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ON THE LENGTH OF FAITHFUL MODULES OVER ARTINIAN LOCAL RINGS

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Let R be an Artinian local ring with residue field $k=R/\mathfrak{m}$. Let M be any faithful R-module, that is $rM=0$ implies $r=0$ for all $r \in R$. Then for a large class of rings R one has the inequality

$$(*) \qquad l(M) \geqq l(R) ,$$

l denoting classical length. It is easily seen that the inequality is valid whenever R is self injective, that is when $\dim_k \operatorname{Hom}_R(k,R)=1$; see (2.8) in [1]. The purpose of the present note is to generalize this fact by showing that (*) is valid for all faithful R-modules M whenever $\dim_k \operatorname{Hom}_R(k,R) \leqq 3$. This result is in a way the best possible, in fact for each integer $s \geqq 4$ we can give an example of a local ring R and a faithful R-module M such that

$$l(M) < l(R) \quad \text{and} \quad \dim_k \operatorname{Hom}_R(k,R) = s .$$

We shall use the following notation.

R will always be an Artinian local ring with maximal ideal $\mathfrak{m}$. R-modules are assumed to be unitary and finitely generated. If M is an R-module we define the annihilator

$$\operatorname{an}(M) = \{r \in R \mid rM=0\} ,$$

and the socle

$$s(M) = \{x \in M \mid \mathfrak{m}x=0\} .$$

Observe that $s(M) \approx \operatorname{Hom}_R(R/\mathfrak{m},M)$.

By $l(M)$ we denote the length of M. If $\operatorname{an}(M)=\mathfrak{m}$ then $\dim M$ will denote the dimension of M as a vectorspace over $R/\mathfrak{m}$. By E we denote the injective hull of the R-module $R/\mathfrak{m}$. We let M^* denote the dual of M, that is

$$M^* = \operatorname{Hom}_R(M,E) .$$

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Recall that the functor $\mathrm{Hom}_R(-,E)$ defines a duality on the category of finitely generated R-modules, cf. [2]. Note that

$$\mathrm{an}(M) = \mathrm{an}(M^*), \quad s(M^*) \approx M/\mathfrak{m}M .$$

M will be called a faithful R-module if $\mathrm{an}(M)=0$. Observe that E is, up to isomorphism, the only faithful R-module with one-dimensional socle.

LEMMA 1. *Let M be a faithful R-module. Suppose that M/N is not faithful for any submodule $N \neq 0$. Then $s(M)=s(R)M$.*

PROOF. Let N be a submodule of M such that

$$s(M) = (s(R)M)\oplus N .$$

We are going to show that $N=0$. Suppose $N \neq 0$. Then by the minimality of M there exists an element $r \neq 0$ in R such that $rM \subset N$. We may as well assume that $r \in s(R)$. It follows that $rM \subset (s(R)M)\cap N=0$. Hence $r=0$, which is a contradiction.

LEMMA 2. *Let M be a faithful R-module. Assume that neither N nor M/N is faithful for any submodule N such that $0 \neq N \neq M$. Then we have*

$$\text{(i)} \qquad \dim M/\mathfrak{m}M \leqq \dim s(R)$$

$$\text{(ii)} \qquad \dim s(M) \leqq \dim s(R) .$$

Moreover, if $M \neq R$ then at least one of the inequalities is strict.

PROOF. We will first prove (i). Let $m=\dim M/\mathfrak{m}M$ and let $g_1,\ldots,g_m$ be a minimal set of generators for M. Since (i) is obvious if $m=1$, we may assume that $m \geqq 2$. For $1 \leqq i \leqq m$ let M_i be the submodule generated by all $g_1,\ldots,g_m$ except g_i. Put $\mathfrak{c}_i=\mathrm{an}(M_i)$. By the minimality of M we have $\mathfrak{c}_i \neq 0$ hence $\mathfrak{c}_i \cap s(R) \neq 0$ for all i. Choose one non-zero element u_i in $\mathfrak{c}_i \cap s(R)$ for each i. Since M is faithful, the elements u_i are clearly linearly independent over the field $R/\mathfrak{m}$. It follows that $m \leqq \dim s(R)$.

