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WEAK COMPACTNESS AND TIGHTNESS OF SUBSETS OF $M(X)$

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1. Introduction.

Let X be a completely regular Hausdorff space, then $C(X)$ denotes the space of all real bounded continuous functions on X, $\mathscr{B}(X)$ denotes the Borel σ-algebra, that is the σ-algebra generated by the closed subsets of X, and $M(X)$ denotes the space of all bounded real measures on $(X,\mathscr{B}(X))$, which are *regular*, that is

$$|m|(A) = \sup\{|m|(K) \mid K \text{ compact}, K \subseteq A\} \quad \forall A \in \mathscr{B}(X)$$

whenever $m \in M(X)$, where $|m|$ denotes the total variation of m. $C^+(X)$ and $M^+(X)$ denote the positive parts of $C(X)$ respectively $M(X)$. The bilinear form

$$(f,m) = \int_X f\,dm$$

$f \in C(X)$ and $m \in M(X)$, makes $(C(X),M(X))$ a dual pair. The weak topology on $M(X)$ coming from this duality is denoted w^*, and the Mackey topology on $M(X)$ will be denoted τ^*. The weak topology on $C(X)$ arising from this duality will be denoted w.

We shall occasionally deal with *the uniform topology* on $M(X)$, which we shall denote u^*, and which is the topology generated by the norm

$$\|m\| = |m|(X)\,.$$

On $C(X)$ we shall occasionally deal with *the strict topology*, denoted β, which is the topology generated by the seminorms

$$q(f) = \sup\{a_n |f(x)| \mid x \in K_n,\ n \geqq 1\}$$

where $\{K_n\}$ runs through all sequences of compact subsets of X, and $\{a_n\}$ runs through all sequences of strictly positive numbers converging to zero (for properties of the strict topology see [8]).

We shall in this paper study the structure of the locally convex linear space $(M(X),w^*)$, in particular we shall study the w^*-compact subsets of $M(X)$ and of $M^+(X)$. If L is a subset of $M(X)$, then we shall say that L is *uniformly tight*, if L satisfies

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(1.1) $\forall \varepsilon > 0$, $\exists K$ compact, such that $|m|(X \setminus K) < \varepsilon \ \forall m \in L$.

It is well known that a u^*-bounded uniformly tight subset of $M(X)$ is relatively w^*-compact. Conversely if X is either locally compact or metrizable under a complete metric, then it is well known that every compact subset of $(M^+(X), w^*)$ is uniformly tight (see for example [19] p. 205). A completely regular Hausdorff space X, satisfying

(1.2) every compact subset of $(M^+(X), w^*)$ is uniformly tight,

is called a *Prohorov space*. Notice that the property is only demanded for subsets of $M^+(X)$ and not for subsets of $M(X)$. The statement above says that every locally compact space and every space, which is metrizable under a complete metric, is a Prohorov space. In [19] Varadarajan states that every metric space is a Prohorov space, but as pointed out by Topsøe in [18] Varadarajan's proof is only valid for locally compact metric spaces. R. O. Davies has given an example of a σ-compact subset of the unit square, which is not a Prohorov space (private communication).

We shall remind the reader of some topological notions. In all what follows X is supposed to be a completely regular Hausdorff space, and βX is the Stone–Čech compactification of X.

X is said to be *complete in the sense of Čech*, if X is a G_δ-set in βX. It is well known that every locally compact space is complete in the sense of Čech, and a metrizable space is complete in the sense of Čech, if and only if it is metrizable under a complete metric (see for example [6, pp. 142–146 and p. 190].)

If A is a subset of X, then we shall say that *X is complete at A*, if there exists a compact set $B \supseteq A$, such that X has a countable base at B (that is, there exist open sets $\{V_n\}$ such that $B \subseteq V_n$ for all $n \geq 1$, and if U is an open set containing B, then for some $n \geq 1$ we have $V_n \subseteq U$). We shall say that *X is σ-complete at A*, if there exists subsets A_n of X, such that X is complete at A_n for all $n \geq 1$, and $A \subseteq \bigcup_1^\infty A_n$.

X is said to be *complete at compact sets*, if X is complete at every compact subset of itself. These spaces have also been studied by Arhangel'skiĭ [1]–[2], Vaughan [20] and others under the name *spaces of countable type*. In [20] it has been shown that X is complete at compact sets if and only if X is a *generalized G_δ-set* in βX (that is, every compact subset of X is contained in a subset of X, which is a G_δ-set in βX). From this it follows, that if X is complete in the sense of Čech, then X is complete at every compact set. It is evident that every metrizable space is complete at compact sets.

X is said to be *complete at points*, if X is complete at $\{x\}$ for all $x \in X$. In [1] and [20] these spaces are studied under the name *spaces of point countable type*. In [20] it has been shown that X is complete at points if and only if X is a *point generalized G_δ-set in βX* (that is, every point in X is contained in subset of X, which is a G_δ-set in βX). If X is complete at compact sets, then X is complete at points, and evidently every space satisfying the first countability axiom is complete at every point. In [1] it has been shown that if X is complete at every point, then X is a k-space (a topological space is called a *k-space*, if a set is closed, whenever its intersection with every compact subset of X is closed; a topological space is called a *k^*-space*, if a real function f on X is continuous, whenever its restriction to every compact subset of X is continuous).

In section 3 and section 4 we shall see that there is a close connection between the Prohorov property (1.2) and the completeness properties defined above.

X is called *Radonian*, if every finite measure on $(X,\mathscr{B}(X))$ is regular. It is well known that every analytic space is Radonian (a topological space is *analytic*, if it is a continuous image of a complete separable metrizable space). For this result and further properties of Radonian spaces see for example [3, section 3.3 and exercises to section 3].

X is called *quasi-Radonian*, if every finite measure m, on $(X,\mathscr{B}(X))$ satisfies the following condition

$$(1.3) \qquad \forall \varepsilon > 0,\ \exists K \text{ compact, such that } |m|(X \setminus K) < \varepsilon .$$

Every Radonian and every σ-compact space is quasi-Radonian. A countable union of quasi-Radonian spaces and a countable intersection of quasi-Radonian spaces are again quasi-Radonian. A perfectly normal quasi-Radonian space is Radonian. All these statements are trivial, and we shall leave the verification to the reader.

X is called *semi-Radonian*, if every τ-smooth finite positive measure on $(X,\mathscr{B}(X))$ is regular. A positive measure m on $(X,\mathscr{B}(X))$ is called *τ-smooth* if

$$(1.4) \qquad m(F) = \inf_{i\in I} m(F_i) ,$$

whenever $\{F_i \mid i\in I\}$ is a family of closed sets filtering down to F. The reader can easily check the following statement: Every locally semi-Radonian space, every locally quasi-Radonian space, every Borel subset of a semi-Radonian space, every space, which is locally complete in the sense of Čech, every countable union of semi-Radonian spaces, every countable intersection of semi-Radonian spaces, and every countable product of semi-Radonian spaces are again semi-Radonian.

