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CONTENT MODULES AND ALGEBRAS

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0. Introduction.

Let R be a commutative ring with identity, and let M be a unitary R-module. We define a content function c from M to the ideals of R by

$$c(x) = \bigcap \{A \mid A \text{ is an ideal of } R \text{ and } x \in AM\},$$

and we call M a content module if for every $x \in M$, $x \in c(x)M$. Every free module or, more generally, every projective module is a content module. In the free case, merely write x as a linear combination of basis elements (for any basis), and then $c(x)$ is the ideal generated by the coefficients of x. For a projective module a similar observation holds by first writing the module as a direct summand of a free module. Another characterization of a content module is that $\bigcap (A_i M) = (\bigcap A_i)M$ for every set of ideals $\{A_i\}$ of R. (Recall that flat modules have this property for *finite* sets of ideals [2, p. 32, Prop. 6].)

Our interest in these matters stems from [9], where we frequently worked with the content function for polynomial rings. When one localizes at a multiplicative system (m.s.) of such a ring, the properties of the content are usually retained but one loses the freeness, thus suggesting that a more general setting would make for better exposition.

Also, the content function plays a role similar to that of the trace function; indeed, for a projective module they coincide. One is therefore led to hope that some new insight might be obtained by using the content instead of the trace, especially since there exist flat content modules for which the functions do not agree.

In Section 1 we characterize flat content modules as those content modules M for which $c(rx) = rc(x)$ for all $r \in R$, $x \in M$. We also characterize the submodules of a content module which are content modules with respect to the restricted content function. In Section 2 we give examples of content modules which are not torsion-free and which are flat but not torsionless. We study localizations of content modules in

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Section 3 and show there that a finitely generated flat content module has its content ideal generated by an idempotent. In Section 4 we study the content function for modules over an absolutely flat ring R; and we prove that for any such R-module M, $c(x) = \operatorname{Ann}\operatorname{Ann}x$, and that M is a content module if and only if $\operatorname{Ann}x$ is finitely generated for every $x \in M$. Section 5 is devoted to the question of when finitely generated flat content modules are projective. We prove that they are projective when R is a product of fields or when M is a direct sum of cyclic modules, but are not in general, even when R is absolutely flat. In Section 6 we define content algebras, that is, R-algebras which are faithfully flat content R-modules and which also satisfy the usual multiplicative property that the content function of a polynomial ring satisfies, and we give examples to show such algebras need not be projective. Finally, in Section 7 we define a trace module to be a content module for which the trace and content coincide, and we prove that finitely generated trace modules are projective.

Our notation and terminology will be that of Zariski–Samuel [12]. All our rings are commutative rings with identity, our modules are unitary, and our ring homomorphisms map the identity to the identity.
If M is an R-module, $\operatorname{Hom}_R(M,R)$ denotes the set of R-module homomorphisms of M into R,

$$\mathscr{Z}(M) = \{r \in R \mid rx = 0 \text{ for some } x \neq 0 \in M\},$$

and by $\operatorname{rk} M$, we shall mean the maximum of the ranks of free submodules of M. M is called *torsion-free* if for every regular $r \in R$ and every $x \in M$, $rx = 0$ implies $x = 0$. For $x \in M$, the *trace* of x, denoted $T(x)$, is the ideal of R consisting of

$$\{h(x) \mid h \in \operatorname{Hom}_R(M,R)\},$$

and M is called *torsionless* if $T(x) = 0$ implies $x = 0$ for all $x \in M$. If N is a subset of M, we denote

$$\operatorname{Ann} N = \{r \in R \mid rN = 0\}.$$

Note that $\operatorname{Ann} N$ is an ideal of R. Also we denote

$$(A:B)_D = \{d \in D \mid dB \subset A\},$$

whenever A, B, D are sets for which this makes sense. Finally (R,m) signifies that R is a quasi-local ring with maximal ideal m.

1. Basic properties of content modules.

Let R be a commutative ring with identity, and let M be an R-module. For any $x \in M$, we define the *content* of x, denoted $c_{R,M}(x)$, by

$$c_{R,M}(x) = \bigcap \{A \mid A \text{ is an ideal of } R \text{ and } x \in AM\}.$$

Similarly, if N is any non-empty subset of M, we define $c_{R,M}(N)$ to be the ideal of R generated by $\{c_{R,M}(x) \mid x \in N\}$. Either or both of the subscripts R, M will be omitted whenever we feel it is possible to do so without causing confusion.

1.1 Definition. M will be called a *content R-module* if for every $x \in M$, $x \in c(x)M$.

One sees easily from the definition that the following are equivalent:

(1.2) (i) M is a content R-module.
(ii) For every set of ideals $\{A_i\}$ of R, $(\bigcap A_i)M = \bigcap (A_iM)$.
(iii) For every set of finitely generated ideals $\{A_i\}$ of R, $(\bigcap A_i)M = \bigcap (A_iM)$.
(iv) There exists a function f from M to the set of ideals of R such that for every $x \in M$ and every ideal A of R, $x \in AM$ if and only if $f(x) \subset A$.

Moreover, when (iv) holds, then the function f of (iv) is unique and is just the content function c. Note also that if M is a content module and $x \in M$, then $c(x)$ is a finitely generated ideal; for $x \in c(x)M$ implies there exists a finitely generated ideal $A \subset c(x)$ such that $x \in AM$, and then $c(x) = A$.

Any free module M is a content module, for $c(x)$ is then merely the ideal generated by the coefficients of x when x is written as a linear combination of basis elements. Corollary 1.4 of the next theorem shows that any projective module is also a content module and that its content function is obtained by writing the module as a direct summand of a free module and restricting the content function.

1.3 Theorem. *Let L be a content R-module, and let K be a submodule of L. Then the following are equivalent:*
(i) *$AL \cap K = AK$ for every ideal A of R.*
(ii) *For every $x \in K$, $x \in c_L(x)K$.*
(iii) *K is a content module and c_L restricted to K is c_K.*

PROOF. (i) $\Rightarrow$ (ii): Let $x \in K$. Then $x \in c_L(x)L$ since L is a content module. Therefore $x \in c_L(x)L \cap K = c_L(x)K$.

