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COMMUTATIVE n-GORENSTEIN RINGS

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1. Introduction.

The purpose of this article is to expand on Bass' theme that Gorenstein rings are ubiquitous (Bass [4]). The article arose out of observations by the first author who examined several connections between various results which have appeared in the literature in the last few years.

In this section we introduce common terminology and give some results about the class of rings which we will consider. The general standing assumption is that all rings are commutative with identity, that all modules are unitary. Except in the last section, all rings are Noetherian, and most modules are of finite type, the exceptions being easy to identify. With these preliminaries in mind, recall that a local ring A is Gorenstein if $\operatorname{inj\,dim}_A A < \infty$. Bass [4] has given other equivalent conditions which will be used throughout. Also recall that the depth of the local ring A, $\operatorname{depth} A$, is the length of a maximal A-sequence in the maximal ideal of A.

Now Ischebeck [11] discussed a family of rings which he called G_n-rings (no doubt G for Gorenstein). He says a ring A is G_n if A possesses the two properties

(S_n) $\quad \operatorname{depth} A_\mathfrak{q} \geqq \inf(n, \operatorname{ht}\mathfrak{q})$

(T_{n-1}) $\quad$ *If* $\operatorname{ht}\mathfrak{q} \leqq n-1$, *then* $A_\mathfrak{q}$ *is Gorenstein*

for all prime ideals in $\operatorname{Spec} A$.

We also say that A is S_n or A is T_{n-1}, respectively.

Ischebeck proved the following statements to be equivalent (Theorem 3.15 in [11]):

a) $\quad A$ *is* G_n (that is, (S_n) and (T_{n-1})).

b) $\quad$ *For each A-sequence $x_1,\ldots,x_i$, $i \leqq n-1$, the total quotient ring of $A/\Sigma Ax_i$ is quasi-Frobenius.*

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c) *If* $E^0 \to E^1 \to \ldots$ *is a minimal injective resolution of* A, *then* $\operatorname{grade} E^j \geqq \inf(n,j)$.
(Here, by definition, $\operatorname{grade} E = \inf_{m \in E} \operatorname{grade}(A/\operatorname{Ann} Am)$.)

c¹) *In the situation of* c), $\operatorname{grade} E^j \geqq j$ for $j \leqq n$.

d) *For all* M, $\operatorname{grade} \operatorname{Ext}_A{}^j(M,A) \geqq \inf(n,j)$.

Other statements equivalent to these were discovered by Auslander and Bridger [1]. Let M be an A-module (of finite type) and

$$F_1 \to F_0 \to M \to 0$$

an exact sequence with F_i free as an A-module for each i. Let $M^\vee = \operatorname{Hom}_A(M,A)$. Then the sequence induces an exact sequence

$$0 \to M^\vee \to F_0{}^\vee \to F_1{}^\vee \to D(M) \to 0\ ,$$

$D = D(M)$ being the cokernel of $F_0{}^\vee \to F_1{}^\vee$. Auslander and Bridger say that M is *k-torsionfree* if $\operatorname{Ext}_A{}^j(D,A) = 0$ for $j = 1,\ldots,k$. This notion is independent of the resolution $F_1 \to F_0$. In fact, it can be shown that there is induced an exact sequence

$$0 \to \operatorname{Ext}_A{}^1(D,A) \to M \to M^{\vee\vee} \to \operatorname{Ext}_A{}^2(D,A) \to 0\ ,$$

which shows that M 2-torsion free means M is reflexive.

Now say M is an nth *syzygy* if there is an exact sequence

$$0 \to M \to F_{n-1} \to F_{n-2} \to \ldots \to F_0$$

with each F_i free (of finite type). If M is n-torsion free, then M is an nth syzygy. The equivalence of the two concepts is another characterization of G_n-rings. The following, in fact, are equivalent (Auslander–Bridger [1]):

d¹) *For all* M, $\operatorname{grade} \operatorname{Ext}_A{}^j(M,A) \geqq j$ *for* $j \leqq n$.

e) *For all* M, *if* $0 \to K_{n+1} \to F_n \to \ldots \to F_0 \to M \to 0$ *is exact with each* F_j *free, then* K_{n+1} *is* $(n+1)$*-torsion free*
(write $\Omega^{n+1}M = K_{n+1}$ so $\Omega^0 M = M$, $\Omega^n M = \Omega(\Omega^{n-1}M)$).

f) *For all* M *and all* $r, 1 \leqq r \leqq n$, *if*

$$0 \to K_{r+1} \to F_r \to \ldots \to F_0 \to M \to 0$$

is exact with each F_j *free, then* K_{r+1} *is* $(r+1)$*-torsion free.*

g) $\operatorname{depth} A_\mathfrak{q} < n$ *implies* $A_\mathfrak{q}$ *is Gorenstein.*

There is also another characterization due to Auslander, obtained from [6, VI (5.3)].

h) *In a minimal injective resolution of* A, $\operatorname{flat\,dim} E^i \leqq i$ for $i < n$.

It is h) which Auslander has taken as a basis for his generalization to the non-commutative case. We will use his terminology, and call the rings *n-Gorenstein rings*.

Let us examine the situation for low n. If A is 0-Gorenstein, then (i) $\operatorname{depth} A_{\mathfrak{q}} \geqq \inf(\operatorname{ht}\mathfrak{q}, 0) = 0$, and (ii) $\operatorname{ht}\mathfrak{q} < 0$ implies $A_{\mathfrak{q}}$ is Gorenstein. So every ring is 0-Gorenstein.

If A is 1-Gorenstein, then A has the properties:

(i) $\operatorname{depth} A_{\mathfrak{q}} \geqq \inf(\operatorname{ht}\mathfrak{q}, 1)$ and (ii) $\operatorname{ht}\mathfrak{q} < 1$ implies $A_{\mathfrak{q}}$ is Gorenstein.

But perhaps it is easier to look at condition b). It says that A is 1-Gorenstein if and only if the total quotient ring of A is quasi-Frobenius.

