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DIRECT LIMITS OF POWER SERIES RINGS[1]

ELIZABETH A. MAGARIAN

All rings considered in this paper are assumed to be commutative and to contain an identity element.

Definition. A subring R of a ring S is said to have *property* (C) *with respect to* S provided $(AS)\cap R=A$ for every ideal A of R.

Let R be a ring, and let $\{R_n\}_{n\in D}$ be a directed family of subrings of R whose union is R. We shall be interested in the question:

When is $\bigcup R_n[[X_1,\ldots,X_t]]$ Noetherian?

Theorem 1 gives a description of a finite basis for each prime ideal of $\bigcup R_n[[X]]$, where each R_n has property (C) with respect to R. Theorem 2 shows that if R is Noetherian and if for each n, there is an R_n-module T_n for which $R=R_n\oplus T_n$, then $\bigcup(R_n[[X_1,\ldots,X_t]])$ is Noetherian. In Theorems 3 and 4, Theorem 2 is applied to the question: When is the tensor product of a local ring containing a field k and an algebraic extension field K of k Noetherian? Examples 1 and 2 show that if R is a local ring containing a field k and if K is an extension field of k, then $R\otimes_k K$ is not necessarily semilocal.

Theorem 1. *Let $\{R_n\}$ be a directed family of subrings of a Noetherian ring R whose union is R and such that each R_n has property* (C) *with respect to R. If $A=\bigcup R_n[[X]]$, then A is Noetherian. In fact, if P is a prime ideal of A and*

$$P_0 = \{d : d\in R \text{ and } f(0)=d \text{ for some } f\in P\} = (a_1,\ldots,a_k),$$

let f_i be an element of P for which $f_i(0)=a_i$. Then $P=(f_1,\ldots,f_k,X)$ or $P=(f_1,\ldots,f_k)$ according as X is in P or not.

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PROOF. Let P be a prime ideal of A. If

$$P_0 = \{d : d \in R \text{ and } f(0)=d \text{ for some } f \in P\},$$

then P_0 is an ideal of the Noetherian ring R and hence has a finite basis $\{a_1,\ldots,a_k\}$. Since each $a_i \in P_0$, there exist $f_i \in P$ such that $f_i(0)=a_i$. For each i, choose such an f_i and fix it.

For each $R_n \in \{R_n\}$,

$$P_0 \cap R_n = ((a_1,\ldots,a_k)R) \cap R_n = ((a_1,\ldots,a_k)R_n)R \cap R_n = (a_1,\ldots,a_k)R_n$$

provided $\{a_1,\ldots,a_k\} \subseteqq R_n$ since R_n has property (C) with respect to R.

If $\{f_1,\ldots,f_k\} \subseteqq R_n[[X]]$, let

$${}_nP_0 = \{d : d \in R \text{ and } f(0)=d \text{ for some } f \in P \cap R_n[[X]]\}.$$

Then ${}_nP_0$ is an ideal of R_n, and

$${}_nP_0 \subseteqq P_0 \cap R_n = (a_1,\ldots,a_k)R_n .$$

Moreover, $f_i \in P \cap R_n[[X]]$; hence, $a_i \in {}_nP_0$ for every i. Therefore, ${}_nP_0 \supseteqq (a_1,\ldots,a_k)R_n$, and

$${}_nP_0 = P_0 \cap R_n = (a_1,\ldots,a_k)R_n .$$

It will be shown that

$$\begin{aligned} P &= (f_1,\ldots,f_k,X) && \text{if } X \in P \\ P &= (f_1,\ldots,f_k) && \text{if } X \notin P . \end{aligned}$$

Since $f_i \in P$, we have $P \supseteqq (f_1,\ldots,f_k)$ and if $X \in P$,

$$P \supseteqq (f_1,\ldots,f_k,X) .$$

Let $g \in P$. Then there is an n such that $g, f_1,\ldots,f_k$ belong to $R_n[[X]]$. Note that $f_i \in R_n[[X]]$ implies that $a_i \in R_n$. Now $g(0) \in {}_nP_0$. Hence, there exist $r_{01},\ldots,r_{0k}$ in R_n such that $g(0)=\sum_{i=1}^{k} r_{0i}a_i$. Moreover,