(ii) $\Rightarrow$ (iii): Let $x \in K$. Then $x \in c_L(x)K$ implies $c_K(x) \subset c_L(x)$. Since the reverse inclusion is always true, $c_K(x) = c_L(x)$. Thus, c_K is the restriction of c_L; and $x \in c_L(x)K = c_K(x)K$ implies K is a content module.

(iii) $\Rightarrow$ (i): Let $x \in AL \cap K$. Then $c_L(x) \subset A$, and hence $x \in c_K(x)K = c_L(x)K \subset AK$.

Theorem 1.3 is a routine generalization of [2, p. 65, Ex. 23]. The condition $AL \cap K = AK$ for every ideal A of R has been extensively studied by Besserre [1]. If L/K is flat, then this condition is satisfied (and conversely, if L is flat and the condition is satisfied, then L/K is flat) [2, p. 33, Cor.].

1.4 COROLLARY. *Let M be an R-module and $\{M_i\}_{i \in I}$ be a set of submodules such that $M = \oplus M_i$. Then M is a content module if and only if M_i is a content module for every $i \in I$. Moreover when this occurs, then for every $x = \sum x_i$, $x_i \in M_i$, $c_M(x)$ is the ideal of R generated by $\{c_{M_i}(x_i)\}_{i \in I}$.*

PROOF. If M is a content module, then M_i is a content module by 1.3. Conversely, suppose each M_i is a content module, and let $x = \sum x_i$, where $x_i \in M_i$. If $x \in AM$ for some ideal A of R, then $x_i \in AM_i$. Thus, $c_{M_i}(x_i) \subset c_M(x)$. On the other hand, since M_i is a content module, $x_i \in c_{M_i}(x_i)M_i$. Therefore $x \in BM$, where B is the ideal of R generated by $\{c_{M_i}(x_i)\}_{i \in I}$. This implies $c_M(x) \subset B$, which proves the second assertion of the corollary. Now to conclude that M is a content module, observe that $x_i \in c_{M_i}(x_i)M_i$ for all i implies $x \in c_M(x)M$.

We shall now characterize (in 1.6) flat and faithfully flat content modules in terms of their content functions.

1.5 THEOREM. *Let M be a content R-module, and let $r \in R$. Then the following are equivalent:*

(i) *$rc(x) = c(rx)$ for all $x \in M$.*

(ii) *$(A:r)_R M = (AM:r)_M$ for every ideal A of R.*

(iii) *$(0:r)_R M = (0M:r)_M$.*

PROOF. (i) $\Rightarrow$ (ii): The inclusion $\subset$ is always true, so suppose $x \in AM:r$. Then $rx \in AM$, and hence $c(rx) \subset A$. But $c(rx) = rc(x)$ by (i), so then $c(x) \subset A:r$. Therefore $x \in (A:r)M$ since M is a content module.

(ii) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (i): Clearly $x \in c(x)M$ since M is a content module. Therefore $rx \in rc(x)M$, which implies $rc(x) \supset c(rx)$. Conversely, $rx \in rM$ implies $c(rx) \subset (r)$. Therefore $c(rx) = (r)B$, where $B = c(rx):r$. Since M is a content module, $rx \in c(rx)M = rBM$. Therefore $rx = rz$, for some element $z \in BM$, which implies $x - z \in 0M:r$. But $0M:r = (0:r)M$ by (iii), so then $x - z \in (0:r)M \subset BM$. Since $z \in BM$, then $x \in BM$. Therefore $c(x) \subset B$, and hence $rc(x) \subset rB = c(rx)$.

Note that if $r \notin \mathscr{Z}(M)$, then $(0M:r)_M = 0$ and hence (iii) is valid. Also observe that the set of all $r \in R$ which satisfy the conditions of 1.5 form a m.s. (which will enter into the localization considerations of Section 3). The following corollary shows that this m.s. equals R if and only if M is flat.

1.6 Corollary. *Let M be a content R-module. Then the following are equivalent*:

(i) *M is flat.*

(ii) *For every $r \in R$ and $x \in M$, $rc(x) = c(rx)$.*

(iii) *$(A:B)_R M = (AM:B)_M$ for every pair of ideals A, B of R.*

Moreover, M is faithfully flat if and only if M is flat and $c(M) = R$.

Proof. M is flat if and only if for every ideal A of R and every $b \in R$, $(A:b)M = AM:b$ [2, p. 65, Ex. 22]. Therefore (i) $\Leftrightarrow$ (ii) follows from 1.5.

(iii) $\Rightarrow$ (i) is immediate; and (i) $\Rightarrow$ (iii) since

$$\begin{aligned}(A:B)M &= [\bigcap\{(A:b) \mid b \in B\}]M = \bigcap\{(A:b)M \mid b \in B\} \\ &= \bigcap\{AM:b \mid b \in B\} = AM:B,\end{aligned}$$

the second equality resulting from 1.2–(ii) and the third equality from M being flat.

For the second assertion of the corollary, suppose M is flat and $c(M) = R$. Then $mM \neq M$ for every maximal ideal m of R, since otherwise $c(M) \subset m$. Therefore M is faithfully flat by [2, p. 44, Prop. 1]. Conversely, if M is faithfully flat, then M is flat and for every maximal ideal m of R, $mM \neq M$. Therefore for every maximal ideal m, there exists $x \in M$ such that $x \notin mM$ and hence such that $c(x) \not\subset m$. Therefore $c(M) \not\subset m$ for every maximal ideal m of R, and consequently $c(M) = R$.

Thus, a module M is a flat content module if and only if there exists a function c from M to the set of finitely generated ideals of R such that

(i) for every ideal A of R, $x \in AM$ if and only if $c(x) \subset A$,

(ii) for every $x \in M$ and $r \in R$, $c(rx) = r\,c(x)$.

These are the axioms proposed in [9]. It also follows from 1.5 and 1.6 that if M is a content module and $\mathscr{Z}(M) = 0$, then M is flat; in particular, if R is a domain, then M is flat if (and only if) M is torsion-free. Actually, Jensen [3, p. 943, Thm. 1] has proved that if R is a domain and M is torsion-free, then M is flat if (and only if) $\bigcap (A_i M) = (\bigcap A_i)M$ for every *finite* set $\{A_i\}$ of ideals of R, so the full assumption that M is a content module is not needed for this result.

