(1+|x_k|)^{N_k} \right\|_{\vec{t}'} \left\| \prod_{k=1}^{n}(1+|x_k|)^{-(\varepsilon+1/t_k)} \right\|_{\vec{t}}.$$

Now it is easy to observe that

$$\left\| \prod_{k=1}^{n}(1+|x_k|)^{-(\varepsilon+1/t_k)} \right\|_{\vec{t}} = \prod_{k=1}^{n} \left\| (1+|x_k|)^{-(\varepsilon+1/t_k)} \right\|_{t_k} \le c.$$

On the other hand, since

$$\prod_{k=1}^{n}(1+|x_k|)^{N_k} \le (1+|x|)^N \le c_N \sum_{|\gamma|\le N} |x^\gamma|,$$

we conclude that

$$\begin{aligned} I \le c\left\| \mathscr{F}^{-1}(g_{(j)})(x) \sum_{|\gamma|\le N} |x^\gamma| \right\|_{\vec{t}'} &\le c \sum_{|\gamma|\le N} \left\| \mathscr{F}^{-1}(g_{(j)})(x)x^\gamma \right\|_{\vec{t}'} \\ &=: c \sum_{|\gamma|\le N} I_\gamma. \end{aligned} \tag{3.9}$$

Estimation of I_γ. We have for every multi-index $|\gamma| \le N$,

$$\begin{aligned} I_\gamma &= \left\| \mathscr{F}^{-1}(g_{(j)})(x)x^\gamma \right\|_{\vec{t}'} = \left\| \mathscr{F}^{-1}(\partial^\gamma g_{(j)})(\cdot) \right\|_{\vec{t}'} \\ &= \left\| \mathscr{F}(\partial^\gamma g_{(j)})(-\cdot) \right\|_{\vec{t}'} = \left\| \mathscr{F}(\partial^\gamma g_{(j)})(\cdot) \right\|_{\vec{t}'} . \end{aligned}$$

By assumption, $\vec{t}$ is admissible, so we may apply the mixed Hausdorff-Young inequality (2.2) to obtain

$$I_\gamma = \left\| \mathscr{F}(\partial^\gamma g_{(j)})(\cdot) \right\|_{\vec{t}'} \le \left\| \partial^\gamma g_{(j)}(\cdot) \right\|_{\vec{t}} . \tag{3.10}$$

From the definitions of $m_{(j)}$ and $g_{(j)}$, it follows that

$$(\partial^\gamma g_{(j)})(x) = 2^{j\vec{a}\gamma}(\partial^\gamma m_{(j)})(2^{j\vec{a}}x).$$

Moreover, by a change of variables, we observe that

$$\left\| \partial^\gamma m_{(j)}(2^{j\vec{a}}\cdot) \right\|_{\vec{t}} = 2^{-j\vec{a}\cdot 1/\vec{t}} \left\| \partial^\gamma m_{(j)}(\cdot) \right\|_{\vec{t}}.$$

Additionally, by Leibniz's product rule,

$$\begin{aligned} |\partial^\gamma m_{(j)}(x)| &\le 2^{-j\alpha} \sum_{k=j-M}^{j+M} \left| \partial^\gamma (m\widehat{\varphi_k})(x) \right| \\ &\le 2^{-j\alpha} \sum_{k=j-M}^{j+M} \sum_{\beta\le\gamma} \binom{\gamma}{\beta} \left| (\partial^\beta m)(x)(\partial^{\gamma-\beta}\widehat{\varphi_k})(x) \right| \\ &\le c2^{-j\alpha} \sum_{\beta\le\gamma} |\partial^\beta m(x)|, \end{aligned}$$

since $|(\partial^{\gamma-\beta}\widehat{\varphi_k})(x)| = 2^{-k\vec{a}\cdot(\gamma-\beta)}|(\partial^{\gamma-\beta}\hat{\varphi})(2^{-k\vec{a}}x)| \le c$ using $\beta \le \gamma$ and $\varphi_k \equiv 0$, for $k < 0$.

