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GENERALIZED V-RINGS

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1. Introduction.

Kaplansky [11] noticed that a commutative ring is (von Neumann) regular if and only if it is a V-ring, that is, each simple module over the ring is injective. However in the general case, regularity of a ring is neither necessary nor sufficient to ensure that it is a left or a right V-ring, ([4], [9]). Here we consider right V-rings and also the generalized right V-rings which are written for short as GV-rings. These are the rings over which each simple right module is either projective or injective. We begin with the simple observation that every right V-ring possesses the property that to each element a in R there exists an element x in RaR such that $a=ax$. Our investigation of GV-rings generalizes the work of [1], [3], [13]. For instance a ring R is a right GV-ring if and only if $J(R) \cap Z(R) = 0$, and every proper large right ideal of R is the intersection of all maximal right ideals containing it. In the presence of commutativity, von Neumann regularity is equivalent to the GV-ring condition. The right noetherian semiprime right GV-rings are precisely the rings R such that each semisimple right R-module is injective.

2. Preliminaries.

All the rings that we consider are associative rings with identity and all the modules, unitary right modules. A ring R will be called a *right* V-*ring* if each simple right R-module is injective and, a *generalized right* V-*ring* or for short, a GV-*ring* if each simple right R-module is either projective or injective. Observe that a GV-ring with zero socle is a V-ring. We shall require the following characterisation of right V-rings [5]: The following are equivalent for any ring R:

1) R is a right V-ring,
2) each right ideal of R is an intersection of maximal right ideals,
3) each right R-module has zero radical.

A ring R is called (von Neumann) regular if for each element a in R there exists an element x in R such that $a=axa$. A semiprime ring is

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one which has no nonzero nilpotent ideals and a Goldie ring is a ring R which satisfies the conditions

1) R has maximum condition on annihilator right ideals,
2) R contains no infinite direct sum of right ideals.

A right R-module M is called singular if each element of M is annihilated by a large right ideal of R. It can be checked that a ring R is a right GV-ring if and only if each singular simple right R-module is injective. For any right R-module M, $Z(M)$, $\mathrm{Soc}\, M, J(M)$ will denote respectively the singular submodule of M, the sum of all simple submodules of M, and the intersection of all maximal submodules of M. A submodule S will be called an absolute summand if for any submodule T of M such that T is maximal with respect to $S \cap T = 0$, we have $S \oplus T = M$. A submodule S is fully invariant if S is invariant under every R-endomorphism of M. The fully invariant submodules of injective modules are called quasi-injective. Observe that any simple R-module is quasi-injective. An R-module is π-quasi-injective if each direct product of copies of M is quasi-injective. An R-module is semisimple if it is a direct sum of simple modules.

3. Characterising Properties.

PROPOSITION 3.1. *If R is a right V-ring, then for each element $a \in R$ there is an element $x \in RaR$ such that $a = ax$.*

PROOF. If there is no element x in RaR such that $a = ax$, then $a \notin aRaR$. Hence by Villamayor (see [5, p. 130]), there is a maximal right ideal M of R which contains $aRaR$ but not a. Since $R = aR + M$ we have $1 = as + m$ for some $s \in R$ and $m \in M$, so that $a = asa + ma$ is an element of M, a contradiction. Hence the assertion.

We call rings with the property of proposition 3.1 *right weakly regular* rings. We shall need the following characterisations of such rings (due to Ramamurthi [10] and Vanaja [12]).

PROPOSITION 3.2. *Any ring R has all or none of the following properties*:

1) *R is right weakly regular.*
2) *The quotient R/I is left R-flat for any two-sided ideal I of R.*
3) *The equation $I^2 = I$ holds for every right ideal I of R.*

We now give a characterisation of GV-rings, and this generalizes theorem 1.1 of [1].