2. Some examples.

The next proposition provides a useful characterization of content modules over a discrete rank 1 valuation ring.

2.1 PROPOSITION. *Let (R,m) be a discrete rank 1 valuation ring, and let M be an R-module. Then M is a content module if and only if*

$$\bigcap \{m^i M \mid i = 1, 2, \ldots\} = 0 .$$

PROOF. $\Rightarrow$: $0 = \bigcap m^i$ since R is noetherian [12–I, p. 216, Cor. 1]. Therefore $0 = (\bigcap m^i)M = \bigcap (m^i M)$ by 1.2 (ii).

$\Leftarrow$: Let $x \neq 0 \in M$. Since every ideal of R is of the form m^i,

$$c(x) = \bigcap \{m^i \mid x \in m^i M\} .$$

But $\bigcap \{m^i M \mid i = 1, 2, \ldots\} = 0$ implies there exists an n such that $c(x) = m^n$ and $x \in m^n M$. Then $x \in c(x)M$.

We have seen in Section 1 that every projective module is a content module, and we have given a characterization of flat content modules. To get now an example of a content module which is not flat and in fact which is not even torsion-free, one need only take a discrete rank 1 valuation ring (R,m) and let $M = R/m^2$.

Recall that if M is an R-module and $x \in M$, then the trace of x, denoted $T(x)$, is the ideal of R consisting of

$$\{h(x) \mid h \in \operatorname{Hom}_R(M; R)\} .$$

It is immediate that $T(x) \subset c(x)$, but in general the two functions are not equal since $T(rx) = rT(x)$, $r \in R$, is true for any R-module M while the corresponding equality for c is valid for a content module if and only if the module is flat. However, it follows from our previous considerations that these functions coincide for a projective module, and one might ask if they also coincide for a flat content module. The next example shows that this is not the case. We remind the reader that M is said to be torsionless if $x \neq 0$ implies $T(x) \neq 0$.

2.2 EXAMPLE of a flat content module M of finite rank which is not torsionless.

Let (R, m) be the discrete rank 1 valuation ring $k[t]_{(t)}$, where k is a field and t an indeterminate. Let $R[X]$ be the polynomial ring in an indeterminate X, and choose $f \in R[X]$ such that f is irreducible and the constant term and leading coefficient of f are in m and such that $c(f) = R$ (here $c(f)$ denotes the ideal of R generated by the coefficients of f), e.g. take $f = t + X + tX^2$. Let $I = fR[X]$, and let M denote the ring $R[X]/I$. Since $R[X]$ is a UFD, the ideal I is prime, and hence M is a domain. M is then a torsion-free and hence flat R-module [2, p. 29, Prop. 3]. $mM \neq M$ since $1 \notin I + mR[X]$ because the constant term of f is in m. Therefore $\bigcap (m^i M) = 0$ since M is a noetherian domain and mM is a proper ideal of M [12–I, p. 216, Cor. 1]. Thus, M is a content module by 2.1. Since the leading coefficient of f is in m, the ideal I does not contain a monic polynomial; and hence M is not an integral extension of R and is consequently not a finite R-module. However, M does have finite rank since $M \otimes_R K$, where K is the quotient field of R, is a free K-module of rank 2. Now observe that a module of finite rank over a domain is torsionless (if and) only if it is a submodule of a finitely generated free module (Recall that "rank" here means the maximum of the ranks of free submodules of M). This follows from the observation that the dual of a module of finite rank is a submodule of a finitely generated free module and a torsionless module is embedded in its double dual (see [6, Prop. 1.3]). Thus, since the above module has finite rank but is not finitely generated and hence certainly cannot be a submodule of a finitely generated module (since R is noetherian), it follows that this module is not torsionless.

2.3 REMARKS. (a) Example 2.2 also shows that a flat content module need not be a submodule of a free module, since a submodule of a torsionless module is torsionless.

(b) A variation of 2.1 can be stated as follows. Let R be a ring such that any $r \neq 0 \in R$ is in at most finitely many ideals (e.g. let R be a Dedekind domain), and let M be a flat R-module. Then M is a content module if and only if $\bigcap (A_i M) = 0$ whenever $\{A_i\}$ is an infinite set of ideals of R. This follows from 1.2 (ii) and the fact that finite intersections distribute through flat modules.

(c) It would be interesting to have a characterization of the R-algebras of the type $R[X]/I$, for I an ideal of $R[X]$, which are flat content modules. It can be seen that $R[\xi]$, for ξ in the total quotient ring of R, is such a module if and only if $\xi \in R$.

(d) Another way of getting content modules is by means of the following observation: If M is a content R-module and A is an ideal of R, then M/AM is a content (R/A)-module (with $c_{M/AM}(x+AM)=c_M(x)+A$ for any $x \in M$). Finally, we shall give in Section 6 further examples of flat content modules which are not projective.

3. Localization of content modules.

Let M be a content R-module. We shall denote by $\mathscr{S}_M$ the m.s. of R consisting of

$$\{r \in R \mid c(rx)=rc(x) \text{ for all } x \in M\} .$$

We have shown in 1.5 and 1.6 that a content module is flat if and only if its corresponding m.s. $\mathscr{S}_M$ is the whole ring, and that $R \setminus \mathscr{Z}(M) \subset \mathscr{S}_M$. Thus, in many cases $\mathscr{S}_M$ is large. In order for a content module M to localize well, it seems necessary to restrict one's attention to multiplicative systems contained in $\mathscr{S}_M$.

3.1 THEOREM. *Let M be a content R-module and S be a* m.s. *of R such that $S \subset \mathscr{S}_M$. Then M_S is a content R_S-module; and for any $x \in M$ and $s \in S$ we have $c_M(x)R_S=c_{M_S}(x/s)$.*

PROOF. For $x \in M$, $x \in c_M(x)M$ implies $x/s \in (c_M(x)R_S)M_S$. Therefore $c_{M_S}(x/s) \subset c_M(x)R_S$, and the proof will be complete if we show the reverse inclusion. Let A' be any ideal of R_S such that $x/s \in A'M_S$, and let $A=A' \cap R$ (here $A' \cap R$ denotes the inverse image of A' under the canonical homomorphism of R into R_S). Then there exists an element $s' \in S$ such that $s'x \in AM$. Therefore $c_M(s'x) \subset A$; and since $S \subset \mathscr{S}_M$, it follows that $s'c_M(x)=c_M(s'x) \subset A$. But then $c_M(x)R_S \subset A'$, and hence $c_M(x)R_S \subset c_{M_S}(x/s)$.

