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ON A PROBLEM OF BOONE

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In his earlier paper, *Some undecidable problems in group theory* [7], the author discussed the question of when the decision problem $(?\boldsymbol{x})\varphi(\boldsymbol{x})$, $\varphi(\boldsymbol{x})$ any formula of first order group theory all of whose free variables are among the finite sequence $\boldsymbol{x}$, can be undecidable in some finitely presented group. The results of that paper were of the following general type: If $\varphi(\boldsymbol{x})$ meets certain hypotheses, then there exist finite group presentations $\pi_0(\varphi)$ and $\pi_1(\varphi)$ where π_1 presents a group with insolvable word problem, such that

$$[(?\boldsymbol{x})\varphi(\boldsymbol{x}) \text{ in } G_{\pi_0}]\ {}_T\!\geq [(?x)[x=1] \text{ in } G_{\pi_1}]\,.$$

Boone has asked whether one could show that the problems $(?\boldsymbol{x})\varphi(\boldsymbol{x})$ have recursively enumerable (r.e.) degree of unsolvability in all finitely presented groups for all choices of the first order formula $\varphi(\boldsymbol{x})$.

There is a substantial amount of evidence for a positive answer to Boone's question. First, all of the undecidabilities constructed so far have had r.e. degree. Secondly, many of the constructions used in the embedding theorems for groups have the property that they reduce complex problems to much simpler problems such as the word problem. For example, if one could construct a countably generated group G in which $(?x)(\forall y)[xy^2=y^2x]$ has a high degree of unsolvability (say ${}_T\!> \boldsymbol{O}'$), an attempt to embed G into a finitely generated group H, either by the Higman–Neumann–Neumann embedding [2] or the Neumann–Neumann wreath product construction [5], would result in embedding G into a group in which the problem in question is equivalent to the word problem. The final bit of evidence for the positive side is the Higman embedding theorem which, in effect, says that the properties of being recursively enumerable and of being finitely presented are very closely intertwined.

This paper is devoted to showing that the answer to Boone's problem is "No". Specifically, we shall construct a finitely presented group G and a first order decision problem whose degree in G is exactly $\boldsymbol{O}''$.

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1.

Let L be the first order language of group theory with individual variables $x_1, y_1, z_1, x_2, \ldots$, an individual constant 1, operation symbols $\cdot$ and $^{-1}$, the predicate $=$, and the logical symbols &, v, $\sim$, $\rightarrow$, $\forall$, and $\exists$. Given a group G, the language L^G is obtained from L by adding a new constant 'g' (name of g) for each element g of G. (We will use the same symbols to denote elements of groups and their names in the corresponding languages.)

A *group presentation* π consists of a set S of distinct letters and a set D of freely reduced words on the letters of S (and their inverses). The *group presented by* π, G_π, is the quotient of the free group F on S modulo the normal closure in F of D. If S is finite, D recursively enumerable, or both S and D are finite, we call G_π *finitely generated, recursively presented,* or *finitely presented,* respectively.

Given a group presentation π and a formula $\varphi(\boldsymbol{x})$ of L, all of whose free variables are among $\boldsymbol{x}$, the decision problem $(?\boldsymbol{x})\varphi(\boldsymbol{x})$ for G_π is the problem of deciding for arbitrary tuples of words $\boldsymbol{u}$ on S whether or not the sentence $\varphi(\boldsymbol{u})$ holds in G_π. It is clear that for finitely presented groups G, the Turing degree of unsolvability of this problem is independent of which finite presentation you choose for G. (See [7] for details.)

A useful construction in the theory of groups is the Higman–Neumann–Neumann (henceforth HNN) construction: Let G be a group, let A and B be subgroups of G, and let δ be an isomorphism of A onto B. Then G is embedded in the group H obtained by adding a new letter t to the generators of G and adding the relation $t^{-1}at=\delta(a)$, for each a in A. The letter t is called a *stable letter* for this extension of G. A "normal form" for elements of H is given by the following lemma.

LEMMA (Britton). *Let W be a word on the generators of H. If $W=1$ in H, then either W does not involve t or else W has a subword of one of the following two forms*:

(i) $t^{-1}W't$, *where W' is t-free and an element of A*
(ii) $tW't^{-1}$, *where W' is t-free and an element of B.*

A word W on the generators of H having no subwords of form (i) or (ii) above will be called *t-reduced.*

In addition, we will require the following lemma due to Graham Higman.