The perfect kernel of X is the largest subset having no isolated points. It is well known that the perfect kernel allways exists and is a closed subset of X. If the perfect kernel of X is empty, that is, X does not contain any nonempty subset without isolated points, then X is called *scattered*. Every subset of a scattered set is scattered. Every locally compact, countable Hausdorff space is scattered. (See for example [10, pp. 77–80].)

The perfect kernel of X may be constructed from outside by means of *the coherence sets*, which are defined by transfinite induction:

$$\begin{aligned} a_0(X) &= X\,, \\ a_1(X) &= \{x \in X \mid x \text{ is not isolated in } X\}\,, \\ a_{\gamma+1}(X) &= a_1(a_\gamma(X))\,, \\ a_\gamma(X) &= \textstyle\bigcap_{\pi<\gamma} a_\pi(X) \text{ if } \gamma \text{ is a limit ordinal.} \end{aligned}$$

It is well known that there exists a unique ordinal number $d(X)$, with the following properties

$$a_\gamma(X) \neq a_\pi(X) \quad \text{if } \gamma < \pi \leqq d(X)\,, \tag{1.5}$$

$$a_{d(X)}(X) = a_\gamma(X) \quad \text{if } \gamma \geqq d(X)\,, \tag{1.6}$$

$$a_{d(X)}(X) \text{ is the perfect kernel of } X\,, \tag{1.7}$$

$$a_\gamma(X) \subseteqq a_\pi(X) \quad \text{if } \gamma \geqq \pi\,. \tag{1.8}$$

We shall need the following easy fact:

(1.9) If U is a open subset of X, then $a_\gamma(U) = U \cap a_\gamma(X)$ for all $\gamma \geqq 0$.

2. The structure of $(M(X), w^*)$.

We shall in this section study the structure of $(M(X), w^*)$ as a locally convex linear space. We shall give conditions under which the Mackey topology is complete (see Theorem 1), and we shall show that under fairly mild conditions the closed balanced convex hull of a w^*-compact set is w^*-compact (see Theorem 2). In Theorem 3 we shall give conditions, which assure that the different notions of compactness in $(M(X), w^*)$ coincide.

PROPOSITION 1. *Let X be a completely regular Hausdorff space and let f be a bounded real Borel function on X. Let us consider the maps*

$$F_1(m) = \int_X f\,dm, \qquad F_2(m) = \int_X f\,dm^+\,,$$
$$F_3(m) = \int_X f\,dm^-, \qquad F_4(m) = \int_X f\,d|m|\,.$$

Then F_1, F_2, F_3, and F_4 are Borel maps on $(M(X),w^*)$. Furthermore if f is lower semi-continuous, non negative and bounded then F_2, F_3 and F_4 are lower semi-continuous (F_1 is not lower semi-continuous in general, and the statement is not true in general if f is not non negative).

PROOF. Let us first notice that

$$F_3(m) = F_2(-m), \quad F_1 = F_2 - F_3, \quad F_4 = F_2 + F_3 .$$

Hence it suffices to prove that F_2 is lower semi-continuous, whenever $f = 1_U$ for some open subset U of X (see for example [14, chap. I, Theorem 20]). This statement however will follow immediately from the following equality

$$(2.1) \quad m^+(U) = \sup\{\textstyle\int_X g\,dm \mid g \in C(X),\ 0 \leqq g \leqq 1_U\} \quad \forall m \in M(X) .$$

To prove this let $m \in M(X)$, and let $\varepsilon > 0$ be given. Let (X^+, X^-) be the Hahn decomposition of X with respect to m. Let us choose a compact set K and an open set V, such that $|m|(V \setminus K) < \varepsilon$ and $K \subseteq U \cap X^+ \subseteq V \subseteq U$. From the complete regularity of X we can find a function $g \in C(X)$, such that $1_K \leqq g \leqq 1_V$. Then we have

$$\begin{aligned}\textstyle\int_X g\,dm &= \textstyle\int_K g\,dm + \int_{V\setminus K} g\,dm \\ &\geqq \textstyle\int_K g\,dm^+ - |m|(V \setminus K) \\ &\geqq m^+(U) - 2|m|(V \setminus K) \geqq m^+(U) - 2\varepsilon .\end{aligned}$$

This shows that $m^+(U)$ is less than the right hand side of (2.1). Since the converse inequality is obvious we have proved Proposition 1.

LEMMA 1. *Let X be a normal, countably paracompact Hausdorff space, and let F be a linear functional on $C(X)$, satisfying*

(2.2) *if $\{f_n\}$ is a sequence in $C(X)$, which decreases pointwise to 0, then $\lim_{n\to\infty} F(f_n) = 0$.*

Then there exists a finite measure m on $(X, \mathscr{B}(X))$, such that

$$(2.3) \quad F(f) = \textstyle\int_X f\,dm \quad \forall f \in C(X) ,$$

$$(2.4) \quad |m|(A) = \sup\{|m|(D) \mid D \text{ is closed in } X,\ D \subseteq A\} \quad \forall A \in \mathscr{B}(X) .$$

PROOF. Let us define the positive and the negative part of F in the usual way

$$\begin{aligned} F^+(f) &= \sup\{F(g) \mid g \in C(X),\ g \leqq f\} \quad \forall f \in C^+(X) . \\ F^- &= (-F)^+ . \end{aligned}$$

Then it is easily seen, that F^+ and F^- are linear functionals on $C^+(X)$ both satisfying (2.2). Also it can be shown in the usual way that $F = F^+ - F^-$. From Theorem II-7-1 p. 57 in [15] it follows that there exist positive finite measures m' and m'' on $(X, \mathcal{B}a(X))$, where $\mathcal{B}a(X)$ is the Baire σ-algebra, that is the σ-algebra generated by $C(X)$, such that

$$F^+(f) = \int_X f\, dm' \quad \forall f \in C^+(X)\,,$$
$$F^-(f) = \int_X f\, dm'' \quad \forall f \in C^+(X)\,.$$

Since X is normal and countably paracompact it follows from Theorem 5.3 in [13] that we may assume that m' and m'' are defined on $\mathcal{B}(X)$ and that they satisfies (2.4). Putting $m = m' - m''$ we have proved Lemma 1.

THEOREM 1. *Let X be a normal, countably paracompact Hausdorff space. If X is quasi-Radonian, then the Mackey topology τ^*, on $M(X)$ is complete.*

REMARKS. (a) For a long time it was not known whether every normal space is countably paracompact (see for example chapter 5 section 2 of [6], where other properties of countably paracompact spaces may be found). However M. E. Rudin has recently shown in [17], that there exists normal non-countably paracompact spaces.