We next prove a theorem which describes the local behavior of the content of those submodules of a content module which are described in 1.3. We first need a lemma which generalizes [2, p. 66, Ex. 23-d].

3.2. LEMMA. *Let L be a content R-module, and let K be a content submodule of L having restricted content function. If $K \subset JL$, where J is the Jacobson radical of R, then $K=0$.*

PROOF. For any $x \in K$, we have $x \in c_K(x)K$ since K is a content module. Therefore $x \in c_L(x)JL$ since $c_K(x)=c_L(x)$ and $K \subset JL$. But then $c_L(x) \subset c_L(x)J$ by definition of the content function. Since $c_L(x)$ is a finitely generated ideal, it follows from Nakayama's lemma that $c_L(x)=0$. Therefore $x=0$.

3.3 THEOREM. *Let L be a content R-module, and let K be a content submodule of L having restricted content function. Then for any prime ideal P of R such that $P \supset R \setminus \mathscr{S}_L$, either $c(K)_P=0$ or $c(K)_P=R_P$.*

PROOF. By localization at P via 3.1, it suffices to prove the theorem for the case that R is quasi-local with maximal ideal P. If $c(K) \neq R$, then $c(K) \subset P$. But then $K \subset PL$; and hence by 3.2, we conclude $K=0$.

If L is a flat content module, then by 1.5 we have $\mathscr{S}_L=R$, and hence $c(L)_P=0$ or $c(L)_P=R_P$ for every prime P of R. Thus, we have the following:

3.4 COROLLARY. *Let L be a flat content R-module. Then for any prime ideal P of R, $c(L)_P=0$ or $c(L)_P=R_P$. If in addition $c(L)$ is a finitely generated ideal, then $c(L)$ is principal and is generated by an idempotent.*

PROOF. The first assertion follows from the preceding remark, and the second is a consequence of the observation that an ideal which is locally idempotent is idempotent and a finitely generated idempotent ideal is generated by an idempotent (see [2, p. 83, Cor. 3]).

In connection with the above corollary, note that if L is a finitely generated content module, then $c(L)$ is finitely generated; and hence if L is also flat, then $c(L)$ is generated by an idempotent. More precisely, it follows from the definition that if M is a content module and T a subset which generates M, then $c(T)=c(M)$. A partial converse exists:

3.5 THEOREM. *Let M be a flat content R-module, and let T be a subset of M such that $c(T)=c(M)$. If M_P is a cyclic R_P-module for every prime P of R, then T generates M.*

PROOF. Since it suffices to see that T generates M locally [2, p. 112, Cor. 1], we may localize at a prime P (using 3.1) and thus are reduced to proving the theorem in the case that R is a quasi-local ring with maximal ideal m and M is a cyclic R-module. If $c(M) \subset m$, then $M = mM$; and hence by Nakayama's lemma $M = 0$, and the theorem is trivially true. If $c(M) \not\subset m$, then $c(T) \not\subset m$; so there exists $t \in T \setminus mM$. But then $t + mM$ generates M/mM as an (R/m)-module since M/mM is cyclic. Therefore by Nakayama's lemma, t generates M.

3.6 COROLLARY. *Let M be a flat content R-module. If M is locally cyclic, then M is finitely generated if (and only if) $c(M)$ is a finitely generated ideal.*

Theorem 3.5 and corollary 3.6 are generalizations of a lemma of Vasconcelos [11, p. 430, lem. 1.2], who assumes that M is projective and then uses the trace function in essentially the same way as we have used the content function. This illustrates a procedure that should be useful. In particular, it would be interesting to know if one could similarly generalize some of the other results of Vasconcelos's paper.

We would also like to mention here that P. Eakin and J. Silver, in a forthcoming work, have used the notion of content module to study a class of rings closely related to polynomial rings.

4. Content modules over absolutely flat rings.

An absolutely flat ring (sometimes called a regular ring) is a ring R for which every R-module is flat. Equivalent properties are that $a \in (a^2)$ for every $a \in R$ [2, p. 64, Ex. 16–17] or that for every prime ideal P of R, the localized ring R_P is a field. It follows that every ideal of an absolutely flat ring is idempotent and hence that every finitely generated ideal is principal and generated by an idempotent. By 1.3 and the remark following 1.3, if R is an absolutely flat ring, then every submodule of a content R-module is also a content module with restricted content function; and by writing a given R-module as a homomorphic image of a free module, the converse also results from 1.3 and the remark following it. Thus, the following are equivalent:

(i) R is an absolutely flat ring.
(ii) Every submodule of a content R-module is a content module with restricted content function.
(iii) Every submodule of a free R-module is a content module with restricted content function.

We shall now proceed to give a very explicit description of content functions and modules for an absolutely flat R. We do this in more generality than is needed since the only property of an absolutely flat ring that is used is that for every finitely generated ideal A of R, the ideal $\operatorname{Ann} A$ is finitely generated and $\operatorname{Ann}\operatorname{Ann} A = A$.

4.1 THEOREM. *Let M be a flat R-module (R any ring), and let $x \in M$. Then $c(x) \subset \operatorname{Ann}\operatorname{Ann} x$. Moreover, if $\operatorname{Ann} x$ is finitely generated, then $x \in (\operatorname{Ann}\operatorname{Ann} x)M$.*

PROOF. If $ax = 0$ for $a \in R$, then $x \in (0M : a)_M$. By the flatness hypothesis, $(0M:a)_M = (0:a)_R M$, which equals $(\operatorname{Ann} a)M$. Therefore

$$c(x) \subset \bigcap \{\operatorname{Ann} a \mid a \in \operatorname{Ann} x\} = \operatorname{Ann}\operatorname{Ann} x .$$

Suppose now $\operatorname{Ann} x = (a_1, \ldots, a_n)$. Then

$$x \in \bigcap (0M : a_i)_M = \bigcap [(0:a_i)_R M] = [\bigcap (0:a_i)_R]M = (\operatorname{Ann}\operatorname{Ann} x)M ,$$

the first and second equalities being consequences of the flatness assumption.