If A is 2-Gorenstein, then A has the properties:

(i) $\operatorname{depth} A_{\mathfrak{q}} \geqq \inf(\operatorname{ht}\mathfrak{q}, 2)$ and (ii) $\operatorname{ht}\mathfrak{q} < 2$ implies $A_{\mathfrak{q}}$ is Gorenstein.

Vasconcelos [19] has called a 2-Gorenstein ring a *quasinormal* ring.

There is yet another characterization of n-Gorenstein rings which comes about from considering the Cousin complex $C(A)$ [17].

i) *$C(A)$ is a minimal injective resolution of A up to $C^{n-2}(A)$* [18].

Finally we give a characterization in terms of quotient categories: Let $\boldsymbol{C}$ be the Serre subcategory of the category of A-modules determined by the injective modules $E(A/\mathfrak{p})$, where $\operatorname{depth} A_{\mathfrak{p}} < n$. That is

$$\boldsymbol{C} = \{M\,;\ \operatorname{Hom}_A(M, E(A/\mathfrak{p})) = 0\,;\ \operatorname{depth} A_{\mathfrak{p}} < n\}\,.$$

Let $\mathscr{C}$ denote the corresponding quotient category (see [9]). Now it is easily seen that $\boldsymbol{C} = \{M\,;\ M_{\mathfrak{p}} = 0\,;\ \operatorname{depth} A_{\mathfrak{p}} < n\}$. Using the fact that minimal injective resolutions are preserved when passing to the quotient category, we get as another equivalent statement:

j) $\operatorname{id}_{\mathscr{C}} A < n$.

2. Change of rings.

In this section we will consider changes of rings, and determine cases in which the property of being n-Gorenstein is preserved. The first result is general in nature, while the result about power series extensions does not seem to follow directly from it.

Some notation is necessary in order to state the first result. Let $\mathfrak{p} \in \operatorname{Spec} A$, denote by $k(\mathfrak{p})$ the residue class field of the local ring $A_{\mathfrak{p}}$. If $\varphi\colon A \to B$ is a ring homomorphism, the *fibre* of φ at $\mathfrak{p}$ is the ring $k(\mathfrak{p}) \otimes_A B$.

PROPOSITION 1. *Let φ: $A \to B$ be a ring homomorphism making B into a flat A-module.*

(i) *If B is a faithfully flat A-module and is n-Gorenstein, then A is n-Gorenstein.*
(ii) *If A is n-Gorenstein and the fibres of φ are n-Gorenstein, then B is n-Gorenstein.*

PROOF. Statements (i) and (ii) are known to be true with Gorenstein instead of n-Gorenstein; (ii) by Hartshorne [10, V (9.6)]), and (i) from the formula

[3] $$\operatorname{Ext}_A{}^i(M,A) \otimes {}_A B = \operatorname{Ext}_B{}^i(M \otimes_A B, B) .$$

A different proof is given in [21]. Further, if $\mathfrak{p} \subset A$, choose $\mathfrak{P} \subset B$ minimal such that $\varphi^{-1}(\mathfrak{P}) = \mathfrak{p}$. Then $\operatorname{depth} A_{\mathfrak{p}} = \operatorname{depth} B_{\mathfrak{P}}$ [14, p. 154]. Hence (i) follows from the property g). To see (ii), let $\mathfrak{P} \subset B$, and set $\mathfrak{p} = \varphi^{-1}(\mathfrak{P})$, and use the formula

$$\operatorname{depth} A_{\mathfrak{p}} + \operatorname{depth} (B_{\mathfrak{P}}/\mathfrak{p} B_{\mathfrak{P}}) = \operatorname{depth} B_{\mathfrak{P}}$$

and property g).

COROLLARY. a) *If A is an n-Gorenstein ring, then $A[t]$ is n-Gorenstein* (where t is an indeterminate).

b) *If A is an n-Gorenstein ring and* S *is a multiplicatively closed subset of A, then* $\mathrm{S}^{-1}A$ *is n-Gorenstein.*

c) *If m is an ideal in the radical of A and the m-adic completion $\hat{A}$ is n-Gorenstein, then A is n-Gorenstein.*

PROOF. We prove c) first. As m is in the radical of A, the completion $A \to \hat{A}$ is faithfully flat. Hence c) follows from part (i) of Proposition 1. To see the two other statements, it is enough to consider the fibres of the homomorphism. The fibres of $A \to A[t]$ are regular rings and hence certainly n-Gorenstein. The fibre of $A \to \mathrm{S}^{-1}A$ at $\mathfrak{p}$ is either $k(\mathfrak{p})$ or zero depending on whether $\mathrm{S} \cap \mathfrak{p}$ is empty or not.

Part a) could also be proved directly, using localization arguments and the corresponding result with n-Gorenstein replaced by Gorenstein. In fact, A Gorenstein implies $A[t]$ Gorenstein is proved in [15]. This is also a special case of the following result of I. Beck:

PROPOSITION 2. *Let N be an $A[t]$-module. Then* $\operatorname{id}_{A[t]} N \leqq 1 + \operatorname{id}_A N$.

The case of power series adjunction is not so simple. The problem in trying to apply Proposition 1 appears when one attempts to show that the fibres of $A \to A[[t]]$ are n-Gorenstein. This can be avoided by a direct attack on the problem. (Sharp remarked that the fibres behave nicely when A is a quotient of a Gorenstein ring.) This is done in our next result.

PROPOSITION 3. *Let A be a local ring with maximal ideal m. Let x be a regular element in m. If $A/(x)$ is n-Gorenstein, then so is A.*

PROOF. We first show that if $A/(x)$ has the property S_n, then each prime ideal $\mathfrak{q}$ in A, with $\mathrm{ht}\,\mathfrak{q} \geqq n$, contains an A-sequence of length n. Let $\mathrm{ht}\,\mathfrak{q} \geqq n$.