(b) If the Mackey topology is complete, then it is well known, that the closed balanced convex hull of every w^*-compact subset of $M(X)$ is again w^*-compact. Our next result states that this fact still holds without assuming normality and countably paracompactness of X.

(c) Notice that the w^*-topology is complete, if and only if X is finite (a complete space is a Baire space, hence if $(M(X), w^*)$ is complete, then there must exists some ball having nonempty w^*-interior, but this implies that $w^* = u^*$, which implies that X is finite). I have no idea what the Mackey topology on $M(X)$ looks like. It would be interesting to have a concrete description of the Mackey topology on $M(X)$, or more modest to construct a complete topology on $M(X)$, which lies between w^* and τ^*.

PROOF OF THEOREM 1. Since the τ^*-equicontinuous subsets of $C(X)$ are by the definition of τ^* exactly those subsets of $C(X)$, which are contained in a w-compact convex subset of $C(X)$, it follows from Corollary 1 to Theorem 3 on p. 106 in [15] that it suffices to show the following statement:

(2.5) If F is a linear functional on $C(X)$, such that $F|K$ is w-continuous for all w-compact convex subsets K, of $C(X)$, then there exists a measure $m \in M(X)$, such that $F(f)=\int_X f\,dm$ for all $f \in C(X)$.

Let F be a linear functional on $C(X)$ satisfying the conditions of (2.5). We shall first show that F satisfies condition (2.2) in Lemma 1. So let $\{f_n\}$ be a sequence in $C(X)$, which decreases pointwise to 0. By Lebesgue's dominated convergence theorem, it follows that $\{f_n\}$ converges to 0 in the w-topology. Now we shall define a linear map T, from l_1 into $C(X)$; let $a=(a_n)\in l_1$, then we define

$$Ta(x) = \textstyle\sum_{n=1}^{\infty} a_n f_n(x) \quad \text{for } x \in X .$$

Since $\{f_n\}$ is uniformly bounded, we have that Ta is a bounded continuous function on X. If $m \in M(X)$, then

$$(Ta,m) = \textstyle\sum_{n=1}^{\infty} a_n \int_X f_n\, dm .$$

Since $\lim_{n\to\infty} \int_X f_n dm = 0$, this equality shows that T is a continuous map from $(l_1,\sigma(l_1,c_0))$ into $(C(X),w)$. If B is the closed unit ball in l_1 then B is $\sigma(l_1,c_0)$-compact and convex. Hence TB is w-compact and convex and so the restriction of F to TB is w-continuous. Since $f_n \in TB$ for all $n \geqq 1$ and $w\text{-}\lim_{n\to\infty} f_n = 0$, we have that $\lim_{n\to\infty} F(f_n)=0$. That is, F satisfies the conditions of Lemma 1, so we can find a finite measure m on $(X,\mathscr{B}(X))$, which satisfies (2.3) and (2.4). Since X is quasi-Radonian, we have that m satisfies (1.3), which in connection with (2.4) implies that $m \in M(X)$.

This proves (2.5), and so Theorem 1 is proved.

THEOREM 2. *Let X be a completely regular Hausdorff space and L a w^*-compact subset of $M(X)$. If X is semi-Radonian, and if K is the closed convex balanced hull of L, then K is w^*-compact.*

REMARK. David Fremlin has given an example of a subset X of the unit square, and a compact subset L of $M^+(X)$, such that the closed convex hull of L is not compact. This shows that the condition of X being semi-Radonian in Theorem 2 cannot be relaxed.

PROOF. Let $v \in M(L)$, and let $A \in \mathscr{B}(X)$. Then by Proposition 1 we have that the maps

$$F(m) = m(A) \quad \text{and} \quad G(m) = |m|(A)$$

are Borel maps on L, and since L is u^*-bounded we know that F and G are bounded functions on L. Hence we may define

$$n(A) = \int_L m(A)\, v(dm), \quad c(A) = \int_L |m|(A)\, |v|(dm)\,.$$

Then n and c are finite measures on $(X, \mathscr{B}(X))$, such that $|n| \leqq c$. We shall now show that c is τ-smooth. So let $\{U_i\}$ be a family of open sets, which filters upwards to U. Let

$$G_i(m) = |m|(U_i) \quad \text{and} \quad G(m) = |m|(U)$$

for $m \in L$. Then G_i is lower semi-continuous on L, and since every measure in L is τ-smooth, we have that $G_i \uparrow G$. So by τ-smoothness of $|v|$, we find that $c(U_i) \uparrow c(U)$, that is c is τ-smooth, and so c is regular, since we have assumed that X is semi-Radonian. But this clearly implies that $n \in M(X)$. Hence if we define T by

$$Tv(A) = \int_L m(A)\, v(dm) \quad \text{for } v \in M(L) \text{ and } A \in \mathscr{B}(X)\,,$$

then T is a map from $M(L)$ into $M(X)$. Now a standard argument shows that if f is a bounded Borel function on X, then

$$\int_X f\, dTv = \int_L \{\int_X f(x) m(dx)\}\, v(dm)\,.$$

If $f \in C(X)$, then the function

$$H(m) = \int_X f\, dm \quad \text{for } m \in L$$

belongs to $C(L)$, and so T becomes a continuous map from $(M(L), w^*)$ into $(M(X), w^*)$.

If B is the unit ball in $M(L)$, then B is convex and balanced, and by Alaoglu's theorem B is w^*-compact. Hence TB is convex balanced and w^*-compact, but $L \subseteqq TB$ and so $K \subseteqq TB$, which shows that K is w^*-compact.

THEOREM 3. *Let X be a completely regular Hausdorff space and L a subset of $M^+(X)$. If there exists a continuous injective map from X into some metrizable space, then the following four statements are equivalent:*

(2.6) *L is relatively ω-compact in $(M^+(X), w^*)$.*

(2.7) *L is relatively sequentially compact in $(M^+(X), w^*)$.*

(2.8) *L is relatively compact in $(M^+(X), w^*)$.*

(2.9) *The closure of L is compact and metrizable in $(M^+(X), w^*)$.*

REMARKS. (a) If Y is a Hausdorff space and A is a subset of Y then we called A *relatively ω-compact* (*relatively sequentially compact*) if every ordinary sequence in A has a generalized (an ordinary) subsequence, which converges to a point in Y. It should be noticed that the closure of a relative ω-compact set is not necessarily relatively ω-compact, and that even if A is relatively ω-compact in itself, then A is not necessarily closed in Y. The same remark applies to relatively sequentially compact subsets of Y.