$$g-\sum_{i=1}^{k} r_{0i}f_i \in P$$

and has zero constant term so that

$$g-\sum_{i=1}^{k} r_{0i}f_i \in (X) .$$

If $X \in P$, then $g \in (f_1,\ldots,f_k,X)$ so that

$$P \subseteqq (f_1,\ldots,f_k,X)$$

and

$$P = (f_1,\ldots,f_k,X) .$$

On the other hand if $X \notin P$, let

$$Xg_1 = g - \sum_{i=1}^{k} r_{0i} f_i \ .$$

Since P is prime, $g_1 \in P$. For $0 \leqq j \leqq m$ assume that elements $r_{j1}, \ldots, r_{jk}$ of R_n have been found so that

$$g - \left(\sum_{j=0}^{m} \sum_{i=1}^{k} X^j r_{ji} f_i\right) = X^{m+1} g_{m+1} \in P \ .$$

Since P is prime and $X^{m+1} \notin P$, it follows that $g_{m+1} \in P$. Now $g_{m+1}(0) \in {}_nP_0$ so there are elements $r_{m+1,1}, \ldots, r_{m+1,k}$ of R_n such that

$$g_{m+1}(0) = \sum_{i=1}^{k} (r_{m+1,i}) a_i \ .$$

Then $g_{m+1} - \sum_{i=1}^{k} (r_{m+1,i}) f_i \in P$ and has zero constant term so that

$$g - \sum_{i=1}^{k} \left(\sum_{j=0}^{m+1} r_{ji} X^j\right) f_i = X^{m+2} g_{m+2} \in P \ .$$

Hence,

$$g = \sum_{i=1}^{k} \left(\sum_{j=0}^{\infty} r_{ji} X^j\right) f_i \quad \text{and} \quad \sum_{j=0}^{\infty} r_{ji} X^j \in R_n[[X]] \ .$$

Therefore,

$$g \in (f_1, \ldots, f_k) R_n[[X]] \subseteqq (f_1, \ldots, f_k) A$$

and $P \subseteqq (f_1, \ldots, f_k)$; so $P = (f_1, \ldots, f_k)$.

Since every prime ideal of A is finitely generated, by a classical result of I. S. Cohen [8, Theorem 3.4, p. 8] the ring A is Noetherian.

If P is a prime ideal of $\bigcup R_n[[X_1, \ldots, X_t]]$ and

$$P_0 = \{d : d \in R \text{ and } f(0) = d \text{ for some } f \in P\} = (a_1, \ldots, a_k) \ ,$$

it is not known whether P has a basis of the form $\{f_1, \ldots, f_k\} \cup X$, where $f_i \in P$, $f_i(0) = a_i$, and X is the maximal subset of $\{X_1, \ldots, X_t\}$ contained in P.

Corollary. *Let $\{R_n\}$ be a directed family of subrings of R whose union is R and such that each R_n has property* (C) *with respect to R. If R is a principal ideal domain, then $\bigcup R_n[[X]]$ is a unique factorization domain.*

Proof. The ideal

$$(X) = \{f : f \in R[[X]] \text{ and } f(0) = 0\}$$

is prime in $R[[X]]$ since $R[[X]]/(X) \cong R$. Let $A = \bigcup R_n[[X]]$. If a and $b \in A$ and

$$ab \in (X) \cap A = \{f : f \in A \text{ and } f(0) = 0\} \ ,$$

then a or $b \in (X)$; so a or $b \in (X) \cap A$ and $(X) \cap A$ is prime. Let P be a prime ideal in A. If $X \in P$, then P contains the principal prime ideal (X). If $X \notin P$, then P is itself a principal prime. Hence, A is a unique factorization domain [6, Theorem 1.2, p. 2].

Krull is generally credited with the result that if R is a principal ideal domain, then $R[[X]]$ is a unique factorization domain. In 1938 he proved this for the case where R/M is infinite for each maximal ideal M of R [7, § 5, Satz 17, p. 780]. Note, however, that $\bigcup R_n[[X]]$ need not equal $R[[X]]$. For example, if R is an infinite algebraic extension of a field k and if $\{R_n\}$ is the collection of finite algebraic extensions of k contained in R, then $R = \bigcup R_n \cong \text{inj}\lim R_n$ and

$$\begin{aligned} R \otimes_k k[[X]] &\cong (\text{inj}\lim R_n) \otimes_k k[[X]] \\ &\cong \text{inj}\lim (R_n \otimes_k k[[X]]) \\ &\cong \text{inj}\lim R_n[[X]] \\ &\cong \bigcup R_n[[X]] \subsetneqq R[[X]] \end{aligned}$$

since the coefficients of a series in $R[[X]]$ may generate an infinite extension of k while the coefficients of a series in $\bigcup R_n[[X]]$ only generate a finite extension of k.