4.2 COROLLARY. *Let R be a ring with the property that $\operatorname{Ann}\operatorname{Ann} A = A$ for every finitely generated ideal A of R. If M is a flat R-module and $x \in M$, then $c(x) = \operatorname{Ann}\operatorname{Ann} x$.*

PROOF. If A is a finitely generated ideal of R such that $x \in AM$, then $\operatorname{Ann} A \subset \operatorname{Ann} x$; and hence $A = \operatorname{Ann}\operatorname{Ann} A \supset \operatorname{Ann}\operatorname{Ann} x$. Therefore

$$\operatorname{Ann}\operatorname{Ann} x \subset c(x)$$

since

$$c(x) = \bigcap \{\text{finitely generated ideals } A \mid x \in AM\} .$$

The reverse inclusion follows from 4.1.

4.3 COROLLARY. *Let R be a ring with the property that for every finitely generated ideal A of R, the ideal $\operatorname{Ann} A$ is finitely generated and $\operatorname{Ann}\operatorname{Ann} A = A$. Let M be a flat R-module. Then M is a content module if and only if for any $x \in M$, the annihilator $\operatorname{Ann} x$ is finitely generated.*

PROOF. $\Rightarrow$: By 1.6, we have $c(rx) = r\,c(x)$ for all $r \in R$. Therefore $rx = 0$ implies $rc(x) = 0$, so $\operatorname{Ann} x \subset \operatorname{Ann} c(x)$. Conversely $x \in c(x)M$ implies $\operatorname{Ann} c(x) \subset \operatorname{Ann} x$; so $\operatorname{Ann} c(x) = \operatorname{Ann} x$. Since $c(x)$ is finitely generated, by hypothesis $\operatorname{Ann} c(x)$ is also.

$\Leftarrow$: Apply 4.1 and 4.2.

Note that there exist rings other than absolutely flat rings which satisfy the hypothesis of 4.3, for example let $R = V/m^2$, where (V, m) is a discrete rank 1 valuation ring. We now summarize the situation for an absolutely flat ring.

4.4 COROLLARY. *Let R be an absolutely flat ring, and let M be an R-module. Then for any $x \in M$, we have $c(x) = \operatorname{Ann}\operatorname{Ann} x$; and M is a content R-module if and only if for any $x \in M$, the ideal $\operatorname{Ann} x$ is finitely generated.*

5. When are finitely generated flat content modules projective?

Since finitely generated flat modules are often projective, one must resort to something like the absolutely flat rings for a good class on which to test the above question. We shall prove in 5.3 that when R is a product of fields, then any finitely generated content R-module is projective; and we shall give an example in 5.4 of an absolutely flat ring R which does not have this property.

Henceforth in Section 5 let M denote a finitely generated flat content module. Then M is projective if and only if the rank function is locally constant [2, p. 138, Thm. 1]. (The rank function is the map ϱ from $\operatorname{Spec} R$ to the non negative integers, defined by $\varrho(P) = \operatorname{rk} M_P$ for $P \in \operatorname{Spec} R$.) Moreover, $M_P = 0$ if and only if $c(M_P) = 0$; and by 3.1, $c(M_P) = c(M)R_P$. Thus, since by 3.4 $c(M)$ is locally either 0 or the unit ideal, $M_P = 0$ if and only if $c(M) \subset P$. Moreover, since by 3.4 $c(M)$ is generated by an idempotent,

$$\{P \in \operatorname{Spec} R \mid c(M) \subset P\}$$

is an open and closed subset of $\operatorname{Spec} R$, and hence

$$\{P \in \operatorname{Spec} R \mid \operatorname{rk} M_P = 0\}$$

is an open and closed set. We conclude that if the rank function assumes at most one non-zero value, then M is projective; and in particular, if M is cyclic, then M is projective. More generally, it follows from 1.4 that if M is a direct sum of cyclic modules, then M is projective. We summarize some of these remarks in the next proposition. One further fact needed for the proposition is that if every cyclic flat R-module is projective, then every finitely generated flat R-module is also projective [7, p. 1356, Cor. 3.1].

5.1 PROPOSITION. *Any finitely generated flat content module which is a direct sum of cyclic modules is projective. Moreover, the following properties of a ring R are equivalent:*

(i) *Finitely generated flat R-modules are projective.*
(ii) *Finitely generated flat R-modules are content modules.*
(iii) *Cyclic flat R-modules are content modules.*
(iv) *Cyclic flat R-modules are projective.*

Now consider an exact sequence

$$0 \to K \to L \to M \to 0$$

of R-modules, with L finitely generated free and having a basis of n elements. Another characterization for M projective is that K is finitely generated [2, p. 140, Cor. 2]; and since K is a content module by 1.3, this implies $c(K)$ is finitely generated. On the other hand, if $c(K)$ is finitely generated, then by applying to K the same reasoning as applied above to M, one sees that the set of primes P of R for which $K_P = 0$ is an open and closed set. But this is just the set of primes at which M has rank n. In particular, it follows that if M is generated by at most 2 elements and $c(K)$ is finitely generated, then M is projective. We shall now prove that $c(K)$ is finitely generated when R is a product of fields (which is a special kind of absolutely flat ring).

5.2. THEOREM. *Let $\{k_i\}_{i\in I}$ be a collection of fields, let $R = \prod k_i$, and let $0 \to K \to L \to M \to 0$ be an exact sequence of R-modules with L finitely generated free. If M is a content module, then $c(K)$ is principal.*

PROOF. Let $\{m_\alpha\}$ be the image in M of a finite basis of L. Since finitely generated ideals of R are principal, $c(K)$ is generated by

$$\mathcal{G} = \{r \neq 0 \in R \mid \text{there exists } \textstyle\sum r_\alpha m_\alpha \in M \text{ such that } r(\sum r_\alpha m_\alpha) = 0 \text{ and } \sum r_\alpha R = R\};$$

and since for every $r \in R$ there exists an idempotent $r' \in R$ such that $(r) = (r')$, we may assume that the elements of $\mathcal{G}$ are idempotents. Let a_i be the element of R defined by $a_i(j) = \delta_{ij}$, and let

$$\mathcal{G}' = \{a_i \mid \text{there exists } r \in \mathcal{G} \text{ such that } a_i r = a_i\}.$$