If $x \in \mathfrak{q}$, consider a maximal A-sequence $x, x_2, \ldots, x_i$ in $\mathfrak{q}$. Then $x_2, \ldots, x_i \bmod (x)$ is a maximal sequence in $\mathfrak{q}/(x)$. Since $A/(x)$ is S_n, $\mathfrak{q}/(x)$ is a minimal prime ideal over the ideal generated by $x_2, \ldots, x_i \bmod (x)$, if $i < n$. Hence $\mathfrak{q}$ is minimal over $(x, x_2, \ldots, x_i)$. Since A is Noetherian, $\mathrm{ht}\,\mathfrak{q} \leqq i$, and we conclude that $i \geqq n$.

If $x \notin \mathfrak{q}$, let $\mathfrak{a} = (x) + \mathfrak{q}$ and choose $\mathfrak{p}$ minimal over $\mathfrak{a}$ such that $\mathrm{grade}\,A/\mathfrak{p} = \mathrm{grade}\,A/\mathfrak{a}$. Then $\mathrm{ht}\,\mathfrak{p}/(x) \geqq n$. For, otherwise let $x, x_2, \ldots, x_i$ be a maximal A-sequence in $\mathfrak{p}$, with $i \leqq n$. Then $x_2', \ldots, x_i'$ would be a maximal A'-sequence in $\mathfrak{p}'$ (where $'$ denotes reduction modulo (x)). By property (S_n) for $A/(x)$, $\mathfrak{p}'$ is minimal over the ideal $(x_2', \ldots, x_i')$. Hence $\mathfrak{p}$ is minimal over $(x, x_2, \ldots, x_i)$, so $\mathrm{ht}\,\mathfrak{p} \leqq i \leqq n$. But $\mathfrak{p} \supset \mathfrak{q}$, and $\mathrm{ht}\,\mathfrak{q} \geqq n$, a contradiction.

Now $A/(x)$ has property S_n, so there is an A-sequence $x, x_2, \ldots, x_{n+1}$ in $\mathfrak{p}$. By choice of $\mathfrak{p}$, the elements $x_2, \ldots, x_{n+1}$ can be taken from $\mathfrak{a}$. Let $x_i = r_i x + f_i$, with $f_i \in \mathfrak{q}$. Then $x, f_2, \ldots, f_{n+1}$ is an A-sequence. For suppose $y f_2 \in (x)$. Then $y x_2 \in (x)$, hence $y \in (x)$. Now go by way of induction, using the fact that any permutation of an A-sequence is an A-sequence for a local Noetherian ring A. We then conclude that $f_2, \ldots, f_{n+1}$ is an A-sequence in $\mathfrak{q}$. This completes the proof of the first part.

Assuming that $A/(x) = A'$ is n-Gorenstein we want to show that $A_{\mathfrak{q}}$ is Gorenstein for $\mathrm{ht}\,\mathfrak{q} < n$. Let $\mathfrak{q}$ be a prime ideal with $\mathrm{ht}\,\mathfrak{q} < n$. If $x \in \mathfrak{q}$, then $\mathrm{ht}(\mathfrak{q}/(x)) < n-1$. Hence $A'_{\mathfrak{q}'} = A_{\mathfrak{q}}/xA_{\mathfrak{q}}$ is Gorenstein. This implies $A_{\mathfrak{q}}$ Gorenstein.

If $x \notin \mathfrak{q}$, let $\mathfrak{a} = (x) + \mathfrak{q}$, and let $\mathfrak{p}$ be minimal over $\mathfrak{a}$ such that $\mathrm{grade}\,A/\mathfrak{p} = \mathrm{grade}\,A/\mathfrak{a}$. If $\mathrm{ht}\,\mathfrak{p}' \geqq n$, we can find an A-sequence $x, x_2, \ldots, x_n$ of length $n+1$ in $\mathfrak{p}$, hence in $\mathfrak{a}$ and by the same argument as above, an A-sequence $f_2, \ldots, f_{n+1}$ of length n in $\mathfrak{q}$, which is a contradiction to $\mathrm{ht}\,\mathfrak{q} < n$. Hence $\mathrm{ht}\,\mathfrak{p}' < n$, which implies $A'_{\mathfrak{p}'} = A_{\mathfrak{p}}/xA_{\mathfrak{p}}$ is Gorenstein. Hence $A_{\mathfrak{p}}$ is Gorenstein, and consequently also $A_{\mathfrak{q}}$. This completes the proof of Proposition 3.

COROLLARY. *If A is n-Gorenstein, then $A[[t]]$ is n-Gorenstein* (and conversely).

PROOF. The property of being n-Gorenstein is local. Furthermore t is in the radical of $A[[t]]$. Let $\mathfrak{M}$ be a maximal ideal of $A[[t]]$, $\mathfrak{m}=\mathfrak{M}\cap A$. Then $A[[t]]_{\mathfrak{M}}/tA[[t]]_{\mathfrak{M}}=A_{\mathfrak{m}}$. As $A_{\mathfrak{m}}$ is n-Gorenstein, $A[[t]]_{\mathfrak{M}}$ is n-Gorenstein.

The other change of rings results are rather easy, and also expected.

PROPOSITION 4. a) *If $A/\sum x_j A$ is 1-Gorenstein for all A-sequences $x_1,\ldots,x_i$ of length at most $n-1$, then A is n-Gorenstein.*

b) *If A is local,* $\operatorname{depth} A \geqq n-1$, *and if $A/(x_1,\ldots,x_{n-1})$ is 1-Gorenstein for all A-sequences of length $n-1$, then A is n-Gorenstein.*

c) *If A is n-Gorenstein, then A/Ax is $(n-1)$-Gorenstein whenever x is regular in A.*

These statements follow immediately from condition b) in the first section.

It should be remarked that the proofs of Proposition 1 and 3 yield, with little modification, the results that power series or polynomial adjunction preserves normality. In particular, A is S_2 implies $A[[t]]$ and $A[t]$ are both S_2 by the proofs. It is well-known that A regular implies $A[t]$ regular, so the corresponding result follows for prime ideals of codimension one.