THEOREM 3.3. *The following are equivalent for any ring R.*

1) *Every proper large right ideal of R is an intersection of maximal right ideals of R, and $Z(R)\cap J(R)=0$.*

2) *R is a right* GV*-ring.*

3) *The module $J(M)$ vanishes for any right R-module M with $Z(M)$ large in M.*

4) *If M is any right R-module, then every proper large submodule of M is an intersection of maximal submodules of M and $Z(M)\cap J(M)=0$*

PROOF. $1 \Rightarrow 2$: Let X be a simple singular right R-module and I a large right ideal of R with a nonzero R-homomorphism $f\colon I \to X$. If K $(\neq I)$ is the kernel of f and K is not a large R-submodule of I, then K is a summand of I, and so $I=K\oplus S$. Here $S\cong X$ is simple so that $S^2=0$ or $S=eR$ where $e^2=e$. Since S is singular, $S\subset Z(R)$ and hence $S\neq eR$. Thus $S\subset J(R)\cap Z(R)=0$, a contradiction. Thus K is large in I and hence in R. Let M be a maximal right ideal of R containing K but not I. Then $R/K=M/K\oplus I/K$. Let g be the natural map $R\to R/K$, let h be the projection $R/K\to I/K$ and k be the isomorphism $I/K\to X$. Then the composite map khg extends f from I to R. Thus X is injective.

$2 \Rightarrow 3$: Let $0\neq m\in M$. Since $Z(M)$ is large in M, we get $0\neq mr=x\in Z(M)$ for some $r\in R$. Let L be a submodule of M maximal with respect to $x\notin L$. Then the module $(xR+L)/L$ is simple singular and hence is an injective submodule of the subdirectly irreducible module M/L, so that $M/L=(xR+L)/L$. This implies that L is a maximal submodule of M. Clearly $m\notin L$. Thus $J(M)=0$.

$3 \Rightarrow 4$: If M is any right R-module and $S\neq M$ is large in M, then, by hypothesis, $J(M/S)=0$ so that S is the intersection of all the maximal submodules of M containing S. Suppose

$$0 \neq m\in Z(M)\cap J(M)\,.$$

Let L be a submodule of M maximal with respect to $m\notin L$. Clearly M/L is subdirectly inreducible and is not simple since L is not a maximal submodule of M so that $J(M/L)\neq 0$. But $(mR+L)/L$ is singular so that $Z(M/L)$ is large in M/L and hence, by hypothesis, $J(M/L)=0$, a contradiction.

$4 \Rightarrow 1$: Trivial.

REMARK 1. The condition $Z(R) \cap J(R) = 0$ in (1) cannot be dropped. In fact, if R is the ring of integers modulo p^2 where p is a prime, then trivially each proper large right ideal of R is an intersection of maximal right ideals, but R is not a GV-ring, since R/pR is neither injective nor projective.

REMARK 2. The condition (4) implies that if R is a GV-ring, then for any right R-module M, we get $J(M) \subset \operatorname{Soc} M$. This is because $\operatorname{Soc} M$ is the intersection of all large submodule of M and $J(M/S) = 0$ for every large submodule of M. In particular, $J(R) \subset \operatorname{Soc} R$ and hence $(J(R))^2 = 0$.

PROPOSITION 3.4. *If R is a right* GV*-ring in which every primitive idempotent is central, then R is a right* V*-ring.*

PROOF. Let S be a simple projective right R-module so that $S \cong eR$ where e is an idempotent. Let I be an arbitrary right ideal of R and $f: I \to eR$ any nonzero epimorphism with kernel K. Then $I = K \oplus T$ where $T \cong eR$. Since e is central, eR is a fully invariant summand of the right R-module R and hence $eR = T$. Clearly $K \subset (1-e)R$ and hence the map given by $g = 0$ on $(1-e)R$ and $g = f$ on eR is an extension of f. Hence S is injective.

COROLLARY 3.5. *A right* GV*-ring with no nonzero nilpotent elements is a* V*-ring.*

Proposition 3.4 at once yields a characterisation of commutative GV-rings.

THEOREM 3.6. *For any commutative ring R the following are equivalent*

(1) *R is a* GV*-ring,*
(2) *R is a* V*-ring,*
(3) *R is a von Neumann regular ring.*

PROOF. $1 \Rightarrow 2$ by proposition 3.4 and $2 \Rightarrow 3$ by proposition 3.1. Since for any regular ring R, $J(R) = 0$ and furthermore every homomorphic image of a regular ring is regular, we have $3 \Rightarrow 1$ by theorem 3.3.

The following two propositions give sufficient conditions for a right GV-ring and a right weakly regular ring to be a V-ring.