(b) A topological space, which satisfies the condition in Theorem 3, is sometimes called *submetrizable.*

PROOF OF THEOREM 3. The only non trivial statement is that (2.6) implies (2.9). Let Y be a metrizable space, and f an injective continuous map from X into Y. Then the map H, defined by

$$H(m)(A) = m(f^{-1}(A)) \quad \text{for } m \in M^+(X) \text{ and } A \in \mathscr{B}(Y) ,$$

is a continuous injective map from $M^+(X)$ into $M^+(Y)$. From Theorem 13 p. 188 in [19] it follows that $M^+(Y)$ is metrizable. Hence Theorem 3 is a consequence of the following lemma:

LEMMA 2. *Let X be a completely regular Hausdorff space and A a relatively ω-compact subset of X. If X is submetrizable, then the closure of A is compact and metrizable.*

PROOF. Let Y be a metrizable space, and f an injective continuous map from X into Y. First we shall prove

$$f(\mathrm{cl}(A)) = \mathrm{cl}(f(A)) . \tag{2.10}$$

Let $y \in \mathrm{cl}(f(A))$, since Y is metrizable there exists a sequence $\{x_n\} \subseteqq A$, such that $\lim_{n\to\infty} f(x_n) = y$. Since A is relatively ω-compact, we know that $\{x_n\}$ has at least one limit point, say x. Then clearly $x \in \mathrm{cl}(A)$ and $f(x) = y$. That is $\mathrm{cl}(f(A)) \subseteqq f(\mathrm{cl}(A))$, and since the converse inequality is trivial we have proved (2.10).

Next we shall prove that f is a homeomorphism from $\mathrm{cl}(A)$ onto $\mathrm{cl}(f(A))$. To see this it suffices to show

(2.11) If $\{x_n\}_0^\infty \subseteqq \mathrm{cl}(A)$ and $\lim_{n\to\infty} f(x_n) = f(x_0)$, then $\lim_{n\to\infty} x_n = x_0$.

Suppose that this is not true for some sequence $\{x_n\} \subseteqq \mathrm{cl}(A)$. Then by regularity of X, there exists an open neighborhood Y of x_0 and integers $n_1 < n_2 < \ldots < n_j < \ldots$, such that $x_{n_j} \notin \mathrm{cl}(U)$ for any $j \geqq 1$. If $F = \mathrm{cl}(U)$, then $\mathrm{cl}(A) \setminus F \subseteqq \mathrm{cl}(A \setminus F)$, since F is closed. Hence by (2.10) we have

$$f(x_{n_j}) \in f(\mathrm{cl}(A) \setminus F) \subseteqq \mathrm{cl}(f(A \setminus F)) .$$

That is, there exists $z_j \in A \setminus F$, such that $d(f(x_{n_j}), f(z_j)) < j^{-1}$ for all $j \geqq 1$, where d is a metric for Y. From this it follows that $\lim_{j\to\infty} f(z_j) = f(x_0)$. Since f is injective and continuous, we have that x_0 is the only possible limit point for $\{z_j\}$. On the other hand $z_j \notin U$ for any $j \geqq 1$, and so $\{z_j\}$ can have no limit points at all. But this contradicts the assumption that A is relatively ω-compact.

That is (2.14) holds and f is a homeomorphism from $\mathrm{cl}(A)$ onto $\mathrm{cl}(f(A))$. Now $f(A)$ is obviously relatively ω-compact in Y, and since Y is metrizable this actually implies that $f(A)$ is relatively compact in Y, and so $\mathrm{cl}(f(A))$ is compact and metrizable. But $\mathrm{cl}(A)$ is homeomorphic to $\mathrm{cl}(f(A))$ and so Lemma 2 is proved.

3. Closure properties for Prohorov spaces.

It is well known that complete metric spaces and locally compact spaces are Prohorov spaces. We shall in this section show that every space, which is locally complete in the sense of Čech, is a Prohorov space, a result, which contains the results mentioned above. Furthermore we shall study operations which preserve the Prohorov property. First we shall show a characterization of spaces, which are complete in the sense of Čech, a result of interest in itself.

PROPOSITION 2. *Let X be a completely regular Hausdorff space. Then X is complete in the sense of Čech, if and only if there exist families $\{\mathscr{V}_n\}$ satisfying*

(3.1) *$\mathscr{V}_n$ is an open covering of X for each $n \geqq 1$,*

(3.2) *if $\{V_{jn} \mid 1 \leqq j \leqq k(n)\} \subseteq \mathscr{V}_n$ for all $n \geqq 1$, then the set*

$$A = \bigcap_{n=1}^{\infty} \bigcup_{j=1}^{k(n)} V_{jn}$$

is relatively compact in X.

PROOF. Let us first prove the "if" part. So let $\{\mathscr{V}_n\}$ be a sequence of families satisfying (3.1) and (3.2). We shall now prove that $\{\mathscr{V}_n\}$ has the following completeness property:

(3.3) If $\{F_n\}$ is a decreasing sequence of nonempty closed sets, such that for all $n \geqq 1$ we have $F_n \subseteq W_n$ for some $W_n \in \mathscr{V}_n$, then the set $F = \bigcap_1^\infty F_n$ is nonempty and compact.

Since F is closed it follows from (3.2) that F is compact, so we are only left with proving that $F \neq \emptyset$. Now let $x_n \in F_n$. Then by (3.1) we can find $V_{jn} \in \mathscr{V}_j$, such that $x_n \in V_{jn}$ for all $n \geqq 1$. Now put

$$V_n = W_n \cup \bigcup_{j=1}^{n-1} V_{jn} .$$

Then $x_j \in V_{jn} \subseteqq V_n$ for all $j < n$ and $x_j \in F_j \subseteqq F_n \subseteqq W_n \subseteqq V_n$ for all $j \geqq n$. That is, the sequence $\{x_j\}$ is contained in the set $\bigcap_1^\infty V_n$, which is relatively compact by (3.2). Hence $\{x_j\}$ has a limit point, which of course must belong to F.

Next we shall show that the countability assumption in (3.3) can be removed:

(3.4) Let $\mathscr{F}$ be a family of closed sets, which has the finite intersection property. If there exists $W_n \in \mathscr{V}_n$ and $F_n \in \mathscr{F}$ for each n, such that $F_n \subseteqq W_n$ for all $n \geqq 1$, then $H = \bigcap \{F \mid F \in \mathscr{F}\}$ is compact and nonempty.

Since H is closed we deduce from (3.2) that H is compact. Let F_n be chosen according to (3.4) and let $H_0 = \bigcap_1^\infty F_n$. Then H_0 is compact and nonempty by (3.3). If $G_1, \ldots, G_p \in \mathscr{F}$, then by (3.6) we have that

$$H_0 \cap \bigcap_{j=1}^{p} G_j \neq \emptyset .$$

So by compactness of H_0 we find that

$$H = \bigcap \{H_0 \cap F \mid F \in \mathscr{F}\} \neq \emptyset ,$$

which proves (3.4).