Then $\mathcal{G}' \subset \mathcal{G}$; and it follows from the defining properties of the a_i and $\mathcal{G}'$ that if there exists an ideal (a) such that every element of $\mathcal{G}'$ is in (a), then every element of $\mathcal{G}$ is also in (a). Thus, it remains to show there exists an element $a \in c(K)$ such that every element of $\mathcal{G}'$ is in (a). For any $a_i \in \mathcal{G}'$, there exist $r_{i\alpha} \in R$ such that $a_i(\sum r_{i\alpha} m_\alpha) = 0$ and $\sum r_{i\alpha} R = R$. Define r_α by $r_\alpha(i) = r_{i\alpha}(i)$ if $a_i \in \mathcal{G}'$ and $r_\alpha(i) = 1$ if $a_i \notin \mathcal{G}'$. Then $\sum r_\alpha R = R$ and $a_i(\sum r_\alpha m_\alpha) = 0$ for every $a_i \in \mathcal{G}'$. But since M is a content module, the annihilator of every element of M is finitely generated (by 4.4) and hence principal; so $\text{Ann}(\sum r_\alpha m_\alpha) = (a)$ for some $a \in R$. Then $a \in \mathscr{G}$ and hence is in $c(K)$, and $a_i \in (a)$ for all $a_i \in \mathscr{G}'$.

5.2 COROLLARY. *If R is a product of fields and M is a finitely generated content R-module, then M is projective.*

PROOF. By induction on n = the minimum number of generators for M. If $n = 1$, the corollary follows from 5.1, so assume $n > 1$. Let $0 \to K \to L \to M \to 0$ be an exact sequence of R-modules with L free of rank n. By 5.2, we have $c(K) = (e)$ for some idempotent e of R. Let

$$V(e) = \{P \in \text{Spec}\, R \mid e \in P\}.$$

Then

$$V(e) = \{P \in \text{Spec}\, R \mid K_P = 0\} = \{P \in \text{Spec}\, R \mid \text{rk}\, M_P = n\}.$$

Then the rank of $(1-e)M$ is n on $V(e)$ and 0 elsewhere. Thus, the rank function for $(1-e)M$ is locally constant, and hence $(1-e)M$ is projective. On the other hand, eM has rank $\leqq n-1$ at every prime; and hence by [5, p. 197, Prop. 4.2] or [10, p. 75, Prop. 18.2], eM can be generated by $n-1$ elements. Since eM is a content module by 1.4, it follows from the induction hypothesis that eM is also projective. Thus, $M = eM \oplus (1-e)M$ is projective.

The following example is due to W. Heinzer.

5.4 EXAMPLE of an absolutely flat ring R and a finitely generated (flat) content R-module M such that M is not projective.

Let $R' = \mathsf{Q}^{\mathsf{N}}$, where N denotes the positive integers and Q the rationals. We call an element $f \in \mathsf{Q}^{\mathsf{N}}$ eventually constant if there exists an integer i_0 such that $f(i) = f(j)$ for all $i, j \geqq i_0$; and we say that f is eventually a if there exists i_0 such that $f(i) = a$ for all $i \geqq i_0$. Let R be the subring of R' consisting of

$$\{f \in R' \mid f \text{ is eventually constant}\},$$

and let P_∞ denote the ideal of R consisting of

$$\{f \in R' \mid f \text{ is eventually } 0\}.$$

Let x', y' be the elements of R' defined by $x'(i) = 1$ for all i, and $y'(i) = i$ for all i; and let M be the R-module $x'R + y'R$.

First we shall check that M is a content module. By 4.4, we must show $\text{Ann}\, z'$ is finitely generated for every $z' = \alpha x' + \beta y' \in M$, $(\alpha, \beta \in R)$.

If α and β are not both eventually 0, then there exists an integer n_0 such that $z'(i) \neq 0$ for $i \geqq n_0$; and hence $rz' = 0$, $r \in R$, implies $r(i) = 0$ for $i \geqq n_0$. It follows that in this case $\operatorname{Ann} z'$ is finitely generated and is contained in P_∞. On the other hand, if α and β are both eventually 0, then z' is eventually 0; and hence there exists n_0 such that if r denotes the element of R defined by $r(i) = 0$ for $i < n_0$ and $r(i) = 1$ for $i \geqq n_0$, then $r \in \operatorname{Ann} z'$. But every ideal containing such an r is finitely generated, so again $\operatorname{Ann} z'$ is finitely generated.

Now let $0 \to K \to L \to M \to 0$ be an exact sequence of R-modules with L free on two generators x, y which map onto x', y'. We shall show that $c(K) = P_\infty$ and thus that $c(K)$ is not finitely generated; this in turn implies M is not finitely presented and hence is not projective. The ideal $c(K)$ is generated by

$$\{r \in R \mid r(\alpha x' + \beta y') = 0 \text{ and } (\alpha, \beta) = R\}.$$

We have already observed that every such r is in P_∞. On the other hand, if δ_i is defined by $\delta_i(j) = 0$ if $i \neq j$ and $\delta_i(j) = 1$ if $i = j$, then $\delta_i(x' + \beta_i y') = 0$, where β_i is defined by $\beta_i(j) = 0$ if $i \neq j$ and $\beta_i(i) = -1/i$. Thus, $\delta_i \in c(K)$ for all i; and since the δ_i generate P_∞, we have $P_\infty \subset c(K)$.

6. Content algebras.

Let R' be an R-algebra; whenever R' is regarded as an R-module, it will be done so with respect to the scalar multiplication given by the R-algebra structure. We shall call R' a *content R-algebra* if it satisfies the following axioms:

(i) R' is a content R-module.

(ii) (Faithful flatness) For any $r \in R$ and $f \in R'$, the equation $c(rf) = rc(f)$ holds; and $c(R') = R$.

(iii) (Content formula) For any $f, g \in R'$, there exists an integer $n \geqq 0$ such that $c(fg)c(g)^n = c(f)c(g)^{n+1}$.

Note that by 1.6, (i) and (ii) merely assert that R' is a faithfully flat content R-module. The following are consequences of the definition:

(6.1) a) If l' denotes the identity of R', then $c(l') = R$.

b) The defining homomorphism of R into R' is injective.

c) $g \in R'$ is a zero-divisor of R' if and only if there exists $r \neq 0 \in R$ such that $rc(g) = 0$.

d) For any $f, g \in R'$, $c(fg) \subset c(f)c(g)$, and if $c(g) = R$, then $c(fg) = c(f)$.