To prove (ii) we just have to apply (i) to the dual M^*, observing that M^* satisfies the same minimality conditions as M. We get

$$\dim s(M) = \dim M^*/\mathfrak{m}M^* \leqq \dim s(R) .$$

We will now assume that we have equality in both (i) and (ii), and we assume that M is not isomorphic to R. We are going to show that this is impossible.

Since M is faithful, but not isomorphic to R, we have $\dim M/\mathfrak{m}M \geqq 2$. Let $g_1,\ldots,g_m$ and $u_1,\ldots,u_m$ be as above. The equality in (i) gives that $u_1,\ldots,u_m$ is a basis for $s(R)$. Hence by lemma 1 we obtain

$$s(M) = (u_1,\ldots,u_m)(g_1,\ldots,g_m) = (u_1g_1, u_2g_2,\ldots,u_mg_m) .$$

Let $\mathfrak{c}$ be the annihilator of the element $g_1+\ldots+g_m$. By minimality of M we have $\mathfrak{c}\neq 0$ and hence $\mathfrak{c}\cap s(R)\neq 0$. Let u be a non-zero element in $\mathfrak{c}\cap s(R)$. Let $r_1,\ldots,r_m$ be elements in R such that $u=\sum_{i=1}^m r_iu_i$. We have

$$0 = u(g_1+\ldots+g_m) = \sum_{i=1}^m r_iu_ig_i .$$

Since not all r_i are in $\mathfrak{m}$, the equation above shows that $\dim s(M) < m$ contradicting the equality in (ii).

COROLLARY. *Let M be as in lemma 2 and suppose that* $\dim s(R) \leqq 2$. *Then $M\approx R$ or $M\approx E$.*

PROOF. If $M\neq R$ then by lemma 2 we have $\dim s(M)=1$, hence $M\approx E$.

THEOREM 1. *Let R be an Artinian local ring with*

$$\dim_{R/\mathfrak{m}} \operatorname{Hom}_R(R/\mathfrak{m},R) \leqq 3 .$$

Let M be a faithful R-module. Then we have $l(M)\geqq l(R)$.

PROOF. Clearly we may assume that M is a faithful module of minimal length, so that M as well as M^* satisfies the assumption in lemma 2. If $\dim s(R)\leqq 2$ then the theorem follows from the above corollary. We may therefore assume that $\dim s(R)=3$. Moreover we may assume that M is not isomorphic to R. Hence using lemma 2 and the relation

$$\dim M/\mathfrak{m}M = \dim s(M^*) ,$$

we have either

$$\dim s(M^*) \leqq 2 \quad \text{or} \quad \dim s(M) \leqq 2 .$$

There is no loss of generality in assuming that $\dim s(M)\leqq 2$. If $\dim s(M)=1$ then $M\approx E$, and if $\dim M/\mathfrak{m}M=1$ then $M\approx R$. Hence in the rest of the proof we may work under the following assumptions:

$$\dim s(R) = 3, \quad \dim s(M) = 2, \quad \dim M/\mathfrak{m}M \geqq 2 .$$

By the second of these assumptions we can find non-zero irreducible submodules M_1, M_2 in M such that $0 = M_1 \cap M_2$ (see § 2 in [1]). Put $\mathfrak{a}_i = \mathrm{an}(M/M_i)$ for $i = 1,2$. We will first show that

$$(1) \qquad l(M/M_i) = l(R/\mathfrak{a}_i) \quad \text{for } i = 1,2 .$$

Since M_i is irreducible we have $\dim s(M/M_i) = 1$. It follows that $(M/M_i)^*$ is a homomorphic image of R. Moreover we have

$$\mathrm{an}((M/M_i)^*) = \mathrm{an}(M/M_i) = \mathfrak{a}_i ,$$

and hence $(M/M_i)^* \approx R/\mathfrak{a}_i$, so (1) follows.