Proposition 3.7. *A ring R is a* V-*ring if and only if R is a right* GV-*ring and every simple right ideal of R is an absolute summand of R.*

Proof. Let R be a V-ring and S be a simple right ideal of R. Let T be a right ideal of R maximal with respect to $S \cap T = 0$. If $X = S \oplus T$ and $X \to S$ is the obvious projection, then, by the injectivity of S, this extends to an epimorphism $f: R \to S$. Clearly $\ker f \cap S = 0$ and $\ker f \supset T$. Hence $\ker f = T$. Thus $S \oplus T = R$ as $\ker f$ is a maximal right ideal in R. Conversely, let R be a right GV-ring and let every simple right ideal of R be an absolute summand. If X is a projective simple right R-module and $f: I \to X$ is a nonzero morphism with kernel K, then K is a summand of I, that is $I = K \oplus L$ for a right ideal L of R. As L is isomorphic to X, the ideal L is simple and hence, if T is a right ideal of R containing K and maximal with respect to $T \cap L = 0$, then $L \oplus T = R$. The projection $R \to L$, followed by the isomorphism $L \to X$, extends f from I to R. Thus X is injective.

Proposition 3.8. *A ring R is a right* V-*ring if and only if it is right weakly regular and each simple right R-module is π-quasi injective.*

Proof. Let X be a simple right R-module that is π-quasi-injective. Then, by Fuller [6], X is an injective right $R/l(X)$-module where $l(X)$ is the annihilator of X in R. If R is right weakly regular then, by proposition 3.2, the left R-module $R/l(X)$ is flat, so that

$$\text{inj dim}_R X \leqq \text{inj dim}_{R/l(X)} X$$

by Cartan and Eilenberg [2, exercise 9, chapter VI]. Thus X is an injective right R-module. The converse is trivial.

We conclude this section by noting the following.

Proposition 3.9. *A ring R is a* V-*ring if and only if R_n (the ring of all $n \times n$ matrices over R) is a* V-*ring.*

Proof. This follows from the Morita equivalence of the categories of R-modules and R_n-modules.

4. Semiprime GV-rings.

In general a GV-ring need not be semiprime. We give below, conditions under which a right GV-ring is semiprime. Our main result in this section asserts that, noetherian semiprime GV-rings are precisely the rings over which each semisimple module is injective.

LEMMA 4.1. *In any right* GV*-ring each large right ideal is idempotent.*

PROOF. Let I be a large right ideal of the right GV-ring R such that $I^2 \neq I$. Then there is an element $a \in R$, such that $a \in I$ but $a \notin I^2$. By Zorn's lemma, choose a right ideal T, maximal among those containing I^2 but not a. Then $(aR+T)/T$ is a simple singular right R-module and hence injective. Since R/T is an essential extension of $(aR+T)/T$, it follows that T is a maximal right ideal of R. Since $a \notin T$, there are elements $r \in R$, $t \in T$ such that $ar+t=1$. Thus $ara+ta=a$ is an element of T, a contradiction.

PROPOSITION 4.2. *A right* GV*-ring is semiprime if and only if each two-sided ideal of R is idempotent.*

PROOF. Let A be a two-sided ideal of the semiprime right GV-ring R. If B is the right annihilator of A in R, then B is a two-sided ideal and since R is semiprime, $A \cap B = 0$ and $A+B$ is a large right ideal. Hence, by lemma 4.1,

$$A \oplus B = (A \oplus B)^2 = A^2 + AB + BA + B^2 = A^2 \oplus B^2 .$$

Thus $A = A^2$. The converse is obvious.

PROPOSITION 4.3. *A ring R, in which each large right ideal is two-sided, is semiprime right* GV, *if and only if it is right weakly regular.*

PROOF. Suppose that R is a right weakly regular ring in which every large right ideal is two-sided. Then by Ramamurthi [10], we have $J(R)=0$ and $J(R/I)=0$ for each large right ideal I of R, so that, by theorem 3.3, R is a right GV-ring. Conversely, let a be any element of the semiprime right GV-ring R and let H be a right ideal of R, maximal with respect to $aR \cap H = 0$. Then, $aR+H$ is a large right ideal. As this is a two-sided ideal, $RaR \subset (aR+H)$ so that

$$\begin{aligned} aR + (RaR \cap H) &= RaR = (RaR)^2 \\ &= aRaR + aR(RaR \cap H) + (RaR \cap H)aR + (RaR \cap H)(RaR \cap H) . \end{aligned}$$

Thus

$$aR = aRaR + aR(RaR \cap H) = aRaR ,$$

which implies that R is a right weakly regular ring, by proposition 3.2.