The first example of a content R-algebra that comes to mind is that of a polynomial ring over R in any number of indeterminates. More generally, if G is any torsion-free abelian group, then the group algebra of G over R is a content R-algebra [8]. These are all free R-modules (and hence trivially satisfy (i) and (ii)). We shall give in 6.3 further examples which are not free, but first we need a localization theorem.

6.2. THEOREM. *Let R' be a content R-algebra, let S' be a* m.s. *of R', and let $S=S'\cap R$. If for every $s'\in S'$, $c(s')\cap S\neq\varphi$, then $R'_{S'}$ is a content R_S-algebra, and for any $f\in R'$, $s'\in S'$,*

$$c_{R_S}(f/s') = c_R(f)R_S .$$

PROOF. $f\in c_R(f)R'$ because R' is a content R-module. Therefore $f/s'\in c_R(f)R'_{S'}$, and hence $c_{R_S}(f/s')\subset c_R(f)R_S$. Conversely, $f/s'\in AR'_{S'}$, A an ideal of R, implies $s''f\in AR'$ for some $s''\in S'$. Therefore $c_R(s''f)\subset A$, and hence $c_R(s''f)R_S\subset AR_S$. But $c_R(s''f)c_R(s'')^n=c_R(s'')^{n+1}c_R(f)$, and by hypothesis $c_R(s'')R_S=R_S$. Therefore, $c_R(s''f)R_S=c_R(f)R_S$. Thus, $c_R(f)R_S\subset AR_S$. Since any ideal of R_S is of the form AR_S, we then have $c_R(f)R_S\subset c_{R_S}(f/s')$. This proves the equality of the theorem. The axioms for a content R-algebra now follow.

Note that the condition on S' in the above theorem is satisfied whenever $S'=R'\setminus P'$, P' a prime ideal of R'; for then $s'\notin P'$ implies $c_R(s')\nsubseteq P'\cap R$, and hence $c_R(s')\cap S\neq\varphi$.

6.3 EXAMPLES. (a) Let $R'=R[X]$, X an indeterminate, and let

$$S' = \{f\in R' \mid c(f)=R\} .$$

Then $R'_{S'}$ is usually denoted $R(X)$, and 6.2 asserts that $R(X)$ is a content R-algebra. If we take R to be a quasi-local domain whose residue field k is finite or countable and whose quotient field K is uncountable, then the ring $R(X)$ is a content R-algebra which is not a free R-module (and hence which is not even projective since projective implies free over a quasi-local ring). For, $R(X)$ is R-free implies the dimensions of the k-vector space $k(X)$ and the K-vector space $R(X)\otimes_R K$ are equal by [4, p. 418, Prop. 8], and this is not the case by [4, p. 125, Thm. 8].

(b) Let (V,m) and (V',m') be discrete rank 1 valuation rings such that $V<V'$, $mV'=m'$, V/m is canonically isomorphic to V'/m', and V' is integral over V. Such rings exist by the example of F. K. Schmidt [12-II,p. 62]. By 2.1, V' is a content V-module and the content function is defined by

$$c(x') = \bigcap\{m^iV' \mid x'\in m^iV'\} .$$

Since $mV' = m'$, $c(x') = m^n$, where n is the V'-value of x', and it follows that $c(x'y') = c(x')c(y')$ for all $x', y' \in V'$. Thus, V' is a content R-algebra.

On the other hand, V' is not a free V-module; for since the dim of V'/m' over V/m is 1, if V' were free, we would then have $V' = Vx'$ for some $x' \in V'$. But then $1 = rx'$, where $r \in V$, and hence $x' \in V$ and $V = V'$, a contradiction. Moreover, since V' is integral over V, V' cannot be a localization of any intermediate ring ($\neq V'$) between V and V'. Thus, V' is an example of a content R-algebra which is not a localization of a free R-subalgebra. (The above argument arose from a conversation with W. Heinzer, who originally suggested using the completion for V').

It seems difficult in general to determine when an R-algebra of the type $R(X)$ of 6.3 (a) is free or not. Related to these considerations is the question of whether the trace and content coincide for such rings. We shall digress briefly here to prove that they do coincide on a certain subring of $R(X)$.

Let then (R,m) be a quasi-local ring, and let $R' = R[X]_{S'}$, where

$$S' = \{f \in R[X] \mid f = 1 + a_1X + \ldots + a_nX^n,\ a_i \in m\}.$$

Note that by the same argument as in 6.3 (a), R' is not a free R-module if, say, R is a domain and R/m is finite or countable and R is uncountable; and also R' is a content R-algebra by 6.2. In addition R' has the pleasant property that the *R-algebra* homomorphisms of R' into R are exactly the homomorphisms obtained by substituting an element of R for X. Thus, if for $x' \in R'$ we denote by $T^*(x')$ the ideal of R generated by

$$\{h(x') \mid h \in \mathrm{Hom}_R(R',R) \text{ and } h \text{ is an } R\text{-algebra homomorphism}\},$$

then $T^*(x') \subset T(x') \subset c(x')$.

6.4 Theorem. *Let (R,m) be a quasi-local ring, and let $R' = R[X]_{S'}$, where*

$$S' = \{f \in R[X] \mid f = 1 + a_1X + \ldots + a_nX^n,\ a_i \in m\}.$$

Then $T^*(x') = c(x')$ for every $x' \in R'$ if and only if R/m is infinite.

Proof. $\Rightarrow$: If R/m is finite, then there exists an integer $n > 0$ such that $\alpha^n - \alpha = 0$ for all $\alpha \in R/m$. Then $T^*(X^n - X) \subset m$. However,

$$c(X^n - X) = R.$$

$\Leftarrow$: Since $c(f/s') = c(f)$ and $T^*(f/s') = T^*(f)$, for $f \in R[X]$ and $s' \in S'$, it suffices to show $c(f) \subset T^*(f)$. Let

$$f = a_0 + a_1X + \ldots + a_nX^n, \quad a_i \in R.$$

Since R/m is infinite, there exist $r_0,\ldots,r_n \in R$ such that $\prod_{i<j}(r_i - r_j) \notin m$. Consider then the $n+1$ linear equations in $n+1$ unknowns $a_0,\ldots,a_n$ given by $f(r_0)=b_0,\ldots,f(r_n)=b_n$. The coefficient determinant D is just the Vandermonde, which is not in m by the choice of the r_i. Then D is a unit of R, and hence by Cramer's rule [4, p. 330], we have $a_i \in (b_0,\ldots,b_n)$ for $i=0,\ldots,n$. Thus,

$$c(f) = (a_0,\ldots,a_n) \subset (b_0,\ldots,b_n) \subset T^*(f) .$$

We conclude this section with a generalization of Theorem 1.5 of [9].