LEMMA. *Let G be a finitely generated free group and let K be a recursively enumerable set of elements of G. Then the* HNN *extension H of G obtained by adding a stable letter t and the relations $t^{-1}kt=k$ for each k in K is embeddable in a finitely presented group.*

For the remainder of this paper, the letters i, j, k, m, n will be used as variables ranging over the integers or the natural numbers. W_i will denote the ith ($i \geqq 0$) recursively enumerable set of natural numbers in some (fixed) enumeration. Let π_0 be the group presentation

$$\begin{gathered}\langle a,b,c_1,c_2,\ldots ;\\ c_i^{-1}\alpha_j^n\alpha_{j+1}^n\alpha_{j+2}^n\alpha_{j+3}^n c_i \;=\; \alpha_j^n\alpha_{j+1}^n\alpha_{j+2}^n\alpha_{j+3}^n\\ \text{all } n \text{ in } \mathsf{Z},\ |j| \text{ in } W_i\rangle\ ,\end{gathered}$$

where α_k is an abbreviation for $a^{-k}ba^k$. Let $\psi(x_1,x_2,x_3)$ be the formula of L

$$(\exists y_1)(\exists y_2)[y_1x_2=x_2y_1 \;\&\; y_2x_3=x_3y_2 \;\&\; x_1=x_2^{-1}x_3x_2]\ .$$

LEMMA 1. *In G_{π_0} the decision problem*

$$(\,?x_1)(\,?x_2)(\,?x_3)(\forall y)[\exists z)[\psi(z,x_2,x_3) \;\&\; y=zx_2^{-1}zx_2^{-1}zx_2^{-1}zx_2^{3}] \rightarrow x_1y=yx_1]$$

has degree $\boldsymbol{O}''$.

PROOF. Let P denote the above problem. G_{π_0} has recursively enumerable word problem. Thus each atomic formula of L is a recursively enumerable predicate of elements of G_{π_0}. Therefore, by the Tarski–Kuratowski algorithm, $P \leqq_T \boldsymbol{O}''$ for G_{π_0}.

The converse reducibility will be proved in five steps. Let

$$\beta_{n,j} \;=\; \alpha_j^n\alpha_{j+1}^n\alpha_{j+2}^n\alpha_{j+3}^n\ .$$

(i) The subgroup C generated by the $\beta_{n,j}$, for $n \neq 0$ is freely generated by the $\beta_{n,j}$. Observe that each $\beta_{n,j}$ has length $4n$ in the free group

$$K \;=\; \langle \alpha_j,\ j=\ldots,-1,0,1,2,\ldots\rangle$$

and that no more than $\inf\{|m|,|n|\}$ α-factors are cancelled from either factor in the product $\beta_{n,j}\beta_{m,k}^{\varepsilon}$ ($\varepsilon=\pm 1$) unless $k=j$, $\varepsilon=-1$ and $m=n$. Thus the set of elements $\beta_{n,j}$, $n \neq 0$ is Nielsen reduced and freely generates C.

(ii) An immediate consequence is that G_{π_0} is an HNN extension of $\langle a,b\rangle$ with stable letters $c_1,c_2,\ldots$.

(iii) If $ub = bu$ for some u in G_{π_0}, then $u = b^\gamma$ for some integer γ. If u is c-free, the result is clear. Suppose that u involves c's. We may assume that u is c-reduced. Since $u^{-1}b^{-1}ub = 1$ in G_{π_0}, $u^{-1}b^{-1}u$ must contain a subword of the form $c^{-\varepsilon}dc^{\varepsilon}$, $\varepsilon = \pm 1$ and d an element of C, by Britton's lemma. Since u is c-reduced, we may assume that u is of the form $Ac^{\varepsilon}u'$ where A is c-free and $A^{-1}b^{-1}A$ is the required element d of C. But $A^{-1}b^{-1}A$ has exponent sum -1 on b and all elements of C have even exponent sum on b; this contradiction forces us to conclude that $ub \neq bu$.

(iv) If $ua = au$ for some u in G_{π_0}, then for some integer γ, $u = a^\gamma$. The proof is similar to that for (iii); no element of C can be of the form $A^{-1}a^{-1}A$, because all elements of C have exponent sum zero on a.

