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NON-NOETHERIAN RINGS FOR WHICH EACH PROPER SUBRING IS NOETHERIAN

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Let S be a ring and let $\mathscr{F}$ and $\mathscr{S}$ be two families of subrings of S. An important question in the theory of rings is to determine conditions under which each ring in the family $\mathscr{S}$ has a given property P_2 if each ring in the family $\mathscr{F}$ has property P_1. We consider here two special cases of this problem. In order to describe these cases succinctly, we introduce two definitions.

A ring R has *property* (C1) if R does not satisfy the ascending chain condition (a.c.c.) on two-sided ideals, but each proper subring of R satisfies the a.c.c. on two-sided ideals; R has *property* (C2) if R does not satisfy the a.c.c. on left ideals, but each proper left ideal of R satisfies the a.c.c. on left ideals.

The purpose of this paper is to determine all rings with property (C1) or (C2). We prove Theorem 3.2:

In a ring R, the following conditions are equivalent.

(a) *R has property* (C1).

(b) *R has property* (C2).

(c) *R is the zero ring on a p-quasicyclic group.*

The type of problem considered here—that is, that of characterizing rings for which each proper subring (or ideal) satisfies a given ring property P—is the same as that considered in [4]. It is interesting to note that the zero ring on a quasicyclic group was of prime importance in [4], also.

1. Preliminaries.

We use the word *ideal* to mean *two-sided ideal* throughout the paper. If $\{x_\alpha\}$ is a subset of a ring R, then $(\{x_\alpha\})$ will denote the ideal of R generated by $\{x_\alpha\}$. We make frequent use of the fact that the a.c.c. on ideals (left ideals) of R is equivalent to the condition that each ideal (left

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ideal) of R is finitely generated. In particular, if R is a ring with property (C1) (or (C2)), then R is the only ideal (or left ideal) of R that is not finitely generated. We use $\subseteq$ for containment, and $\subset$ for proper containment; Z denotes the set of integers, and ω is the set of positive integers. If $xy=0$ for all x,y in the ring R, we say that R is the *zero* ring on R^+, the additive group of R; we will also say in this case that R has the *trivial multiplication*. Our proof of Theorem 3.2 proceeds in essentially two steps: In Section 2, we prove that the quasicyclic groups are the only abelian groups G for which G is not finitely generated, but every proper subgroup of G is finitely generated. Then in Section 3, we prove that a ring satisfying property (C1) or (C2) has the trivial multiplication.

Before proceeding to Section 2, we give a brief description of a quasicyclic group. (For details, see [1, p. 15], [5, p. 4], or [6, p. 19].) Let p be a prime integer. The p-quasicyclic group, which we denote by $C(p^\infty)$, is an abelian group generated by a set $\{c_i\}_{i\in\omega}$ such that c_i has order p^i, and $pc_{i+1}=c_i$ for each $i\in\omega$; there is, to within isomorphism, exactly one group with these properties. (The group of all complex pth power roots of unity, under multiplication, is a realization of $C(p^\infty)$.) The nonzero proper subgroups of $C(p^\infty)$ are precisely the finite cyclic groups generated by the c_i's. Thus, with the trivial multiplication, the proper subrings (and proper ideals) of $C(p^\infty)$ are just the proper subgroups of $C(p^\infty)$. It then follows that the zero ring on $C(p^\infty)$ satisfies (C1) and (C2).

2. The group case.

In this section, we show that if G is a non-finitely generated abelian group for which each proper subgroup is finitely generated, then G is a quasicyclic group.

LEMMA 2.1. *Let G be an abelian group. Suppose that G is not finitely generated, but each proper subgroup of G is finitely generated.*

(i) *If H is a proper subgroup of G, then G/H is not finitely generated, but each proper subgroup of G/H is finitely generated.*

(ii) *G is not the sum of two of its proper subgroup; in particular, G is indecomposable.*

PROOF. (i) Clear.

(ii) If A and B are subgroups of G such that $G=A+B$, then A or B is not finitely generated, since G is not finitely generated. Hence $G=A$ or $G=B$.