Combining (3.10) with the above estimates, equation (2.10), and the fact that $\operatorname{supp}(m_{(j)}) \subseteq \bigcup_{k=j-M}^{j+M} R_k$, we arrive at

$$\begin{aligned} I_\gamma &\le c2^{j\vec{a}\cdot\gamma}2^{-j\vec{a}\cdot 1/\vec{t}} \left\| \partial^\gamma m_{(j)}(\cdot) \right\|_{L^{\vec{t}}(\bigcup_{k=j-M}^{j+M} R_k)} \\ &\le c2^{-j\alpha}2^{j\vec{a}\cdot\gamma}2^{-j\vec{a}\cdot 1/\vec{t}} \sum_{\beta\le\gamma} \left\| \partial^\beta m(\cdot) \right\|_{L^{\vec{t}}(\bigcup_{k=j-M}^{j+M} R_k)} \\ &\le c \sum_{\beta\le\gamma} \sum_{k=j-M}^{j+M} 2^{-k\alpha}2^{k\vec{a}\cdot\gamma}2^{-k\vec{a}\cdot 1/\vec{t}} \left\| \partial^\beta m(\cdot) \right\|_{L^{\vec{t}}(R_k)} . \end{aligned} \tag{3.11}$$

By (3.9) and (3.11), we have

$$\begin{aligned} I &\le c \sum_{|\gamma|\le N} \sum_{k=j-M}^{j+M} 2^{-k\alpha} 2^{k\vec{a}\cdot\gamma} 2^{-k\vec{a}\cdot 1/\vec{t}} \left\| \partial^\beta m(\cdot) \right\|_{L^{\vec{t}}(R_k)} \\ &\le c A_{\alpha,\vec{t},N}(m) < c, \end{aligned}$$

since m belongs to the family $\mathscr{A}(\alpha, \vec{t}, N)$. Combining the last inequality with (3.7) and (3.8), we deduce that

$$|(\varphi_j * T_m f)(x)| \le c 2^{j\alpha} \mathscr{M}_{\vec{r}}(\varphi_j * f)(x).$$

We now pass to the Triebel-Lizorkin norm, and apply the mixed-Fefferman-Stein maximal inequality (2.12), to conclude that

$$\begin{aligned} \|T_m f\|_{F^s_{\vec{p}q}(\vec{a})} &= \left\| \left(\sum_{j=0}^{\infty} (2^{js} |\varphi_j * T_m f|)^q \right)^{1/q} \right\|_{\vec{p}} \\ &\le c \left\| \left(\sum_{j=0}^{\infty} (2^{js} 2^{j\alpha} \mathscr{M}_{\vec{r}}(\varphi_j * f)^q \right)^{1/q} \right\|_{\vec{p}} \\ &\le c \left\| \left(\sum_{j=0}^{\infty} (2^{j(s+\alpha)} |\varphi_j * f|)^q \right)^{1/q} \right\|_{\vec{p}} \le c \, \|f\|_{F^{s+\alpha}_{\vec{p}q}(\vec{a})}, \end{aligned}$$

which concludes the proof.

Remark 3.7. Let us consider the case $\vec{t} = (2, \ldots, 2)$. As mentioned in Remark 3.4, our condition coincides with the (anisotropic) Hörmander condition with the extra parameter $\alpha \in \mathbb{R}$ (as in [10], [37]). The smoothness level that we require for the multiplier m is $N = [\mu + n/2] + 1$ for $\mu = 1/\mu_1 + \cdots + 1/\mu_n$, where $\mu_j = \min(p_1, \ldots, p_j, q)$. This means that we ask for

$$N \le N_0 := \left[\frac{n}{\min(p_1, \ldots, p_n, q)} + \frac{n}{2} \right] + 1$$

derivatives on m.

When $\vec{p} = (p, \ldots, p)$, the index N_0 becomes the same as the one appearing in Yang et al. [37].