3. Duality.

The second author (Fossum [8]) has generalized some results which Roos first established for regular rings. The techniques used in Fossum [8] apply to n-Gorenstein rings. In some instances the statements are sufficient as well as necessary in order that the ring be n-Gorenstein.

The homomorphism $M \to M^{\vee\vee}$ for any A-module M has an analogue when $\operatorname{Ext}_A{}^i(M,A)$ vanishes for small i. In fact there is a natural map

$$M \to \operatorname{Ext}_A{}^i(\operatorname{Ext}_A{}^i(M,A),A)$$

provided grade $M \geqq i$ (and M is of finite type). The kernel and cokernel of this homomorphism are of the form $\operatorname{Ext}_A{}^{i+1}(D,A)$ and $\operatorname{Ext}_A{}^{i+2}(D,A)$ for a suitable D (see the Introduction where the result for $i=0$ is stated).

If M is an A-module of finite type, there is a unique submodule $M(i)$ of M maximal with respect to the property that grade $M(i) \geqq i$. (Then $M(0)=M$ and in general $M(i) \supseteq M(i+1)$.)

In order to make the connections clear, it may be wise to state the pertinent result in Fossum [8].

PROPOSITION. *Let A be locally Gorenstein. Let*

$$L_i(M) = \mathrm{Ext}_A{}^i(\mathrm{Ext}_A{}^i(M,A),A) .$$

Then

(i) $L_i(L_i(M)) \cong L_i(M)$,
(ii) grade $M \geqq i$ *implies* $M(i+1) = \mathrm{Ker}(M \to L_i(M))$,
(iii) grade $\mathrm{Ext}_A{}^j(L_i(M)) \geqq j+2$ *for* $j > i$,
(iv) $L_i(M(i)) \cong L_i(M)$, *and*
(v) grade $\mathrm{Coker}(M(i) \to L_i(M)) \geqq i+2$.

It is seen that (i) is true if A is $i+1$-Gorenstein. Now by e) a ring A is 1-Gorenstein if and only if each dual is reflexive. This has the obvious generalization.

PROPOSITION 5. *A is an $(n+1)$-Gorenstein ring if and only if for all $i=0,1,\ldots,n$ and for all modules M with* grade $M \geqq i$ *it is the case that* $L_i(\mathrm{Ext}_A{}^i(M,A)) \cong \mathrm{Ext}_A{}^i(M,A)$.

PROOF. If A is an $n+1$-Gorenstein ring then the statement about the modules follows from a generalized version of the proposition above. Suppose that the second statement in the proposition holds. Let $x_1,\ldots,x_i$, $i \leqq n$, be an A-sequence and let M be an $A/\sum Ax_j$-module. Then

$$\mathrm{Ext}_A{}^i(M,A) \cong \mathrm{Hom}_A(M,A') \quad \text{and} \quad L_i(\mathrm{Ext}_A{}^i(M,A)) \cong \mathrm{Ext}_A{}^i(M,A)$$

implies $M^{\vee}$ is a reflexive A'-module ($A' = A/\sum Ax_j$). Since M is arbitrary, any dual is reflexive. Hence A' is a 1-Gorenstein-ring. Hence A is $(n+1)$-Gorenstein.

Statement (ii) in the previous Proposition is seen to hold for n-Gorenstein rings by considering the sequence

$$0 \to \mathrm{Ext}_A{}^{i+1}(D(\Omega^i M), A) \to M \to L_i(M) \to \mathrm{Ext}_A{}^{i+2}(D(\Omega^i M), A) \to 0 ,$$

(where $\Omega^i M$ is the ith syzygy of M as defined in e) of the Introduction), which is exact in case grade $M \geqq i$, and then using d).

This leads to another characterization which generalizes the statement: A is 1-Gorenstein if and only if each torsion free module is torsionless (Vasconcelos [19]).

Proposition 6. *A is an* $n+1$-*Gorenstein-ring if and only if*

$$M(j+1) = \operatorname{Ker}(M \to L_j(M))$$

for all $j=0,1,\ldots,n$, *and all M with* grade $M \geqq j$.

Proof. The one implication follows from statement (ii) as generalized above. Suppose the module statement is true and let us try to establish that A is $n+1$-Gorenstein. Let $A'=A/\sum x_i A$ where $x_1,\ldots,x_j$, $j \leqq n$, is an A-sequence. Let N be a torsion free A'-module, that is, $N_{A'}(1)=0$. (*Torsion free* means no regular element of A' is a zero divisor. So $N_{A'}(1) \neq 0$ would give a submodule of grade $\geqq 1$ which would have a zero divisor. So N is torsion free if and only if $N(1)=0$.) Hence $N_A(j+1)=0$, so $N \to L_j(N)$ is an injection. This gives that $N \to N^{\vee\vee}$ ($^\vee$ with respect to A') is an injection, so N is torsionless. Hence A' is 1-Gorenstein, so A is $n+1$-Gorenstein.

Both Propositions 5 and 6 have stronger statements when A has some unmixedness conditions.

Proposition 7. *Let A be such that* depth $A_{\mathfrak{m}} \geqq n$ *for all maximal ideals* $\mathfrak{m}$ *of A. Then the following are equivalent*:

a) A is $(n+1)$-Gorenstein.
b) grade $M \geqq n$ implies $L_n(\operatorname{Ext}_A{}^n(M,A)) \cong \operatorname{Ext}_A{}^n(M,A)$.
c) grade $M \geqq n$ implies $M(n+1) = \operatorname{Ker}(M \to L_n(M))$.

Corollary. *If* grade $M \geqq i$ *implies* $M(i+1)=\operatorname{Ker}(M \to L_i(M))$ *for all i and all M, then A is Gorenstein.*

In Fossum [8] a modification of a theorem in Auslander–Buchsbaum [2] is given, such modification being related also to results in Vasconcelos [20] and Fossum [8]. However, the result does not cover the Auslander–Buchsbaum nor the Vasconcelos result because it is assumed that the ring is Gorenstein. Using the more general notion of n-Gorenstein rings and the results of the Proposition which hold for, say $n+2$-Gorenstein rings, it is possible to give a most general result of this nature which includes all the previous ones as special cases.