(v) Any element d of G_{π_0} satisfying

$$(\exists z)\psi(z,a,b) \;\&\; d = zb^{-1}zb^{-1}zb^{-1}zb^{-3}$$

must be of the form $\beta_{n,j}$ for some n and j. Thus

$$\begin{aligned}&\{i \mid \langle c_i, a, b\rangle \text{ satisfies} \\ &\qquad (\forall y)(\exists z)[\psi(z,x_2,x_3) \;\&\; y = zx_2^{-1}zx_2^{-1}zx_2^{-1}zx_2^{3} \rightarrow x_1 y = y x_1]\} \\ &= \{i \mid W_i = \mathsf{N}\}\end{aligned}$$

which has degree $\boldsymbol{O}''$. Thus problem P has degree $\boldsymbol{O}''$ in G_{π_0}.

Let s and t be new letters, and let $\sigma_i = s^{-i}ts^i$. Thus the set $\{\sigma_i \mid i \text{ in } \mathsf{N}\}$ freely generates a free subgroup of $\langle s,t\rangle$. Let π_1 present the amalgamated free product of G_{π_0} and $\langle s,t\rangle$ where the amalgamated subgroup is given by $c_i = \sigma_i$.

$$\begin{aligned}\pi_1 &= \langle a,b,s,t,c_1,c_2,\ldots;\ c_i^{-1}\beta_{n,j}c_i = \beta_{n,j},\ c_i = \sigma_i,\ i \text{ in } \mathsf{N}, \\ &\qquad |j| \text{ in } W_i,\ n \text{ in } \mathsf{Z}\rangle \\ &= \langle a,b,s,t;\ \sigma_i^{-1}\beta_{n,j}\sigma_i = \beta_{n,j},\ i \text{ in } \mathsf{N},\ |j| \text{ in } W_i,\ n \text{ in } \mathsf{Z}\rangle .\end{aligned}$$

G_{π_1} is finitely generated and recursively presented, and G_{π_0} is embedded in G_{π_1}.

LEMMA 2. *Let u be an element of G_{π_1}. If $ua = au$ or $ub = bu$, then u is in G_{π_0}. In particular $u = a^\gamma$ or $u = b^\gamma$ for some integer γ.*

PROOF. Neither a nor b is in the amalgamated subgroup. Consequently, if u has free product length more than 1, $ua \neq au$ and $ub \neq bu$ because the words $u^{-1}a^{-1}ua$ and $u^{-1}b^{-1}ub$ have length more than 1. The second conclusion follows from Lemma 1 (iii) and (iv).

COROLLARY 3. *The problem P above has degree $\boldsymbol{O}''$ in G_{π_1}.*

Let D be the normal subgroup of $\langle a,b,s,t\rangle$ generated by the elements $\gamma_{i,n,j}=\sigma_i^{-1}\beta_{n,j}\sigma_i\beta_{n,j}^{-1}$ where i is in N, n is in Z and $|j|$ is in W_i. D is recursively enumerable. Therefore, by Higman's lemma, there exists a finitely presented group H in which the group

$$\langle a,b,s,t,u;\ u^{-1}\gamma_{i,n,j}u=\gamma_{i,n,j},\ \gamma\text{'s in } D\rangle$$

is embedded.

Let $\langle a_1,\dots,a_\mu;\ r_1,\dots r_\nu\rangle=\pi_2$ be a presentation for H. We will use the letters a, b, s, t, and u to denote the images of these elements in H.

Let $\bar{a}$, $\bar{b}$, $\bar{s}$, $\bar{t}$, and $\bar{p}$ be new letters. Let $\bar{\gamma}_{i,n,j}$ be obtained from $\gamma_{i,n,j}$ by replacing each letter in the latter by the corresponding letter with a bar. We will now carry out a modified version of the Higman embedding of G_{π_1} into a finitely presented group. (See the appendix to [8] for a detailed account of the construction.)