From Theorem 2 on p. 143 in [6] and (3.4) it follows that X is complete in the sense of Čech.

Now let us prove the "only if" part. So let X be complete in the sense of Čech. Then X is a G_δ-set in βX. Let U_n be open sets in βX, such that $X = \bigcap_1^\infty U_n$. If $x \in X$ and $n \geqq 1$, then we choose a neighborhood $U_n(x)$, in βX of x, such that

$$x \in U_n(x) \subseteqq \mathrm{cl}(U_n(x)) \subseteqq U_n .$$

Let $V_n(x) = U_n(x) \cap X$ and $\mathscr{V}_n = \{V_n(x) \mid x \in X\}$. Then it is easily checked that $\{\mathscr{V}_n\}$ satisfies (3.1) and (3.2), and so Proposition 2 is proved.

THEOREM 4. *Let X be a completely regular Hausdorff space and D a closed subset of X, which is locally complete in the sense of Čech. If L is a w^*-compact subset of $M^+(X)$ and $\varepsilon > 0$, then there exists a compact set $K \subseteqq D$, such that $m(D \setminus K) < \varepsilon$ for all $m \in L$.*

REMARK. This theorem generalizes a result of X. Fernique (see [7, Lemma 6.5.b]). As pointed out to me by F. Topsøe, Theorem 4 follows from Corollary 4 and (3.9).

PROOF OF THEOREM 4. For each $x \in D$ we can choose a neighborhood $U(x)$ of x in D and a set $C(x) \subseteq D$, such that $U(x) \subseteq C(x)$ and $C(x)$ is complete in the sense of Čech. By regularity of X we can choose an open neighborhood $V(x)$ of x, in D, such that $V(x) \subseteq \mathrm{cl}(V(x)) \subseteq U(x)$.

Since $\{V(x) \mid x \in D\}$ is an open covering of D, a standard argument shows that there exists a finite subset $\{x_1, \ldots, x_p\} \subseteq D$, such that

$$(3.5) \qquad m(D \setminus \textstyle\bigcup_{j=1}^{p} V(x_j)) < \frac{1}{2}\varepsilon \quad \forall m \in L .$$

Let $D_j = \mathrm{cl}(V(x_j))$, then D_j is closed in $C(x_j)$, and so D_j is complete in the sense of Čech. So by Proposition 4 there exist families $\{\mathscr{V}_n^j\}$ satisfying (3.1) and (3.2) relatively in D_j. From (3.1) one deduces exactly as above, that for each $n \geqq 1$ and $j \geqq 1$ there exists

$$V_k(j,n) \in \mathscr{V}_n^j \quad \forall k = 1, \ldots, k(j,n),$$

such that

$$(3.6) \qquad m(D_j \setminus \textstyle\bigcup_{k=1}^{k(j,n)} V_k(j,n)) < p^{-1}2^{-n-1}\varepsilon \quad \forall m \in L .$$

Now let

$$K_j = \mathrm{cl}(\textstyle\bigcap_{n=1}^{\infty} \bigcup_{k=1}^{k(j,n)} V_k(j,n)) \quad \text{for } 1 \leqq j \leqq p ,$$
$$K = \textstyle\bigcup_{j=1}^{p} K_j .$$

Then by (3.2) K is compact, and if $m \in L$, then we find from (3.5) and (3.6) that

$$\begin{aligned} m(D \setminus K) &\leqq m(D \setminus \textstyle\bigcup_{j=1}^{p} D_j) + \sum_{j=1}^{p} m(D_j \setminus K_j) \\ &\leqq \tfrac{1}{2}\varepsilon + \textstyle\sum_{j=1}^{p} \sum_{n=1}^{\infty} m(D_j \setminus \bigcup_{k=1}^{k(j,n)} V_k(j,n)) < \varepsilon . \end{aligned}$$

Hence Theorem 4 is proved.

COROLLARY 1. *If X is a completely regular Hausdorff space, which is locally complete in the sense of Čech, then X is a Prohorov space.*

REMARK. If X is metrizable and locally complete, in the sense of Čech, then X is actually metrizable under a complete metric. Hence the result of Corollary 1 does not give anything new in the category of metrizable spaces.

THEOREM 5. *Let X be a completely regular Hausdorff space, and let X_n be a Prohorov space and f_n a continuous map from X into X_n, such that*

(3.7) $\bigcap_{n=1}^{\infty} f_n^{-1}(K_n)$ *is compact in* X, *whenever* K_n *is compact in* X_n *for all* $n \geqq 1$.

Then X is a Prohorov space.

PROOF. Let L be a w^*-compact subset of $M^+(X)$, and let $\varepsilon > 0$ be given. Let

$$F_n(m)(A) = m(f_n^{-1}(A)) \quad \text{for } m \in M(X) \text{ and } A \in \mathscr{B}(X_n)\,.$$

Then F_n is a continuous map from $(M(X), w^*)$ into $(M(X_n), w^*)$. Hence $F_n(L)$ is a w^*-compact subset of $M^+(X_n)$. Since X_n is a Prohorov space there exists a compact subset K_n of X_n, such that

$$m(X \setminus f_n^{-1}(K_n)) < \varepsilon 2^{-n} \quad \forall m \in L\,.$$

Now let $C = \bigcap_{n=1}^{\infty} f_n^{-1}(K_n)$, then C is compact by the assumption (3.7), and

$$m(X \setminus C) \leqq \sum_{n=1}^{\infty} m(X \setminus f_n^{-1}(K_n)) < \varepsilon \quad \forall m \in L\,.$$

Hence L is uniformly tight, and so Theorem 5 is proved.