Sharp multiplier results on Triebel-Lizorkin spaces have been proved by Triebel [34]. The main tool used by Triebel [34] is complex interpolation, and currently such tools are not available in the mixed-norm setting.

4. Special Cases

4.1. Fourier multipliers on anisotropic mixed-norm Sobolev spaces

One of the main motivation for studying Triebel-Lizorkin and Besov spaces is that for specific choices of parameters many well-known spaces of harmonic analysis can be recovered. Let us mention the special case mixed-norm Sobolev and generalized Sobolev spaces, see [27]. Such spaces provide a natural setting for the study of partial differential operators.

Let $\vec{p} \in (1,\infty)^n$ and $\vec{k} \in \mathbb{N}_0^n$, the *mixed-norm Sobolev space* $W_{\vec{p}}^{\vec{k}}$ is defined, as the set of all $f \in \mathscr{S}'$ such that

$$\|f\|_{W_{\vec{p}}^{\vec{k}}} := \|f\|_{\vec{p}} + \sum_{j=1}^{n} \left\| \frac{\partial^{k_j} f}{\partial x_j^{k_j}} \right\|_{\vec{p}} < \infty.$$

Note that $W_{\vec{p}}^{\vec{0}} = L^{\vec{p}}$, for every $\vec{p} \in (1,\infty)^n$.

Now for $\vec{p} \in (1,\infty)^n$, $s \in \mathbb{R}$ and $\vec{a} \in [1,\infty)^n$, the *anisotropic mixed-norm generalized Sobolev space* $H_{\vec{p}}^{s}(\vec{a})$ is defined, as the set of all $f \in \mathscr{S}'$ such that

$$\|f\|_{H_{\vec{p}}^{s}(\vec{a})} := \left\| \mathscr{F}^{-1}\big((1+|\xi|_{\vec{a}}^2)^{s/2}\hat{f}\big)(\cdot) \right\|_{\vec{p}} < \infty.$$

We have the following *identifications*:

(1) when $\vec{p} \in (1,\infty)^n$, $s \in \mathbb{R}$ and $\vec{a} \in [1,\infty)^n$, then

$$F_{\vec{p}2}^{s}(\vec{a}) = H_{\vec{p}}^{s}(\vec{a}), \quad \text{with equivalent norms;}$$

(2) when $\vec{p} \in (1,\infty)^n$, $s \in \mathbb{R}$, $\vec{k} \in \mathbb{N}_0^n$ and $\vec{a} \in [1,\infty)^n$, satisfying $k_j = s/a_j$, for all $j = 1,\dots,n$, then

$$F_{\vec{p}2}^{s}(\vec{a}) = W_{\vec{p}}^{\vec{k}}, \quad \text{with equivalent norms.}$$

Especially $F_{\vec{p}2}^{0}(\vec{a}) = L^{\vec{p}}$.

Before we present our corollaries, we fix an admissible $\vec{t}$ and we keep $q = 2$. Then for every $\vec{p}$, we have $\mu_j = \min(p_1,\dots,p_n,2)$, $j = 1,\dots,n$. Finally recall that $t = 1/t_1 + \cdots + 1/t_n$ and $\mu = 1/\mu_1 + \cdots + 1/\mu_n$.

By all the above, Theorem 3.6 implies the following:

Corollary 4.1. *Let $\alpha, s \in \mathbb{R}$, $\vec{p} \in (1,\infty)^n$, $N \in \mathbb{N}$ and $\vec{a} \in [1,\infty)^n$. If $m \in \mathscr{A}(\alpha,\vec{t},N)$, for $N > \mu + t$, then the Fourier multiplier T_m is bounded from $H_{\vec{p}}^{s+\alpha}(\vec{a})$ to $H_{\vec{p}}^{s}(\vec{a})$.*

By the identification of mixed-norm Triebel-Lizorkin with Sobolev spaces, Theorem 3.6 also offers the following corollary.