REMARK 3. It is known that a self-injective regular ring need not be a V-ring [9]. It is easy to deduce from the above proposition, that, any self-injective regular ring in which each large right ideal is two-sided (these are precisely the semiprime q-rings of [7]) is a V-ring.

Next we consider noetherian GV-rings. The following lemma is proved in [10].

LEMMA 4.4. *Every ring R contains a largest two-sided ideal W, which is right weakly regular, such that* 0 *is the only right weakly regular ideal in R/W. W is called the* RWR*-radical of R.*

PROPOSITION 4.5. *Any prime right* GV*-ring R is right weakly regular.*

PROOF. If R has zero socle then it is a V-ring and hence right weakly regular by proposition 3.1. Let R have a nonzero socle S. Then S is a regular ideal of R (see the proof of theorem 1.8 of [1]) so that the RWR-radical of R is nonzero. Since R is a prime ring and W is a two-sided ideal of R, it is a large right ideal of R. Hence by lemma 4.1, the ring R/W is right weakly regular, which implies that $R/W=0$ or $R=W$. Hence the proposition.

COROLLARY 4.6. *Any prime Goldie right* GV*-ring is simple.*

PROOF. Follows from proposition 4.5 and the fact that any prime right weakly regular Goldie ring is simple (see [10]).

We need the following result, due to Ornstein [8], for our next theorem.

LEMMA 4.7. *Let R be a non-prime, semiprime ring. Then R is a right Artin ring if and only if R/A is right noetherian for every nonzero two-sided ideal A of R, and R/P is a simple ring for every nonzero prime ideal P of R.*

THEOREM 4.8. *The following are equivalent for any ring R.*

(1) *R is a semiprime right noetherian right* GV*-ring.*

(2) *R is either a simple right noetherian* V*-ring with zero socle or is a semisimple Artin ring.*

(3) *Every semisimple right R-module is injective.*

PROOF. $1 \Rightarrow 2$: If R is prime, then the implication follows by corollary 4.6. Let R be not prime. If A is any nonzero two-sided ideal of R, then R/A is right noetherian because R is so. If P is any nonzero prime ideal of R then R/P is a prime right noetherian right GV-ring so that by corollary 4.6, the ring R/P is simple. Thus by lemma 4.7, R is a semi-simple Artin ring.

$3 \Rightarrow 1$: It is sufficient to prove that R is right noetherian. Let

$$I_1 \subset I_2 \subset \ldots \subset I_n \subset \ldots$$

be an ascending sequence of distinct right ideals of R. As R is a V-ring, there are maximal right ideals M_k $(k=1,2,\ldots)$ such that $I_k \subset M_k$ but $I_{k+1} \not\subset M_k$. Let p_k denote the natural projection $R \to R/M_k$. If $I = \bigcup_{k=1}^{\infty} I_k$, define a morphism

$$f\colon\ I \to \textstyle\sum_{k=1}^{\infty} \oplus R/M_k\,,$$

by $f(x)=\sum_{k=1}^{\infty} p_k(x)$ (the summation on the right being meaningful since every $x \in I$ belongs to all but a finite number of the M_k's). Then f extends to a morphism

$$g\colon\ R \to \textstyle\sum_{k=1}^{\infty} \oplus R/M_k\,,$$

since

$$\textstyle\sum_{k=1}^{\infty} \oplus R/M_k$$

is a semi-simple R-module and hence injective. Since R has an identity, $g(R)$ and hence $g(I)$ is contained in

$$\textstyle\sum_{k=1}^{t} \oplus R/M_k$$

for some positive integer t. This implies that the assumed chain of right ideals is finite.

$2 \Rightarrow 3$: Since R is right noetherian, each direct sum of injective right R-modules is injective. Because R is furthermore a V-ring, 3 follows.

COROLLARY 4.9. ([3]) *A commutative ring R is Artin semisimple if and only if every semisimple R-module is injective.*

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