COROLLARY 2. *Let X be a completely regular Hausdorff space and X_n a subset of X for $n = 1, 2, \ldots$. If X_n is a Prohorov space for each $n \geqq 1$, then so is $\bigcap_{n=1}^{\infty} X_n$.*

COROLLARY 3. *Let X be a completely regular Hausdorff space and $\{U_i \mid i \in I\}$ an open covering of X. If U_i is a Prohorov space for all $i \in I$, then so is X.*

COROLLARY 4. *Let X be a completely regular Hausdorff space, F a closed subset of X and G a G_δ-subset of X. If X is a Prohorov space, then so is $F \cap G$. In particular every open, every closed and every G_δ-set in a Prohorov space are again Prohorov spaces.*

COROLLARY 5. *Any countable product of Prohorov spaces is again a Prohorov space.*

THEOREM 6. *Let X be completely regular Hausdorff space, which locally is a Prohorov space. Then X is a Prohorov space.*

PROOF. Let L be a w^*-compact subset of $M^+(X)$, and let $\varepsilon > 0$ be given. By assumption there exists an open set $U(x)$ and a Prohorov subspace $A(x)$ of X, such that $x \in U(x) \subseteq A(x)$ for all $x \in X$. Since X is completely regular there exists a closed set $D(x)$, such that

$$x \in \operatorname{int}(D(x)) \subseteq D(x) \subseteq A(x) .$$

Since $A(x)$ is a Prohorov space and $D(x)$ is a closed subset of $A(x)$, we have that $D(x)$ is a Prohorov space. Now a standard compactness argument shows that there exists points $x_1,\ldots,x_k \in X$, such that

$$m(X \setminus \bigcup_{j=1}^{k} D(x_j)) < \tfrac{1}{2}\varepsilon \quad \forall m \in L . \tag{3.8}$$

From Theorem 9.1 p. 43 in [18], it followst that if D is closed, then

(3.9) The map $m \to m \mid \mathscr{B}(D)$ maps relatively compact subsets of $M^+(X)$ onto relatively compact subsets of $M^+(D)$.

Hence for each $1_j \leqq j \leqq k$ we can find a compact subset K_j of $D(x_j)$, such that

$$m(D(x_j) \setminus K_j) < (2k)^{-1}\varepsilon \quad \forall m \in L .$$

If $K = K_1 \cup \ldots \cup K_k$, then from the inequality above and (3.15) we find that

$$m(X \setminus K) < \varepsilon \quad \forall m \in L ,$$

which proves Theorem 6.

I am indebted to F. Topsøe for pointing (3.9) out to me, which simplified my original proof considerably.

THEOREM 7. *Let X be a complete regular Hausdorff space, $\mathscr{F}$ a family of continuous functions on X and $\{D_n\}$ an increasing sequence of closed subsets of X, such that*

(3.10) *if f and g belong to $\mathscr{F}$, then* $\max(f,g) \in \mathscr{F}$,

(3.11) *if f is map from X into the positive real line, such that f coincides on each D_n with a function from $\mathscr{F}$, then f is continuous,*

(3.12) $X = \bigcup_{n=1}^{\infty} D_n$,

(3.13) $\forall n \geqq 1$, $\forall x \in X \setminus D_n$, $\exists f \in \mathscr{F}$, *such that $f \geqq 0$, $f(y) = 0$ for all $y \in D_n$ and $f(x) \geqq 1$.*

If L is a w^*-compact subset of $M^+(X)$ and $\varepsilon > 0$, then there exists an integer $n \geqq 1$, such that

$$m(X \setminus D_n) < \varepsilon \quad \forall m \in L . \tag{3.14}$$

Furthermore, if D_n is a Prohorov space for all $n \geqq 1$, then X is a Prohorov space.

PROOF. Suppose that (3.14) does not hold, then we can find $m_n \in L$, such that

$$m_n(X \setminus D_n) \geqq \varepsilon \quad \forall n \geqq 1 . \tag{3.15}$$

Let $x \in X \setminus D_n$, then we can find $f^n{}_x \in \mathscr{F}$, such that $f^n{}_x \geqq 0$, $f^n{}_x(y) = 0$ for all $y \in D_n$ and $f^n{}_x(x) \geqq 1$. Let us put

$$V_n(x) = \{y \in X \mid f^n{}_x(y) > \tfrac{1}{2}\} .$$

Then $V_n(x)$ is an open neighborhood of x, for all $x \in X \setminus D_n$. Hence by τ-smoothness of m_n there exists $\{x_{jn}\} \subseteq X \setminus D_n$, such that

$$m_n(\bigcup_{j=1}^{\infty} V(x_{jn})) = m(X \setminus D_n) \geqq \varepsilon .$$

Hence we can find $k(n) \geqq 1$, such that

$$m_n(\bigcup_{j=1}^{k(n)} V(x_{jn})) > \tfrac{1}{2}\varepsilon .$$

Now let us put

$$h_n = \max\{f^n{}_{x_{1,n}}, f^n{}_{x_{2,n}}, \ldots, f^n{}_{x_{k(n),n}}\} .$$

Then $h_n \in \mathscr{F}$ by (3.10), and from the construction of h_n it follows immediately that

$$D_n \subseteqq \{x \in X \mid h_p(x) = 0\} \quad \forall n \leqq p , \tag{3.16}$$

$$0 \leqq h_n(x) \quad \forall n \geqq 1,\ \forall x \in X , \tag{3.17}$$

$$m_n(x \mid h_n(x) > \tfrac{1}{2}) > \tfrac{1}{2}\varepsilon \quad \forall n \geqq 1 . \tag{3.18}$$

If $f_n = \sup\{h_n, h_{n+1}, \ldots\}$, then from (3.16) it follows that

$$f_n(x) = \max\{h_n(x), h_{n+1}(x), \ldots, h_p(x)\} \quad \text{if } x \in D_p,\ p \geqq n .$$

Hence f_n is finite on X, and from (3.11) it follows that f_n is continuous on X for all $n \geqq 1$. Let us now put

$$V_n = \{x \in X \mid f_n(x) < \tfrac{1}{2}\} .$$

Then V_n is open and increases with n. From (3.16) we deduce that $D_n \subseteqq V_n$. That is $V_n \uparrow X$. Since L is w^*-compact a standard argument shows that for some integer $k \geqq 1$ we have that

$$m(X \setminus V_k) < \tfrac{1}{2}\varepsilon \quad \forall m \in L .$$

In particular we find that

$$m_k(x \mid f_k(x) \geqq \tfrac{1}{2}) < \tfrac{1}{2}\varepsilon .$$

But this contradicts (3.18), since $f_k \leqq h_k$. Hence our basic assumption is false and so (3.14) holds. The last part is an immediate consequence of (3.9) and (3.14). Hence Theorem 7 is proved.

COROLLARY 6. *Let X be a completely regular k^*-space. Let $\{D_n\}$ be a sequence of closed subsets of X, which are all Prohorov spaces, and which increases upwards to X. If every compact subset of X is contained in some D_n, then X is a Prohorov space.*

COROLLARY 7. *Let E be a locally convex space, which is an inductive limit of a sequence $\{E_n\}$ of closed subspaces of E, such that $E_n \uparrow E$. If E_n is a Prohorov space for all $n \geqq 1$, then E is a Prohorov space.*

PROOF. Let $\mathscr{F}$ be the class of all functions from E into the real line, which are continuous, positively homogeneous, non negative and sublinear, and let us put $D_n = E_n$. It is then easily checked that (3.10), (3.11), (3.12) and (3.13) are satisfied.

REMARK. (a) Notice that if X and $\{D_n\}$ satisfies (3.10)–(3.13) in Theorem 7, then every compact set in X is contained in some D_n (let L be set of Dirac measures concentrated at x for x belonging to a compact subset of X).