Corollary 4.2. *Let α, $s \in \mathbb{R}$, $\vec{p} \in (1,\infty)^n$, $\vec{a} \in [1,\infty)^n$ be such that*

$$\vec{k} := \left(\frac{s}{a_1}, \dots, \frac{s}{a_n}\right), \quad \vec{\ell} := \left(\frac{\alpha}{a_1}, \dots, \frac{\alpha}{a_n}\right) \in \mathbb{N}_0^n.$$

If $m \in \mathscr{A}(\alpha, \vec{t}, N)$, for $N > \mu + t$, then the Fourier multiplier T_m is bounded from $W_{\vec{p}}^{\vec{k}+\vec{\ell}}$ to $W_{\vec{p}}^{\vec{k}}$.

Let us restrict our attention to the isotropic case: $\vec{a} = (1, \dots, 1)$. Then we recover the following recent results by Antonić and Ivec [1]:

Corollary 4.3. *Let α, $s \in \mathbb{N}_0$ and $\vec{p} \in (1,\infty)^n$. If $m \in \mathscr{A}(\alpha, \vec{t}, N)$, for $N > \mu + t$, then the Fourier multiplier T_m is bounded from $W_{\vec{p}}^{s+\alpha}$ to $W_{\vec{p}}^{s}$. In particular, when $\alpha = 0$, then T_m is bounded on $L^{\vec{p}}$.*

4.2. *Equivalent characterizations*

We conclude the paper by considering one explicit example of a bounded Fourier multiplier that can be used to obtain an equivalent norm on anisotropic mixed-norm Besov and Triebel-Lizorkin spaces.

Let $\vec{a} \in [1,\infty)^n$. We consider the following *anisotropic bracket*,

$$\langle x \rangle_{\vec{a}} := |(1, x)|_{(1,\vec{a})}, \quad x \in \mathbb{R}^n.$$

This quantity has been studied in detail in [6]. It is known that $\langle \cdot \rangle_{\vec{a}} \in \mathscr{C}^\infty(\mathbb{R}^n)$ and that for $\alpha \in \mathbb{R}$, $\gamma \in \mathbb{N}_0^n$, there are constants $c_\gamma, c'_\gamma > 0$ such that for every $\xi \in \mathbb{R}^n$

$$|\partial^\gamma \langle \xi \rangle_{\vec{a}}^{\alpha}| \le c_\gamma \langle \xi \rangle_{\vec{a}}^{\alpha - \vec{a}\cdot\gamma} \le c'_\gamma (1 + |\xi|_{\vec{a}})^{\alpha - \vec{a}\cdot\gamma}.$$

So the multiplier $m_\alpha(\xi) := \langle \xi \rangle_{\vec{a}}^{\alpha}$, $\alpha \in \mathbb{R}$, satisfies the Mihlin condition (3.2) for arbitrary $N \in \mathbb{N}$. By Theorem 3.6, the multiplier T_{m_α} is bounded from $F_{\vec{p}q}^{s+\alpha}(\vec{a})$ to $F_{\vec{p}q}^{s}(\vec{a})$ and from $B_{\vec{p}q}^{s+\alpha}(\vec{a})$ to $B_{\vec{p}q}^{s}(\vec{a})$. Moreover, we observe that $T_{m_\alpha} \circ T_{m_{-\alpha}}$ is the identity on $\mathscr{S}'$ and thus we have the following characterization.

Corollary 4.4. *Let α, $s \in \mathbb{R}$, $\vec{p} = (p_1, \dots, p_n) \in (0,\infty)^n$, $q \in (0,\infty]$ and $\vec{a} \in [1,\infty)^n$. Then $\|T_{m_\alpha} f\|_{B_{\vec{p}q}^{s-\alpha}(\vec{a})}$ and $\|T_{m_\alpha} f\|_{F_{\vec{p}q}^{s-\alpha}(\vec{a})}$ are equivalent quasi-norms on $B_{\vec{p}q}^{s}(\vec{a})$ and $F_{\vec{p}q}^{s}(\vec{a})$, respectively.*

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