Proposition 8. *Let A be* $(n+2)$-*Gorenstein and let N be an A*-module *with* grade $N \geqq n$ *and* $N(n+1)=0$. *Let M be a submodule of N and set* $Q=N/M$.

a) *If* $M = L_n(M)$, *then* $Q(n+2) = 0$.
b) *If* $N = L_n(N)$ *and* $Q(n+2) = 0$, *then* $M = L_n(M)$.

PROOF. a) The module $\mathrm{Ext}^{n-1}(\mathrm{Ext}^{n-1}(Q, A), A)$ is 0 if all the hypotheses hold and A is n-Gorenstein. Further,

$$Q(n+1) = \mathrm{Ker}(Q \to L_n(Q))$$

if A is $(n+1)$-Gorenstein. So the result follows.

b) $\mathrm{grade}(L_n(M)/M) \geqq n+2$ if A is $(n+2)$-Gorenstein. Hence the proposition is valid.

Ischebeck [11] has considered a filtration on an A-module, which in certain cases coincides with the grade filtration. Ischebeck's filtration $U^r M$ is obtained from a convergent spectral sequence and he compares it with the grade filtration, that is, his filtration coincides with the grade filtration up to n if A is an n-Gorenstein-ring. This condition is also sufficient (Corollary 3.26 of Ischebeck [11]) yielding yet another characterization of an n-Gorenstein-ring.

Other results which Ischebeck obtains concern properties of nth syzygies. His Proposition 4.6 asserts that, for an n-Gorenstein ring A, an A-module, M, of finite type is an nth syzygy if and only if each A-sequence of length at most n is an M-sequence. This is related to Vasconcelos' result [19] which says that for any 2-Gorenstein ring, a module is reflexive if, and only if, each A-sequence of length at most 2 is an M-sequence. For a 1-Gorenstein ring A, M is second syzygy if and only if it is reflexive. In case $n = 1$, A is 1-Gorenstein if and only if every finitely generated torsion free module is torsionless (Vasconcelos [19]). A partial converse to Ischebeck's result for $n = 2$ is

PROPOSITION 9. *If A is a ring which satisfies the condition*
"Each A-sequence of length at most $n (n = 1, 2)$ is an M-sequence implies M is an nth syzygy"
then $A_{\mathfrak{q}}$ is Gorenstein if $\mathrm{ht}\,\mathfrak{q} \leqq 1$.

PROOF. Let $\mathrm{ht}\,\mathfrak{q} \leqq 1$. Let $\mathfrak{a}$ be the contraction in A of an ideal $\mathfrak{a}_{\mathfrak{q}}$ in $A_{\mathfrak{q}}$. Then $\mathfrak{p} \in \mathrm{Ass}\,A/\mathfrak{a}$ implies $\mathfrak{p} \subset \mathfrak{q}$. For if $f \in A$ is such that $\mathfrak{p}f \subset \mathfrak{a}$ where $\mathfrak{p} \in \mathrm{Ass}\,A/\mathfrak{a}$ and if $\mathfrak{p} \not\subset \mathfrak{q}$, then $f \in \mathfrak{a}$, since $\mathfrak{a}$ is a contraction. Hence $\mathrm{depth}\,A_{\mathfrak{q}} \geqq 2$ implies $\mathrm{depth}\,(A/\mathfrak{a})_{\mathfrak{q}} \geqq 1$, which implies $\mathrm{depth}\,\mathfrak{a}_{\mathfrak{q}} \geqq 2$. Since $\mathfrak{a}$ is torsionless, $\mathrm{depth}\,A_{\mathfrak{q}} \geqq 1$ implies $\mathrm{depth}\,\mathfrak{a}_{\mathfrak{q}} \geqq 1$. Hence

$$\mathrm{depth}\,\mathfrak{a}_{\mathfrak{q}} \geqq \inf(2, \mathrm{depth}\,A_{\mathfrak{q}}).$$

By Ischebeck [11], we conclude that each A-sequence of length at most 2 is an $\mathfrak{a}$-sequence. So by assumption, $\mathfrak{a}$ is a 2nd syzygy. The assumption for the case $n=1$ yields a 1-Gorenstein ring, so we can conclude that $\mathfrak{a}$ is reflexive. Hence each ideal in $A_{\mathfrak{q}}$ is reflexive, so $A_{\mathfrak{q}}$ is Gorenstein.

4. Non-Noetherian rings.

Many of the results which hold for normal integral domains hold also for Krull domains, especially those results concerning reflexive properties of modules with suitable finiteness conditions. It is our desire to consider a suitable setting in which some of the results in the previous sections hold, even in the non-Noetherian case, and which, at the same time, cover the known results for Krull domains.

Beck [5] has considered modules (and rings) for which certain of the modules in a minimal injective resolution have some finiteness property. Say an injective module E is *Σ-injective* if any direct sum of copies of E is also injective. (This definition must be made for a module over a not necessarily Noetherian ring. For it is a fact that a ring is Noetherian if, and only if, each injective module is Σ-injective.)

Now let M be an R-module. Let $0 \to M \to E^0 \to E^1 \to \dots$ be a minimal injective resolution of M. Say M has property B_n, or is B_n, if E^j is Σ-injective for $0 \leqq j \leqq n$.

For example, every module is B_n for a Noetherian ring and for every n.

Now a B_n-module of finite type has the following properties (Beck [5]).

(i) If x is M-regular and if M is B_n $(n \geqq 1)$, then M/xM is B_{n-1}.

(ii) If M is B_0, then the set of divisors of zero is the union, in R, of the prime ideals of R associated to M and $\operatorname{Ass} M$ is a finite set.