6.5 THEOREM. *Let R' be a content R-algebra, and let I' be an ideal of R'. If for every maximal ideal m' of R', $I'_{m'}$ is a principal ideal of $R'_{m'}$ and $c(I')_{m' \cap R} = 0$ or $R_{m' \cap R}$, then R'/I' is a flat R-module.*

PROOF. Let m' be a maximal ideal of R', and let $m = m' \cap R$. Then $R'_{m'}$ is a content R_m-algebra by 6.2, and $I'_{m'}$ is a principal ideal of $R'_{m'}$ such that $c(I'_{m'}) = 0$ or R_m. If we show $R'_{m'}/I'_{m'}$ (which is isomorphic as an R_m-module to $(R'/I')_{m'}$) is R_m-flat, we will be done by [2, p. 116, Prop. 15]. In other words, it suffices to prove the theorem under the hypothesis that R' is quasi-local with maximal ideal m' and R is quasi-local with maximal ideal m.

If $c(I') = 0$, then $I' = 0$ and R'/I' is R-flat. Therefore assume $c(I') = R$. Then $I' = fR'$, $f \in R'$, implies $c(f) = R$. If $g \in I'$, $g = fh$, $h \in R'$. Then $c(g) = c(h)$. But $g = fh \in f\,c(h)R'$ since R' is a content R-module. Therefore $g \in f\,c(g)R' = c(g)I'$; and hence by 1.3 and the remark following 1.3, R'/I' is R-flat.

7. Trace modules.

In analogy with content modules, we shall define an R-module M to be a *trace module* if for every $x \in M$, we have $x \in T(x)M$. Here $T(x)$ denotes, as before, the ideal of R consisting of

$$\{h(x) \mid h \in \mathrm{Hom}_R(M,R)\} .$$

Since $T(x) \subset c(x)$, the trace modules are exactly the content modules for which the content and trace functions coincide. Moreover, since $T(rx) = rT(x)$ for $r \in R$, it follows from 1.6 that every trace module is a flat content module, while 2.2 is an example of a flat content module which is not a trace module, and (by virtue of Theorem 7.3) example 5.4 gives a finitely generated flat content module which is not a trace module.

The following properties of trace modules are immediate consequences of the definitions and the analogous properties for content modules.

7.1 Properties. a) If $M=\bigoplus M_i$, then M is a trace module if and only if each M_i is. As an application, every projective module is a trace module. Examples of trace modules which are not projective are given by 6.4. However, we shall prove below that every finitely generated trace module is projective.

b) If M is a trace R-module and S is a m.s. of R, then M_S is a trace R_S-module and $T_{R_S}(x/s)=T_R(x)R_S$. (Apply 3.1.)

c) If M is a finitely generated trace module, then $T(M)$ is generated by an idempotent. This follows from 3.4, since trace and content coincide for trace modules.

d) If A is an ideal of R and M is a trace R-module, then M/AM is a trace R/A-module.

We shall now prove a theorem which answers the trace module analogue of the question of Section 5. First, we shall recall some elementary facts.

7.2 Facts. a) Let M be a finitely generated R-module. Then M is projective if (and only if) there exist non-nilpotent elements $a_0,\ldots,a_t\in R$ such that $(a_0,\ldots,a_t)=R$ and such that M_{a_i} is R_{a_i}-projective for $i=0,\ldots,t$ (where M_{a_i} denotes the localization of M at the m.s. of R consisting of powers of a_i). Apply [2, p. 138, Thm. 1].

b) If M is an R-module and $x\in M$, then $T(x)=R$ if and only if xR is free and $M=xR\bigoplus M'$. This follows by observing that the homomorphism $R\to M$ defined by $r\to rx$ splits if and only if $T(x)=R$. Note also that if M_P is R_P-free of rank n at some prime P, then $(M')_P$ will be R_P-free of rank $<n$.

7.3 Theorem. *Every finitely generated trace module is projective.*

Proof. We proceed by induction on $n=\max\varrho_M$, where ϱ_M denotes the rank function of the module M. (One can restrict attention to finitely generated modules having coefficient rings which are quotient rings with respect to multiplicative systems of a fixed ring.) If $n=0$, then $M=0$ and hence is projective. Now let M be a finitely generated trace R-module such that $\max\varrho_M=n\geqq 1$. Now $T(M)=(e)$, where e is idempotent, by 7.1 (c). Therefore there exist $x_1,\ldots,x_t\in M$ and $h_1,\ldots,h_t\in \operatorname{Hom}_R(M,R)$ such that $e=h_1(x_1)+\ldots+h_t(x_t)$. Let

$$a_0 = 1-e,\quad a_1 = h_1(x_1),\ldots,a_t = h_t(x_t)\,.$$

It suffices by 7.2 (a) to show M_{a_i} is R_{a_i}-projective for the a_i which are not nilpotent. Since $T(M)=(e)$, we have $M_{a_0}=0$; and thus M_{a_0} is R_{a_0}-projective. Consider then the R_{a_i}-module M_{a_i}, for $i \geqq 1$ and a_i not nilpotent. Then $T(x_i/1)=T(x_i)R_{a_i}$ by 7.1 (b), and hence $T(x_i/1)=R_{a_i}$ since $a_i \in T(x_i)$. Therefore by 7.2 (b), $M_{a_i}=M'\oplus M''$, where M' is free and $\max \varrho_{M''}<n$. But M_{a_i} is a trace R_{a_i}-module by 7.1 (b), and hence M'' is a trace R_{a_i}-module by 7.1 (a). Therefore by induction hypothesis, M'' is projective; and thus M_{a_i} is also projective.

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