(b) The category of completely regular Hausdorff spaces, which are locally complete in that sense of Čech, is also invariant under the operations defined in Theorems 5 and 6 and in Corollaries 2, 3, 4 and 5. So none of these operations can take us out of the category of spaces, which are locally complete in the sense of Čech.

(c) In example 1 and example 2 below we shall see that the operation defined in Theorem 7 allows us to show that there do exist Prohorov spaces, which are not locally complete in the sense of Čech. However Proposition 3 shows that Theorem 7 and its corollaries do not give anything new in the category of spaces, which are complete at points.

PROPOSITION 3. *Let X be a completely regular Hausdorff space, and let $\{D_n\}$ be an increasing sequence of closed subsets of X, which increases upwards to X, and which satisfies*

(3.19) *every compact subset of X is contained in some D_n.*

Then X is not complete at any point in $\bigcap_{n=1}^{\infty} \mathrm{cl}(X \setminus D_n)$.

Now suppose furthermore that X is complete at points. Then we have:

$$X = \bigcup_{n=1}^{\infty} \mathrm{int}(D_n). \tag{3.20}$$

(3.21) *If D_n is metrizable for all $n \geqq 1$, then so is X.*

(3.22) *If D_n is locally complete in the sense of Čech for all $n \geqq 1$, then so is X.*

(3.23) *If D_n is locally compact for all $n \geqq 1$, then so is X.*

(3.24) *If D_n is metrizable under a complete metric for all $n \geqq 1$, then so is X.*

(3.25) *If D_n is a Prohorov space for all $n \geqq 1$, then so is X.*

PROOF. Let $x \in \bigcap_{n=1}^{\infty} \mathrm{cl}(X \setminus D_n)$ and suppose that X is complete at x. Let K be a compact subset of X, such that $x \in K$ and X has a countable base $\{V_k\}$ at K. Since $V_n \setminus D_n \neq \emptyset$ we can choose points $x_n \in V_n \setminus D_n$. By Lemma 7.5 in [18] we have that $A = \{x_n \mid n \geqq 1\}$ is relatively compact and $A \nsubseteq D_n$ for any $n \geqq 1$, but this contradicts (3.19). That is X is not complete at x.

If X is complete at points, then from the argument above it follows that

$$\bigcap_{n=1}^{\infty} \mathrm{cl}(X \setminus D_n) = \emptyset .$$

And since $\mathrm{int}(D_n) = X \setminus \mathrm{cl}(X \setminus D_n)$, we see that (3.20) holds.

(3.21) follows easily from (3.20) and the metrization theorem of Nagata–Smirnov (see for example chapter II, § 21, section XVII, p. 236 in [10]).

(3.22) and (3.23) are obvious.

(3.24) follows from (3.21) and (3.22).

(3.24) follows from (3.20) and Corollary 2.

EXAMPLE 1. *There exists a Prohorov space, which is a k-space, but which is not complete at any point.*

Let E be a metrizable, locally convex, linear, Hausdorff space and let E' be the dual of E. Let π be the topology on E' of uniform convergence on precompact subsets of E. If $\{V_n\}$ is a neighborhood base at 0 in E, and $V_n{}^\circ$ is the polar of V_n, then it is well known that $V_n{}^\circ \uparrow E'$, $V_n{}^\circ$ is π-compact, and every π-compact subset of E' is contained in some $V_n{}^\circ$.

By a theorem of Banach and Dieudonné (see for example [9, (1) p. 272]) we have that (E', π) is a k-space. Hence (E', π) is a Prohorov space by Corollary 6. If E is infinite dimensional then $V_n{}^\circ$ has no interior points for any n, and so by Proposition 3 we have that E' is not complete at some point $x_0 \in E'$. By translation we find that (E', π) is not complete at any point.

EXAMPLE 2. *There exists a Prohorov space, which is not a k^*-space.*

Grothendieck has shown that there exist separable, locally convex, linear Frechet spaces $\{E_n\}$ and a closed linear subspace H of the direct sum $G=\sum_1^\infty E_n$, such that the quotient G/H is not complete. (See for example § 31,6 in [9].)

Let H° be the polar of H. Then H° is a linear subspace of the dual space G' of G. Let π be the topology on G' of uniform convergence on precompact subset of G. If E'_n is the dual of E_n equipped with the topology of uniform convergence on precompact subsets of E_n, then it is well known that G' is homeomorphic to $\prod_{n=1}^\infty E_n'$. From Example 1, Corollary 4 and Corollary 5 it follows that (H°,π) is a Prohorov space, since H° is π-closed.

From the fact that G/H is not complete and that H° is linearly isomorphic to the dual of G/H, one easily deduces that there exists a linear functional F from H° into the real line, such that F is not continuous in the π-topology, but $F|K$ is a continuous map from (K,π) into the real line for all compact subsets K of (H°,π). That is (H°,π) is not a k^*-space.

4. The Prohorov property for some restricted classes of compact sets in $M^+(X)$.

THEOREM 8. *Let X be a completely regular Hausdorff space and L a u^*-bounded subset of $(M(X),w^*)$, then L is uniformly tight, if and only if L is equicontinuous in the strict topology, β.*

PROOF. If $\{K_n\}$ are compact sets, such that $|m|(X\setminus K_n)\leqq 2^{-2n}$ for all $m\in L$, and all $n\geqq 1$, $K_1=\emptyset\subseteqq K_2\subseteqq\ldots\subseteqq K_n\subseteqq\ldots$, then

$$\left|\int_X f\,dm\right| \leqq \sum_{n=1}^\infty \int_{K_{n+1}\setminus K_n} |f|\,d|m| \leqq 1 \quad \forall m\in L\,,$$

for all $f\in C(X)$ such that $\|f\|_{K_n}\leqq 2^n$ for all $n\geqq 1$. That is L is β-equicontinuous.

Conversely if L is β-equicontinuous, then we can find an increasing sequence $\{K_n\}$ of compact subsets of X, and an increasing sequence of positive real numbers $\{a_n\}$, such that $\lim_{n\to\infty}a_n=\infty$, and

$$\left|\int_X f\,dm\right| \leqq 1 \quad \forall m\in L\,,$$

for all $f\in C(X)$, such that $\|f\|_{K_n}\leqq a_n$ for all $n\geqq 1$. In exactly the same way as in the proof of Theorem 3 in [8] it follows that

$$|m|(X\setminus K_n) \leqq 2a_n-1 \quad \forall n\geqq 1,\ \forall m\in L\,.$$

Which shows that L is uniformly tight.