(iii) R is B_0 as an R-module, if, and only if, the total quotient ring is Noetherian. (Let S be the set of regular elements in R. Then R is B_0 if, and only if, $\mathrm{S}^{-1}R$ is Noetherian.)

(iv) If $0 \to N' \to N \to N'' \to 0$ is an exact sequence of R-modules, and if N is B_{k+r} and N'' is B_k, then N' is $\mathrm{B}_{k+1} (r > 0)$.

(v) If M is B_n, then $M_{\mathfrak{p}}$ is B_n for all prime ideals $\mathfrak{p}$.

(vi) If M is B_n, and if $\operatorname{Ext}_R{}^j(R/\mathfrak{a}, M)=0$ for $0 \leqq j \leqq n$, then $\mathfrak{a}$ contains an M-sequence of length $n+1$.

(vii) if each R-sequence of length at most i is an M-sequence and if R is B_{i-1}, then M is B_{i-1}.

(Beck writes n.d. $M=n$ if M is B_n and not B_{n+1}.)

Now let R be a commutative ring. (Note that we have changed notation slightly. Before rings were denoted by A, and they were understood to be Noetherian. As mentioned before, the rings in this section are not (necessarily) Noetherian.) Say R is G_n' if R is B_{n-1}, if $R_{\mathfrak{q}}$ is Gorenstein whenever $\operatorname{ht}\mathfrak{q} < n$ and if $\operatorname{depth} R_{\mathfrak{q}} \geqq \inf(\operatorname{ht}\mathfrak{q}, n)$.

We now consider B_{n-1}-rings instead of Noetherian rings, and find that some of the characterizations of n-Gorenstein rings hold for these rings as well. Still others would have been valid if we had considered B_n-rings, but then our class of rings would not have contained the Krull rings for $n=2$. For other characterizations, one sided implications are true. We leave out the cases where M of finite type would have to be replaced by several of the F_j in a free resolution being of finite type, with one exception, which is needed for Proposition 12. We then proceed to study the cases $n=1,2$, where better results can be obtained.

Proposition 10. *Let R be B_{n-1}. Then* a), b), g) *and* h) *are equivalent.*

Proof. a) $\Leftrightarrow$ g) is as in the Noetherian case. The proof of a) $\Leftrightarrow$ b) follows Ischebeck's proof [11], since the facts he uses about Noetherian rings still hold for B_{n-1}-rings, by the properties already listed. Finally, g) $\Leftrightarrow$ h) follows by using that $E_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) = E(M)_{\mathfrak{p}}$, if $E(M)$ is Σ-injective (*).

Of the onesided implications we give

Proposition 11. *Let R be a B_{n-1}-ring.*

1) *If* $\operatorname{grade} E^i \geqq i$, $i \leqq n$, *then R is a G_n'-ring.*
2) *If* $\operatorname{grade} E^i \geqq \min(i,n)$, *then* $\operatorname{grade} \operatorname{Ext}^i(M,R) \geqq \min(i,n)$; *$M$ of finite-type.* (*If* $\operatorname{grade} E^i \geqq i$; $i \leqq n$, *then* $\operatorname{grade}\operatorname{Ext}^i(M,R) \geqq i$; $i \leqq n$.)
3) *If R is G_n', then* $\operatorname{grade}\operatorname{Ext}^i(M,R) \geqq i$, $i \leqq n$, *where in*

$$F_i \to \ldots F_1 \to F_0 \to M ,$$

all F_j are of finite type.

Proof. 1) follows as in Bass [4], using again observation (*).

2) Let $0 \to R \to E^0 \to \ldots \to E^n$ be a minimal injective resolution of R. Let $f \in \operatorname{Hom}(M, E^i)$. Then

$$\operatorname{Ann} f = \operatorname{Ann} e_1 \cap \ldots \cap \operatorname{Ann} e_r, \quad e_j \in E^i .$$

By assumption $\operatorname{grade}(\operatorname{Ann} e_j) \geqq \min(n,i)$. If $\operatorname{grade}(\operatorname{Ann} f) < \min(n,i)$, we could use R is B_{n-1} to conclude $\operatorname{grade}(\operatorname{Ann} f) = \operatorname{grade}(\operatorname{Ann} e_j)$ for some j, a contradiction. Since $\operatorname{Ext}^i(M,R)$ is a subquotient of $\operatorname{Hom}(M,E^i)$ we are done.

3) We first show that if for an R-module N, $\operatorname{grade} N = i < n$, then

$$i = \operatorname{grade} N \geqq \min_{\mathfrak{q} \in \operatorname{Supp} N} \operatorname{grade} \mathfrak{q} = \min_{\mathfrak{q} \in \operatorname{Supp} N} \operatorname{depth} R_{\mathfrak{q}} .$$

For let $x_1, \ldots, x_i$ be a maximal R-sequence in $\operatorname{Ann} m$, $m \in N$. Then by i) and ii) above, $R/\sum x_j R$ is B_0. Hence there is only a finite number of associated primes $\mathfrak{p}$, so $\operatorname{Ann} m \subset \mathfrak{p}$, some $\mathfrak{p}$ with $\operatorname{grade} \mathfrak{p} = \operatorname{depth} R_{\mathfrak{p}} = i$.

Hence to show $\operatorname{grade} \operatorname{Ext}(M,R) \geqq i$, $i \leqq n$, it is sufficient to show

$$\operatorname{depth} R_{\mathfrak{p}} < i \leqq n \Rightarrow \operatorname{Ext}^i(M,R)_{\mathfrak{p}} = 0 .$$

By assumption, $R_{\mathfrak{p}}$ is Gorenstein, so $\operatorname{Ext}^i(M_{\mathfrak{p}}, R_{\mathfrak{p}}) = 0$. One can then conclude $\operatorname{Ext}^i(M,R)_{\mathfrak{p}} = 0$, using F_j free of finite type; $j \leqq i$.

We now consider the cases $n = 1, 2$.

