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ANOTHER PROOF FOR A COMBINATORIAL LEMMA IN FLUCTUATION THEORY

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An important problem in fluctuation theory is that of showing that in a random path the number of steps on the positive half-line has the same distribution as the index where the maximum is attained for the first time. This theorem is mentioned by Spitzer [3]. There he refers to a more general proof by H. F. Bohnenblust. Baxter [1] describes a rule due to Richards which was used to prove this theorem by finding an inverse rule. Sparre Andersen gives a proof in this fashion, and Brandt generalizes it (Hobby and Pyke [2].)

The proof that I describe below uses the rule due to Richards, and proves Brandt's generalization by a direct method. The method consists merely of inductive steps based on an inductive definition of the above rule.

1. Notation and definitions.

Let $x_1, x_2, \ldots x_n$ be n real numbers, and let P be the set of the $n!$ permutations of $(1,2,\ldots,n)$. For $\sigma : i_1, i_2, \ldots, i_n$, an element of P, we define the following quantities:

$$s_0(\sigma) = 0, \quad s_k(\sigma) = x_{i_1} + \ldots + x_{i_k} \ (1 \leqq k \leqq n)\,, \tag{1.1}$$

$$R_n(\sigma) = \max\{s_k(\sigma) \mid 0 \leqq k \leqq n\}\,, \tag{1.2}$$

$$N_n(\sigma;\gamma) = \operatorname{card}\{1 \leqq k \leqq n \mid s_k(\sigma) > \gamma\}, \quad \gamma \in R\,, \tag{1.3}$$

$$\begin{aligned} L_n(\sigma;\gamma) &= \min\{0 \leqq k \leqq n \mid s_k(\sigma) \geqq R_n(\sigma) - \gamma\}, \quad \gamma \geqq 0\,, \\ &= \max\{0 \leqq k \leqq n \mid s_k(\sigma) > R_n(\sigma) + \gamma\}, \quad \gamma < 0\,. \end{aligned} \tag{1.4}$$

THEOREM 1.1. *$\{N_n(\sigma;\gamma) \mid \sigma \in P\} \equiv \{L_n(\sigma;\gamma) \mid \sigma \in P\}$, where by $\equiv$ we understand the following two properties:*

i) *If $N_n(\sigma;\gamma) = k$, then there is a permutation τ such that $L_n(\tau;\gamma) = k$; the converse is also true.*

ii) $\operatorname{card}\{\sigma \in P \mid N_n(\sigma;\gamma) = k\} = \operatorname{card}\{\sigma \in P \mid L_n(\sigma;\gamma) = k\}$.

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2. A bijection.

In this section I give an inductive definition of the rule due to Richards, and prove that it is indeed a bijection.

DEFINITION 2.1. For any subset C of the real line, define

$$\Gamma_C\colon P \to P$$

by the following rule. For $\sigma\colon i_1, i_2, \ldots i_n$, write

$$A_C(\sigma) = \{1 \leqq k \leqq n \mid s_k(\sigma) \in C\}, \quad A'_C(\sigma) = \{1,2,\ldots n\} - A_C(\sigma).$$

Let $\operatorname{card} A_C(\sigma) = m$ $(0 \leqq m \leqq n)$; then

$$\Gamma_C(\sigma) = \sigma'\colon i'_1, i'_2, \ldots, i'_n, \quad i'_\nu = i_{k(\nu)} \quad (1 \leqq \nu \leqq n),$$

with

$$\begin{aligned} k(1) &= \max\{k \mid k \in A_C(\sigma)\}, \\ k(\nu) &= \max\{k \mid k \in A_C(\sigma),\ k \neq k(1), \ldots, k \neq k(\nu-1)\} \\ &\qquad\qquad\qquad\qquad\qquad\qquad (2 \leqq \nu \leqq m), \\ k(m+1) &= \min\{k \mid k \in A'_C(\sigma)\}, \\ k(m+\nu) &= \min\{k \mid k \in A'_C(\sigma),\ k \neq k(m+1), \ldots, k \neq k(m+\nu-1)\} \\ &\qquad\qquad\qquad\qquad\qquad\qquad (2 \leqq \nu \leqq n-m). \end{aligned}$$

LEMMA 2.1. *For any subset C of the real line, Γ_C is bijective.*

PROOF. Since P is finite it is sufficient to prove that Γ_C is injective. Let $\sigma\colon i_1, i_2, \ldots, i_n$, and $\tau\colon j_1, j_2, \ldots j_n$, with

$$\Gamma_C(\sigma) = \sigma'\colon i'_1, i'_2, \ldots, i'_n, \quad \Gamma_C(\tau) = \tau'\colon j'_1, j'_2, \ldots, j'_n.$$

Assume $\sigma' = \tau'$, that is $i'_\nu = j'_\nu$ for $1 \leqq \nu \leqq n$; then

$$s_n(\sigma) = \textstyle\sum_{\nu=1}^n x_{i_\nu} = \sum_{\nu=1}^n x_{j_\nu} = s_n(\tau).$$

If $s_n(\sigma) = s_n(\tau) \in C$, then $i_n = i'_1 = j'_1 = j_n$; if $s_n(\sigma) = s_n(\tau) \notin C$, then $i_n = i'_n = j'_n = j_n$.