THEOREM 2. *Let R be a commutative ring, $\{R_n\}$ a directed family of subrings of R whose union is R. For each n, let T_n be an R_n-module such that $R = R_n \oplus T_n$. If R is Noetherian, then $\bigcup (R_n[[X_1, \ldots, X_t]])$ is Noetherian.*

PROOF. Let

$$A_n = R_n[[X_1, \ldots, X_t]], \quad B = R[[X_1, \ldots, X_t]],$$
$$U = \bigcup (R_n[[X_1, \ldots, X_t]]).$$

Since $R = R_n \oplus T_n$,

$$R[[X_1, \ldots, X_t]] = R_n[[X_1, \ldots, X_t]] \oplus T_n[[X_1, \ldots, X_t]]$$

and $T_n[[X_1, \ldots, X_t]]$ is an $(R_n[[X_1, \ldots, X_t]])$-module. A result of Gilmer [5, § 2, Result 3, p. 562] shows that $R_n[[X_1, \ldots, X_t]]$ satisfies property (C) with respect to $R[[X_1, \ldots, X_t]]$.

To show that U satisfies property (C) with respect to B it suffices to show that each finitely generated ideal of U is the contraction of an ideal of B [5, § 2, Result 5, p. 562]. Let D be a finitely generated ideal $(d_1, \ldots, d_m)U$ of U. Let $f \in DB \cap U$. Then $\{d_1, \ldots, d_m, f\}$ is contained in A_n for some n. Since $A_n \subseteqq U \subseteqq B$,

$$((d_1, \ldots, d_m)A_n)B = ((d_1, \ldots, d_m)U)B = DB$$

and $f \in (d_1, \ldots, d_m)A_nB$. Moreover,

$$f \in (d_1, \ldots, d_m)A_nB \cap A_n = (d_1, \ldots, d_m)A_n \subseteqq (d_1, \ldots, d_m)U$$

since A_n satisfies property (C) with respect to B. Hence, $f \in D$ and $D \supseteqq DB \cap U$. Since $DB \cap U$ always contains D [9, p. 219], $DB \cap U = D$ and U has property (C) with respect to B. Therefore, B Noetherian [10, Corollary to Theorem 4, p. 139] implies that U is Noetherian.

Application. Let R be an algebraic extension field of a field k. Then R is the union of the set of all finite algebraic extensions R_n of k contained in R. Since R is a vector space over R_n and R_n is a subspace of R generated by 1, there is a basis $\{1\} \cup B$ for R over R_n. Let T_n be the vector space over R_n generated by B. Then T_n is an R_n-module and $R = R_n \oplus T_n$. Now

$$\begin{aligned} k[[X_1, \ldots, X_m]] \otimes_k R &\cong k[[X_1, \ldots, X_m]] \otimes_k \bigcup R_n \\ &\cong k[[X_1, \ldots, X_m]] \otimes_k (\operatorname{inj\,lim} R_n) \\ &\cong \operatorname{inj\,lim} (k[[X_1, \ldots, X_m]] \otimes_k R_n) \\ &\cong \operatorname{inj\,lim} (R_n[[X_1, \ldots, X_m]]) \\ &\cong \bigcup (R_n[[X_1, \ldots, X_m]]) , \end{aligned}$$

which is Noetherian.

Theorem 3. *Let K be an algebraic extension field of a field k. Let $R = k[[X_1, \ldots, X_n]]$. Then $R \otimes_k K$ is a regular local ring of dimension n, and if $n = 1$, $R \otimes_k K$ is a rank one discrete valuation ring.*

Proof. Let

$$F_\alpha \in \{F : k \subseteqq F \subseteqq K, \ F \text{ a field}, \ [F : k] < \infty\} .$$

Then F_α satisfies property (C) with respect to K, $K = \bigcup F_\alpha$, and $R \otimes_k K$ is Noetherian. Moreover, since $F_\alpha[[X_1, \ldots, X_n]]$ is a local ring with maximal ideal $M_\alpha = \{f \in F_\alpha[[X_1, \ldots, X_n]] : f(0) = 0\} = (X_1, \ldots, X_n)_\alpha$, $M = \bigcup M_\alpha$ is the unique maximal ideal of $\bigcup F_\alpha[[X_1, \ldots, X_n]]$, being the set of all nonunits of the ring. Therefore, $R \otimes_k K$ is a local ring.