PROPOSITION 12. *Let R be a* B_0*-ring.*

a) *R is* G_1' *if and only if each torsionfree module of finite type is torsionless.*

b) *Let R be* G_1'.

i) *If M* (of finite type) *is a* 2nd *syzygy, then M is reflexive.*

ii) *If also $M^\vee$ is of finite type, M reflexive, then M is a* 2nd *syzygy.*

PROOF. a) Since the total quotient ring of R is Noetherian, Vasconcelos' proof [19] shows that the module statement implies that R is G_1'. In order to show that R is G_1' implies the module statement, he uses the fact that $^\vee$ commutes with ring of quotient formation. But since $0 \to M \to S^{-1} M$ is exact (S being regular elements in R, and M torsionfree) and since $S^{-1}M$ is torsionless (because $S^{-1}R$ is Gorenstein of dimension 0), we can find a map f: $M \to R$ taking a given $m \neq 0$ to $f(m) \neq 0$ in R by first finding an $S^{-1}R$-homomorphism $S^{-1}M \to S^{-1}R$ such that the image of $m/1$ is not zero, and then lifting to R, using the hypothesis that M is of finite type.

b) i) Consider $0 \to \Omega^2 M \to F_1 \to \Omega^1 M \to 0$ where F_1 is finitely generated and free. There is an exact sequence [12]

$$0 \to \mathrm{T} \to \mathrm{T}^{\vee\vee} \to \operatorname{Ext}_R{}^1(\Omega^1 M, R) \to 0$$

with $\mathrm{T} = \operatorname{Coker}\left((\Omega^1 M) \to F_1{}^\vee\right)$. Take duals again to get

$$0 \to \operatorname{Ext}_R{}^1(\Omega^1 M, R)^\vee \to \mathrm{T}^{\vee\vee\vee} \to \mathrm{T}^\vee .$$

But by 3) of Proposition 11,

$$\operatorname{grade} \operatorname{Ext}_R{}^1(\Omega^1 M, R) \geqq 1$$

(since F_1, F_2 are of finite type), hence $\mathrm{Ext}_R{}^1(\Omega^1 M, R)^\vee = 0$, and $\mathrm{T}^\vee = \Omega^2 M$ is reflexive.

ii) For the other implication, suppose $F_1 \to F_0 \to M^\vee \to 0$ is exact with F_0 of finite type and F_j free. Then we get the exact sequence

$$0 \to M \to F_0 \to F_1$$

with F_0 free of finite type and F_1, being a direct product of copies of R, torsionless. Hence $K = \mathrm{Coker}(M \to F_0)$ is torsion less. K is of finite type so it is a submodule of a free module of finite type. Hence M is a 2nd syzygy.

Next we verify that Ischebeck's Proposition 4.6 in [11] also holds for G_2'-rings.

Proposition 13. *If R is a G_2'-ring, then the following statements about an R-module M, with M and $M^\vee$ of finite type are equivalent:*

(i) *M is a* 2nd *syzygy.*

(ii) *Each R-sequence of length at most two is an M-sequence.*

Proof. Suppose, more generally, that R is B_n and that M is an $(n+1)$th syzygy. Then there is an exact sequence

$$0 \to M \to F_n \to F_{n-1} \to \ldots \to F_0$$

with each F_j free of finite type. Since each F_j is B_n, it follows, by induction, (iv), and the fact that $\mathrm{Coker}(F_2 \to F_1)$ is B_0, that M is B_n. Further M is B_n, and hence, $\mathfrak{p}R_\mathfrak{p}$ contains an $M_\mathfrak{p}$-sequence of length i, $i \leqq n+1$, if, and only if,

$$\mathrm{Ext}^j((R/\mathfrak{p})_\mathfrak{p}, M_\mathfrak{p}) = 0 \quad \text{for} \quad 0 \leqq j < i\,.$$

Thus depth $M_\mathfrak{p} \geqq \inf(n+1, \operatorname{depth} R_\mathfrak{p})$.

Now we conclude the proof that (i) implies (ii) by modifying a proof by Ischebeck, which itself is a modification of a proof of Samuel. So suppose M satisfies (i).

Let $n=0$. Then depth $M_\mathfrak{p} \geqq \inf(1, \operatorname{depth} R_\mathfrak{p})$. So depth $M_\mathfrak{p} = 0$ implies $\operatorname{depth} R_\mathfrak{p} = 0$. Let $\mathfrak{q}$ be a prime ideal contained in the zero divisors on M. Since M is B_0, $\mathfrak{q}$ is contained in one of the finite numbers of prime ideals $\mathfrak{p}$ of R associated to M. Then depth $M_\mathfrak{p} = 0$, so $\operatorname{depth} R_\mathfrak{p} = 0$. So $\mathfrak{p}$ is a prime ideal contained in the zero divisors of R. Hence $Z(M) \subset Z(R)$ ($Z(M) =$ the set of zero divisors of M). So each R-sequence of length one is an M-sequence.

Now suppose $n > 0$ and that each R-sequence of length at most n is an M-sequence. Let $x_1, \ldots, x_{n+1}$ be an R-sequence. Then $x_1, \ldots, x_n$ is an M-sequence. From depth $M_{\mathfrak{p}} \geqq \inf(n+1, \operatorname{depth} R_{\mathfrak{p}})$ it follows that depth $M'_{\mathfrak{p}'} \geqq \inf(1, \operatorname{depth} R'_{\mathfrak{p}'})$ (where $'$ denotes reduction modulo $x_1, \ldots, x_{n+1}$ and $\mathfrak{p} \supseteqq \sum Rx_j$). Also M is B_n implies M' is B_0. Hence the result follows from the case $n = 0$.