Let

$$\pi_3 = \langle a_1,\dots,a_\mu,\bar{a},\bar{b},\bar{s},\bar{t},\bar{p};\ r_1,\dots,r_\nu,\bar{\gamma}_{i,n,j},\ i \text{ in } \mathsf{N},\ n \text{ in } \mathsf{Z},\ |j| \text{ in } W_i,\ d^{-1}e^{-1}de \text{ for } d \text{ one of } a_1,\dots,a_\mu, \text{ and } e \text{ one of } \bar{a},\bar{b},\bar{s},\bar{t},\bar{p}\rangle\,.$$

Thus π_3 presents the direct product $H\times(\bar{G}_{\pi_1}*\langle\bar{p}\rangle)$. The subgroups A and B of G_{π_3} generated respectively by $\{a,b,s,t,u^{-1}au,u^{-1}bu,u^{-1}su,u^{-1}tu\}$ and $\{a\bar{a},b\bar{b},c\bar{c},d\bar{d},u^{-1}au,u^{-1}bu,u^{-1}su,u^{-1}tu\}$ are isomorphic. Let π_4 be the presentation

$$\langle a_1,\dots a_\mu,\bar{a},\bar{b},\bar{s},\bar{t},\bar{p},v;\ \text{relations of } \pi_3,\ v^{-1}av=a\bar{a},\dots,v^{-1}u^{-1}tuv=u^{-1}tu\rangle\,.$$

Since the relations $\bar{\gamma}_{i,n,j}$ of π_4 follow from the other relations, we may omit them, and obtain a presentation π_5 presenting G_{π_5} isomorphic to G_{π_4}. π_5 is a finite presentation and G_{π_1} is isomorphic to the subgroup of G_{π_5} generated by $\bar{a}$, $\bar{b}$, $\bar{s}$, $\bar{t}$.

Lemma 4. *If w is an element of G_{π_5} and $\bar{a}w=w\bar{a}$ or $\bar{b}w=w\bar{b}$, then either $w\bar{p}=\bar{p}w$ or w is an element of $\bar{G}_{\pi_1}$. In the latter case, w must be a power of $\bar{a}$ or of $\bar{b}$.*

Proof. We argue the case for $\bar{b}$ only, as the case for $\bar{a}$ is similar. If w is v-free the conclusion is obtained at once. If w involves v, we may assume that w is v-reduced and does not begin with any of the letters $a_1,\dots,a_\mu$. Since $w^{-1}\bar{b}^{-1}w\bar{b}=1$ in G_{π_5}, this word must contain a subword of the form $v^{-\varepsilon}dv^{\varepsilon}$ where d is in the appropriate Britton subgroup A or B. Suppose that w is of the form $Mv^{\varepsilon}w'$, where M is v-free and $M^{-1}\bar{b}M$

is the d in question. We may assume that M does not involve $\bar{p}$ or a generator of H. Therefore M is a word on the letters of $\bar{G}_{\pi_1}$. But neither A or B contains an element of the form $M^{-1}\bar{b}M$, for such a word M because $B \cap \bar{G}_{\pi_1} = \{1\}$. Therefore $w\bar{b} \neq \bar{b}w$.

Let $\psi^*(x_1, x_2, x_3, x_4)$ be the following formula of L:

$$(\forall y)[\exists z)(\exists y_1)(\exists y_2)[y_1 x_2 = x_2 y_1 \;\&\; y_2 x_3 = x_3 y_2 \;\&\; y_1 x_4 \neq x_4 y_1 \;\&\; y_2 x_4 \neq x_4 y_2 \;\&\; z = y_1^{-1} y_2 y_1 \;\&\; y = z x_2^{-1} z x_2^{-1} z x_2^{-1} z x_2^{3}] \to x_1 y = y x_1] .$$

THEOREM 5. *The decision problem* $(?x_1)(?x_2)(?x_3)(?x_4)\psi^*(x_1, x_2, x_3, x_4)$ *has degree* $\boldsymbol{O}''$ *in* G_{π_5}.

PROOF. ψ^* is a universal formula of L (to be precise the prenex normal form of ψ^* is universal). The matrix of ψ^* involves both atomic formulas of L and their negations. Since atomic formulas are recursively enumerable predicates in G_{π_5}, the decision problem which we have constructed can have degree no higher than $\boldsymbol{O}''$ by the Tarski–Kuratowski algorithm.

To see the converse reducibility, note that

$$\{i \mid \langle \sigma_i, \bar{a}, \bar{b}, \bar{p} \rangle \text{ satisfies } \psi^*\} = \{i \mid W_i = \mathsf{N}\} ,$$

a set which is maximal in $\boldsymbol{O}''$.

2.

The following conjecture is suggested by theorem 5:

Let $\boldsymbol{D}$ be an arithmetical degree of unsolvability. Then there exists a first order decision problem P and a finitely presented group G such that P has degree $\boldsymbol{D}$ in G.

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