Since K is algebraic over k, $R \otimes_k K$ is integral over R. Therefore, $\dim (R \otimes_k K) = \dim R = n$ [8, I, (10.10), p. 30]. Moreover, the maximal ideal of $R \otimes_k K$ has a basis $\{X_1, \ldots, X_n\}$ of n elements; so $R \otimes_k K$ is a regular local ring of dimension n [10, VIII, § 11, p. 301].

In particular, if n is one, $R \otimes_k K$ is a one-dimensional local ring with principal maximal ideal; so $R \otimes_k K$ is a rank one discrete valuation ring [2, VI, § 3, no. 6, Proposition 9 (d), p. 109].

The following example shows that if the extension field K of k is not an algebraic extension of k, then $R \otimes_k K$ need not be local nor even semilocal.

EXAMPLE 1. Let k be a field, $R = k[[X]]$, $K = k(Y)$, and $S = k[Y] - \{0\}$, where Y is transcendental over k. The ring $k[[X]]$ is a flat k-module [1, § 3, no. 4, Corollary 3 to Theorem 3, p. 70]. Hence,

$$\begin{aligned} R \otimes_k K &= k[[X]] \otimes_k k(Y) = k[[X]] \otimes_k (k[Y])_S \\ &\cong (k[[X]] \otimes_k k[Y])_{1 \otimes S} \cong (k[[X]][Y])_S \,. \end{aligned}$$

Consequently $R \otimes_k K$ is Noetherian [10, Theorem 4, p. 138] and [9, Theorem 1, p. 201; Corollary 1 to Theorem 15, p. 224; and p. 199].

It can be shown that the ideal $(1 - X^i Y)$ in $k[[X]][Y]$ misses S. Let M_i be an ideal of $k[[X]][Y]$ which contains $(1 - X^i Y)$ and is maximal with respect to missing S. Now M_i is prime [8, (2.1), p. 4]. If $i \neq j$, then $M_i \not\supseteq M_j$, for if $M_i \supseteq M_j$, then

$$(1 - X^i Y) - (1 - X^j Y) = (X^j - X^i) Y \in M_i \,.$$

Since $M_i \cap S = \emptyset$, $Y \notin M_i$, and since M_i is prime,

$$\begin{aligned} &X^i(X^{j-i} - 1) \in M_i \quad \text{if } j > i \\ &X^j(1 - X^{i-j}) \in M_i \quad \text{if } j < i \end{aligned}$$

Regardless of whether $j > i$ or $j < i$, X is in M_i, for in each case $X^j - X^i$ can be factored as a product of some power of X and a unit of $k[[X]]$ [10, VII, § 1, Theorem 2, p. 131]. Since M_i is a proper prime ideal, X and consequently $X^i Y \in M_i$. It follows that $1 = X^i Y + (1 - X^i Y)$ is in M_i, which contradicts the fact that $M_i \cap S = \emptyset$.

Consequently, the M_i's which form an infinite collection of prime ideals of $k[[X]][Y]$ which are disjoint from S, extend to an infinite collection of prime ideals of $(k[[X]][Y[)_S$ [9, IV, § 10, Corollary to Theorem 16, p. 224]. In fact, the M_i's extend to an infinite collection of maximal ideals of $(k[[X]][Y])_S$ since each M_i is maximal with respect to missing S.

EXAMPLE 2. Let $\overline{\mathbf{Q}}$ be the field of all complex numbers algebraic over the rational number field $\mathbf{Q}$. We will show that $\overline{\mathbf{Q}} \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}$ has infinitely many maximal ideals and thus is not a semi-local ring.