Since M is faithful we have $\mathfrak{a}_1 \cap \mathfrak{a}_2 = 0$. Since $\dim s(R) = 3$, at least one of the two vectorspaces $s(\mathfrak{a}_1)$ and $s(\mathfrak{a}_2)$ is one-dimensional. We assume that $\dim s(\mathfrak{a}_1) = 1$. In view of (1) it now suffices to show that $l(M_1) \geqq l(\mathfrak{a}_1)$. Since $\mathfrak{a}_1 M \subset M_1$ it will be sufficient to prove that

$$(2) \qquad l(\mathfrak{a}_1 M) \geqq l(\mathfrak{a}_1) .$$

Let $g_1, g_2, \ldots, g_m$ be a minimal set of generators for M. Put $\mathfrak{b}_i = \mathrm{an}(g_i)$ for $1 \leqq i \leqq m$. Then $\bigcap_{i=1}^m \mathfrak{b}_i = 0$. Hence one of the $\mathfrak{b}_i$, say $\mathfrak{b}_1$, does not contain $s(\mathfrak{a}_1)$. Since $\dim s(\mathfrak{a}_1) = 1$ we conclude that $\mathfrak{a}_1 \cap \mathfrak{b}_1 = 0$. We obtain $\mathfrak{a}_1 M \supset \mathfrak{a}_1 g_1 \approx \mathfrak{a}_1(R/\mathfrak{b}_1) \approx \mathfrak{a}_1/\mathfrak{a}_1 \cap \mathfrak{b}_1 = \mathfrak{a}_1$ which yields (2).

THEOREM 2. *Let $s \geqq 4$ be an integer. Then there exists a local Artinian ring R and a faithful R-module M such that*

$$(\mathrm{i}) \qquad \dim_{R/\mathfrak{m}} \mathrm{Hom}_R(R/\mathfrak{m}, R) = s ,$$

$$(\mathrm{ii}) \qquad l(M) < l(R) .$$

PROOF. Let $m \geqq 2$ be an integer and let k be an arbitrary field. Let R_m be the k-algebra of $(m+2) \times (m+2)$-matrices of the form

$$(3) \qquad \left(\begin{array}{ccc|c} & \lambda I_{m,m} & & O_{m,2} \\ \hline a_1 & \ldots & a_m & \\ b_1 & \ldots & b_m & \lambda I_{2,2} \end{array}\right),$$

where λ, $a_1, \ldots, a_m$, $b_1, \ldots, b_m$ run through k and $I_{p,q}$, and $O_{p,q}$ denotes the identity matrix and the zero-matrix of size $p \times q$. Clearly R_m is a commutative local Artinian ring of length $l(R_m) = 2m+1$. In fact the socle of R_m coincides with the maximal ideal which consists of all matrices of the form (3) in which $\lambda = 0$. Hence $\dim s(R_m) = 2m$.

Now let M be the k-vectorspace k^{m+2}. Clearly M becomes a faithful R_m-module in the obvious way. We have

$$l(M) = \dim_k M = m+2 < 2m+1 = l(R_m) .$$

This proves the theorem in the case where s is even.

Assume that s is odd. Write $s = 2m-1$ where $m \geqq 3$. Consider R_m and M as before. Let R'_m be the subring consisting of all matrices of the form (3) in which $a_m = 0$. Clearly R'_m is a local ring of length $2m$ and $\dim s(R'_m) = 2m-1 = s$. Moreover M is a faithful R'_m-module with

$$l(M) = \dim_k M = m+2 < 2m = l(R'_m) .$$

The proof is now complete.

REMARK. Let $R = \mathbf{C}[X, Y]/(X, Y)^4$. It can be shown that $l(M) \geqq l(R)$ for any faithful R-module, inspite of the fact that $\dim s(R) = 4$.

REFERENCES

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