THEOREM 2.2. *Let G be an abelian group satisfying the hypothesis of Lemma* 2.1. *Then $G \cong C(p^\infty)$ for some prime p.*

PROOF. We observe first that G is divisible, for if not, $pG \subset G$ for some prime p, and hence G/pG, as a vector space over $Z/(p)$, is not finitely generated, but each proper subspace of G/pG is finitely generated. This is impossible, and hence G is divisible. From the structure theorem for divisible groups [1; Theorem 19.1], it follows that G is the direct sum of quasicyclic groups and full rational groups. Since G is indecomposable by Lemma 2.1, G is either a quasicyclic group or a full rational group. But a full rational group does not have the property that each of its proper subgroups is finitely generated, and thus $G = C(p^\infty)$ for some prime p.

3. Properties (C1) and (C2).

We first show that any ring satisfying property (C1) or property (C2) has the trivial multiplication.

THEOREM 3.1. *Let R be a ring that satisfies property* (C1) *or property* (C2). *Then R has the trivial multiplication.*

PROOF. We show that if R satisfies (C1), then R has the trivial multiplication. The proof for the case when R satisfies (C2) is essentially the same and we omit it.

Observe that if $\{A_i\}_{i=1}^{\infty}$ is any infinite strictly ascending chain of ideals of R, and if $A = \bigcup_{i=1}^{\infty} A_i$, then A is an ideal of R and $\{A_i\}_{i=1}^{\infty}$ is an infinite strictly ascending chain of ideals of A. Hence $A = R$ by the hypothesis on R.

Let $x \in R$; we show that $Rx = 0$. It will then follow that R has the trivial multiplication. Since $x \in R = \bigcup_{i=1}^{\infty} A_i$, we get $x \in A_m$ for some $m \in \omega$, and hence $Rx \subseteq RA_m \subseteq A_m \subset R$. Therefore, $Rx = 0$ or Rx is a proper subring of R. If $Rx = 0$, we are done; if Rx is a proper subring of R, then, by the hypothesis on R, it follows that Rx is finitely generated as an ideal of Rx. Let $\{r_i x\}_{i=1}^{n}$ be a set of generators for Rx, considered as an ideal of Rx.

If $r \in R$, then

$$rx = \sum_{i=1}^{n} s_i r_i x + \sum_{i=1}^{n} r_i x t_i + \sum_{i=1}^{n} u_i r_i x v_i + \sum_{i=1}^{n} \lambda_i r_i x ,$$

where $s_i, t_i, u_i, v_i \in Rx$, $\lambda_i \in Z$.

Letting $t_i = t_i' x$ and $v_i = v_i' x$ for each i, we have that

$$rx = \sum_{i=1}^{n} s_i r_i x + \sum_{i=1}^{n} r_i x t_i' x + \sum_{i=1}^{n} u_i r_i x v_i' x + \sum_{i=1}^{n} \lambda_i r_i x ,$$

and therefore,

$$[r - \sum_{i=1}^{n} (s_i r_i + r_i x t_i' + u_i r_i x v_i' + \lambda_i r_i)]x = ux = 0 .$$

Therefore, u belongs to the left annihilator L of x, and $r \in (\{r_i\}_{i=1}^{n}) + L$. Therefore, since r was an arbitrary element of R, it follows that $R = (\{r_i\}_{i=1}^{n}) + L$. But L is a left ideal, hence a subring of R, and therefore cannot be finitely generated as an ideal of L, since R is not finitely generated as an ideal of R. Thus $R = L$ so that $Rx = Lx = 0$.

We can now easily prove our main result.

THEOREM 3.2. *Let R be a ring. The following conditions are equivalent.*
(i) *R satisfies property* (C1).
(ii) *R satisfies property* (C2).
(iii) *R is the zero ring on a quasicyclic group.*

PROOF. We have already observed that (iii) → (i) and (iii) → (ii). If R satisfies (C1) or (C2), then R has the trivial multiplication, by Theorem 3.1. Consequently, the subrings (and ideals and left ideals) of R are precisely the subgroups of R^+. Therefore, R^+ is an abelian group satisfying the hypothesis of Theorem 2.2, so that $R^+ \cong C(p^\infty)$ for some prime p, and R is the zero ring on a quasicyclic group.

In conclusion, we state two questions that arise in connection with the results of this paper.

1. *If G is a group which is not finitely generated, but for which each proper subgroup is finitely generated, must G be abelian, and hence quasicyclic?*
2. *Suppose that R is a ring in which the* a.c.c. *for ideals does not hold, and is such that the* a.c.c. *for ideals holds in each proper ideal of R. Is multiplication in R trivial?*

Although we have made some progress toward a solution to each of these questions, we know the answer to neither.

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