If n is rational, then $(n^{\frac{1}{2}} \otimes n^{-\frac{1}{2}})^2 = n \otimes n^{-1} = 1 \otimes 1$. But if $a^2 = 1$, then $\frac{1}{2}(1 - a)$ and $\frac{1}{2}(1 + a)$ are idempotents. In particular,

$$\{\tfrac{1}{2}[n^{-1}(n^{\frac{1}{2}} \otimes n^{\frac{1}{2}}) + (1 \otimes 1)] : n \text{ is a prime integer}\}$$

is an infinite collection of distinct idempotents of $\overline{\mathbf{Q}} \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}$.

If R is a ring with identity 1 and e is an idempotent of R different from 0 and 1, then e and $1-e$ are orthogonal idempotents of R and $R=Re\oplus R(1-e)$. Using this fact and mathematical induction, one can show that if R is a commutative ring with identity and infinitely many idempotents, then, for each positive integer n, the ring R has a direct sum decomposition of the form $R=Re_1\oplus\ldots\oplus Re_n$, where the e_i's are pairwise orthogonal idempotents of R. Each Re_i is a commutative ring with identity and hence has a maximal ideal M_i [9, p. 151]. Therefore, R has n maximal ideals

$$Re_1\oplus\ldots\oplus Re_{i-1}\oplus M_i\oplus Re_{i+1}\oplus\ldots\oplus Re_n\ .$$

Since this is true for every positive integer n, R has infinitely many maximal ideals. Taking R to be $\overline{\mathrm{Q}}\otimes_{\mathrm{Q}}\overline{\mathrm{Q}}$, we see that $\overline{\mathrm{Q}}\otimes_{\mathrm{Q}}\overline{\mathrm{Q}}$ has infinitely many maximal ideals and hence $\overline{\mathrm{Q}}\otimes_{\mathrm{Q}}\overline{\mathrm{Q}}$ is not a semi-local ring.

THEOREM 4. *Let R be a complete local ring with maximal ideal M. Suppose that R and R/M have the same characteristic and that M has a minimal basis of n elements. If K is an algebraic extension of $k\cong R/M$, then $R\otimes_k K$ is Noetherian.*

PROOF. By Cohen's theorem on the structure of complete local rings [4, Theorem 9, p. 72], $R\cong k[[X_1,\ldots,X_n]]/I$, where I is an ideal of $k[[X_1,\ldots,X_n]]$. Hence,

$$\begin{aligned} R\otimes_k K &\cong (k[[X_1,\ldots,X_n]]/I)\otimes_k K \\ &\cong (k[[X_1,\ldots,X_n]]\otimes_k K)/(I,0)(k[[X_1,\ldots,X_n]]\otimes_k K)\ . \end{aligned}$$

Since the homomorphic image of a Noetherian ring is Noetherian, to show that $R\otimes_k K$ is Noetherian, it suffices to show that

$$k[[X_1,\ldots,X_n]]\otimes_k K$$

is. Since K is an algebraic extension field of k, K is the union of the set of finite algebraic extension fields K_α of k contained in K. Since direct limits and tensor products commute [3, Ch. V, § 9, Proposition 9.2*, p. 99],

$$\begin{aligned} k[[X_1,\ldots,X_n]]\otimes_k K &\cong k[[X_1,\ldots,X_n]]\otimes_k(\operatorname{inj}\lim K_\alpha) \\ &\cong \operatorname{inj}\lim(k[[X_1,\ldots,X_n]]\otimes_k K_\alpha) \\ &\cong \bigcup K_\alpha[[X_1,\ldots,X_n]]\ . \end{aligned}$$

Since K_α is a subfield of K, K_α satisfies property (C) with respect to K. Hence, $\bigcup K_\alpha[[X_1,\ldots,X_n]]$ and $R\otimes_k K$ are Noetherian.

EXAMPLE 3. The following example is an exercise in Bourbaki [2, Ch. V, § 1, Exercise 21, p. 74].

Let k_0 be a perfect field of characteristic $p \neq 0$. Let $\{X_n\}_{n\in N}$ be a collection of indeterminates over k_0, and let K be the quotient field of $k_0[\{X_n\}_{n\in N}]$. Let Y and Z be indeterminates over K. Let $k = K^p \subseteqq K$. Then K is an algebraic extension of k. Let $R = k[[Y, Z]]$. Theorems 2 and 4 apply; hence $R \otimes_k K$ is Noetherian.

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STETSON UNIVERSITY, DE LAND, FLORIDA, U.S.A.