Now we wish to show that if each R-sequence of length at most two is an M-sequence, then M is a 2nd module of syzygies. Write $F \to M^{\vee} \to 0$ with F free of finite type. Then $0 \to M^{\vee\vee} \to F$ is exact. Since M is torsion free and hence torsionless, it follows that $M \subset M^{\vee\vee}$ and hence

$$0 \to M \to F^{\vee} \to N \to 0$$

is exact. N is of finite type, and we must show N is torsion free. The ring R is S_2, so depth $M_{\mathfrak{p}} \geqq \inf(2, \operatorname{depth} R_{\mathfrak{p}})$. Furthermore M is B_1, so N is B_0 and hence $\operatorname{depth} N_{\mathfrak{p}} \geqq 1$ (if $\operatorname{depth} R_{\mathfrak{p}} \geqq 2$). If $\operatorname{depth} R_{\mathfrak{p}} = 1$, $R_{\mathfrak{p}}$ is Gorenstein, we are in the Noetherian case, and the usual proof carries through.

There now follows the corollary.

COROLLARY. *Let R be* G_2', *M and $M^{\vee}$ modules of finite type. Then M is reflexive if, and only if, every R-sequence of length at most two is an M-sequence.*

We remark the first part of Proposition 13 could have been proved via "M 2nd syzygy $\Rightarrow$ M reflexive $\Rightarrow$ Each R-sequence of length $\leqq 2$ is an M-sequence", since the last implication is easily seen. But we have proved something more general.

The corresponding result for a Krull domain is a little better.

PROPOSITION 14. *Let R be a Krull domain* (so in fact R is G_2' with the additional assumptions that R is R_1 and is an integral domain). *If M is of finite type, then M is reflexive if, and only if, each R-sequence of length at most* 2 *is an M-sequence.*

REMARK. Here the assumption that both M and $M^{\vee}$ are of finite type is weakened to M of finite type. However, M of finite type implies $M^{\vee}$ is an R-lattice, which is usually good enough for most purposes.

PROOF OF PROPOSITION 14. The necessity follows by using the left exactness of the Hom functor. Suppose, then, that each R-sequence is an M-sequence (length at most two). We know M is torsionless and M is B_1, so $M^{\vee\vee}/M$ is B_0. We want to show $M^{\vee\vee} = M$, so it is sufficient to show $\operatorname{Ass}(M^{\vee\vee}/M)=\emptyset$. If $\operatorname{ht}\mathfrak{p}\geqq 2$, then $\mathfrak{p}\notin\operatorname{Ass}M^{\vee\vee}/M$, as in [19]. If $\operatorname{ht}\mathfrak{p}\leqq 1$, then $(M^{\vee})_{\mathfrak{p}}=(M_{\mathfrak{p}})^{\vee}$ (see [7]), so $(M^{\vee\vee}/M)_{\mathfrak{p}}=0$, since $R_{\mathfrak{p}}$ is Gorenstein. Hence $\operatorname{Ass}M^{\vee\vee}/M=\emptyset$.

5. Conclusion.

We list here the conditions considered in this article and note the equivalences. Once again A denotes a Noetherian ring, $\mathfrak{p}$ is a prime ideal.

S_n: $\operatorname{depth}A\geqq\inf(n,\operatorname{ht}\mathfrak{p})$ *for all* $\mathfrak{p}$.
T_n: $A_{\mathfrak{p}}$ *is Gorenstein for those* $\mathfrak{p}$ with $\operatorname{ht}\mathfrak{p}\leqq n$.
R_n: $A_{\mathfrak{p}}$ *is regular for those* $\mathfrak{p}$ with $\operatorname{ht}\mathfrak{p}\leqq n$.
G_n: A *is* S_n *and* T_{n-1} *or* A *is n-Gorenstein.*
B_n: R *has a minimal injective resolution* $0\to R\to E^0\to E^1\to\ldots$ *and* E^j *is Σ-injective for* $j\leqq n$.

The following conditions are equivalent:

(a) A *is* G_n.

(b) *For each A-sequence* $x_1,\ldots,x_i$, $i<n$, *the total quotient ring of* $A/\sum Ax_j$ *is quasi-Frobenius.*

(c) *If* $E^0\to E^1\to\ldots$ *is a minimal injective resolution of* A, *then* $\operatorname{grade}E^j\geqq\inf(n,j)$ *for all* j.

(d) *For all M of finite type,* $\operatorname{grade}\operatorname{Ext}^j(M,A)\geqq\inf(n,j)$.

(e) *If* $0\to K\to F_{n+1}\to\ldots\to F_0\to M\to 0$ *is exact with each* F_j *free of finite type, then* K *is* $(n+1)$*-torsion free.*

(f) *If* $0\to K\to F_r\to\ldots\to F_0\to M\to 0$ *is exact with* F_j *free of finite type, then* K *is torsion free, for* $0\leqq r\leqq n$.

(g) $\operatorname{depth}A_{\mathfrak{p}}<n$ *implies* $A_{\mathfrak{p}}$ *is Gorenstein.*

(h) $\operatorname{flat\,dim}E^i\leqq i$ *for* $i<n$.

(i) *The Cousin complex* $C(A)$ *is a minimal injective resolution of* A *up to* $C^{n-2}(A)$.

(j) $\operatorname{id}_{\mathscr{C}}A<n$, *where* $\mathscr{C}$ *denotes the quotient category determined by* $E(A/\mathfrak{p})$, *for* $\operatorname{depth}A_{\mathfrak{p}}<n$.

(k) *For all M of finite type with* $\operatorname{grade}M\geqq j$, $0\leqq j\leqq n$,

$$L_j(\operatorname{Ext}_A{}^j(M,A)) = \operatorname{Ext}_A{}^j(M,A)\,.$$

(l) $M(j+1)=\operatorname{Ker}(M\to L_j(M))$ *for all* j, $0\leqq j\leqq n-1$, *where* $\operatorname{grade}M\geqq j$.

(m) *Ischebeck's filtration is the grade filtration up to* $M(n)$.

ADDED IN PROOF. Michel Paugam [C. R. Acad. Sci., Paris, Sér. A 274 (1972), 821–823] has announced results similar to those found in section 2. Also Eagon [Math. Z. 109 (1971), 109–111] has shown that there are n-Gorenstein rings which are not $n+1$-Gorenstein for all integers n.

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