Assume now that $s_\nu(\sigma) = s_\nu(\tau)$, and $i_\nu = j_\nu$ for $k+1 \leqq \nu \leqq n$ $(0 \leqq k < n)$, and let

$$a = \operatorname{card}\{\nu \mid k+1 \leqq \nu \leqq n,\ s_\nu(\sigma) \in C\}.$$

Then

$$s_k(\sigma) = s_{k+1}(\sigma) - x_{i_{k+1}} = s_{k+1}(\tau) - x_{j_{k+1}} = s_k(\tau),$$

and therefore

$$\begin{aligned} i_k &= i'_{a+1} = j'_{a+1} = j_k \quad \text{if} \quad s_k(\sigma) \in C, \\ i_k &= i'_{k+a} = j'_{k+a} = j_k \quad \text{if} \quad s_k(\sigma) \notin C. \end{aligned}$$

3. Proof of theorem 1.1.

To complete the proof of theorem 1.1 it is sufficient to prove the following

LEMMA 3.1. *If* $C=(\gamma,\infty)$ *where* γ *is real, then*

$$N_n(\sigma;\gamma) = L_n(\Gamma_C(\sigma);\gamma).$$

PROOF. Let $\sigma\colon i_1,i_2,\ldots,i_n$, $\Gamma_C(\sigma)=\sigma'\colon i'_1,i'_2,\ldots,i'_n$ and

$$N_n(\sigma;\gamma) = m \quad (0\leqq m\leqq n).$$

I will prove that $L_n(\sigma';\gamma)=m$ if $\gamma\geqq 0$ (the proof for the case $\gamma<0$ is analogous). Let

$$R'_m(\sigma') = \max\{s_m(\sigma'),\ldots,s_n(\sigma')\}.$$

I will prove that

$$s_m(\sigma') \geqq R'_m(\sigma')-\gamma, \tag{3.1}$$

$$s_\nu(\sigma') < R'_m(\sigma')-\gamma \quad \text{for } 0\leqq\nu<m. \tag{3.2}$$

The inequalities (3.1) and (3.2) imply the theorem.

a) PROOF OF (3.1):

$$s_{m+1}(\sigma')-s_m(\sigma') = x_{i_{k(m+1)}} \leqq s_{k(m+1)-1}(\sigma)+x_{i_{k(m+1)}} = s_{k(m+1)}(\sigma) \leqq \gamma.$$

Assume

$$s_{m+\nu}(\sigma')-s_m(\sigma') \leqq \gamma \quad \text{for } 1\leqq\nu<l \quad (l\leqq n-m). \tag{3.3}$$

Consider the sequences

$$y^\nu = (k(m+\nu),\ldots,k(m+l)) \quad \text{for } 1\leqq\nu\leqq l,$$

and let $\bar\nu=\min\{1\leqq\nu\leqq l \mid y^\nu\in J\}$, where

$$J = \{y \mid \exists p\geqq 1,\ p\in \mathsf{Z},\ y=(y_1,\ldots y_p);\ y_{i+1}=y_i+1 \text{ for } 1\leqq i<p\}.$$

If $k(m+\bar\nu)=1$, then

$$s_{m+l}(\sigma')-s_m(\sigma') = s_{k(m+l)}(\sigma)\leqq\gamma.$$

If $k(m+\bar\nu)>1$, then

$$x_{i_{k(m+\bar\nu)}}+\ldots+x_{i_{k(m+l)}}+\gamma < s_{k(m+l)}(\sigma) \leqq \gamma;$$

but by (3.3)

$$x_{i_{k(m+1)}}+\ldots+x_{i_{k(m+\bar\nu-1)}} \leqq \gamma,$$

and hence $s_{m+l}(\sigma')-s_m(\sigma')\leqq\gamma$. Therefore $s_m(\sigma')\geqq R'_m(\sigma')-\gamma$.

b) Proof of (3.2): Let $0 \leqq \nu \leqq m-1$ and

$$\mu = \max\{0 \leqq j \leqq n-m \mid k(m+j) \leqq k(\nu+1)\}.$$

Then

$$s_\nu(\sigma') = s_{m+\mu}(\sigma') - s_{k(\nu+1)}(\sigma) < s_{m+u}(\sigma') - \gamma.$$

Hence $s_\nu(\sigma') < R'_m(\sigma') - \gamma$.

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