LEMMA 3. *Let X be a completely regular Hausdorff space and K a compact subset of X. If X is σ-complete at K, and if U is an open neighborhood of K, then there exist compact sets $\{K_n\}$, such that*

$$K \subseteq \bigcup_{n=1}^{\infty} K_n \subseteq U\,, \tag{4.1}$$

$$K_1 \subseteq K_2 \subseteq \ldots \subseteq K_n \subseteq \ldots\,, \tag{4.2}$$

$$X \text{ has countable base at } K_n \text{ for all } n \geqq 1\,. \tag{4.3}$$

PROOF. Let $\{C_n\}$ be chosen such that $K \subseteq \bigcup_1^\infty C_n$ and X has countable at C_n for all $n \geqq 1$. By regularity of X we can find an open set V_1 and a closed set F_1, such that $K \subseteq V_1 \subseteq F_1 \subseteq U$. Continuing in this way we can construct open sets V_k and closed sets F_k for each $k \geqq 1$, such that

$$K \subseteq V_{k+1} \subseteq F_{k+1} \subseteq V_k \subseteq F_k \subseteq U \quad \forall k \geqq 1\,.$$

Let $F = \bigcap_1^\infty F_k$, and $K'_n = F \cap C_n$, then (4.1) is obviously satisfied.

Let n be fixed for a moment, and let $\{U_k\}$ be a decreasing neighborhood base at C_n. We shall then prove that $\{U_k \cap V_k\}$ is a neighborhood base at K'_n. Suppose that W is an open neighborhood of K'_n, then

$$\bigcup_{k=1}^{\infty} (F_k \cap C_n \setminus W) = \emptyset\,.$$

So by compactness of C_n there exists an integer $p \geqq 1$, such that $F_p \cap C_n \subseteq W$. Hence C_n is contained in the open set $W \cup (X \setminus F_p)$, and so we can find an integer $k \geqq p$, such that $C_n \subseteq U_k \subseteq W \cup (X \setminus F_p)$. Then

$$K'_n = C_n \cap F \subseteq U_k \cap V_k \subseteq W \cup (V_k \setminus F_p) = W$$

since $F \subseteq V_k \subseteq F_p$. But this shows that $\{U_k \cap V_k\}$ is a neighborhood base of K'_n.

Let us now put $K_n = \bigcup_{j=1}^{n} K'_j$, then it is easily seen that $\{K_n\}$ satisfies (4.1), (4.2) and (4.3).

LEMMA 4. *Let X be a completely regular Hausdorff space, which is σ-complete at every compact set. Let $m \in M^+(X)$ and let U be an open subset of X, then for every $\varepsilon > 0$, there exists a compact set $C \subseteq U$, such that $m(U \setminus C) < \varepsilon$ and X has countable base at C.*

PROOF. Let $K \subseteq U$ be a compact set such that $m(U \setminus K) < \varepsilon$. Let $\{K_n\}$ be chosen according to Lemma 3, then

$$\lim_{n\to\infty} m(U \setminus K_n) = m(U \setminus \bigcup_1^\infty K_n) \leqq m(U \setminus K) < \varepsilon\,.$$

Hence we may take $C = K_n$ for n sufficiently large.

THEOREM 9. *Let X be a completely regular Hausdorff space, which is σ-complete at every compact set, L a compact scattered subset of $(M^+(X), w^*)$, Φ an upper semi-continuous function from L into the real line, and f a lower semi-continuous, bounded function from X into the positive real line $[0,\infty)$, such that*

$$\int_X f\,dm \geqq \Phi(m) \quad \forall m \in L\,. \tag{4.4}$$

If $\varepsilon > 0$, then there exists a compact set $K \subseteqq U = \{x \in X \mid f(x) > 0\}$, such that

$$\int_K f\,dm \geqq \Phi(m) - \varepsilon \quad \forall m \in L\,. \tag{4.5}$$

REMARK. (a) It is not obvious that a uniformly tight and compact subset of $(M^+(X), w^*)$ satisfying (4.4) will satisfy (4.5) for a compact set $K \subseteqq U$. But we shall see later that this is actually so (see Proposition 4).

(b) Notice that Theorem 9 does not hold, if we only assume that $\mathrm{cl}(L)$ is compact and scattered.

PROOF OF THEOREM 9. Since L is scattered we have by (1.7) that $a_{d(L)}(L) = \emptyset$. And since L is compact it follows from (1.5) that $d(L)$ is not a limit ordinal, hence we can define $d_0(L)$ to be the predecessor of $d(L)$, that is $d(L) = d_0(L) + 1$.

Now we shall prove Theorem 9 by transfinite induction in $d_0(L)$. If $d_0(L) = 0$, then L is discrete and compact, that is, L is finite. But in this case the theorem is obviously true.

Now suppose that the theorem holds whenever L is compact scattered and $d_0(L) < \pi$, and f and Φ are arbitrary. Suppose that L, f and Φ satisfy the conditions of Theorem 9 and assume $d_0(L) = \pi$.

Then $a_\pi(L)$ is a nonempty discrete compact set. That is $a_\pi(L)$ is finite. Let $m_1, m_2, \ldots, m_p$ be the elements of $a_\pi(L)$, then by Lemma 4 there exists a compact set $K \subseteqq U$, such that X has countable base at K and

$$\int_K f\,dm_j > \Phi(m_j) - \varepsilon \quad \forall\, 1 \leqq j \leqq p\,.$$

Let $\{V_n\}$ be a decreasing neighborhood base at K with $V_1 = X$. Then

$$M_n = \{m \in L \mid \int_{V_n} f\,dm - \Phi(m) > -\tfrac{1}{2}\varepsilon\}$$

is an open subset of L containing $a_\pi(L)$. Since every compact scattered space is zero dimensional we can find an open closed subset $M^\circ{}_n$ of L, such that

$$a_\pi(L) \subseteqq M^\circ{}_n \subseteqq M_n \quad \forall n \geqq 1\,.$$

Since $M_1 = L$ and $\{M_n\}$ is decreasing we can assume that $M^\circ{}_1 = L$ and $\{M^\circ{}_n\}$ is decreasing.

Let $L_n = M^\circ{}_n \setminus M^\circ{}_{n+1}$. Then L_n is an open closed subset of L, and so from (1.9) we find that

$$a_\pi(L_n) = a_\pi(L) \cap L_n = \emptyset .$$

Hence $d_0(L_n) < \pi$, and if $m \in L_n$, then we have

$$\int_X 1_{V_n} f\,dm \geqq \Phi(m) - \tfrac{1}{2}\varepsilon .$$

But this shows that we can use our induction hypothesis for the tripel $(L_n, 1_{V_n} f, \Phi - \frac{1}{2}\varepsilon)$. That is, there exists a compact set $K_n \subseteq V_n \cap U$, such that

$$\int_{K_n} f\,dm \geqq \Phi(m) - \varepsilon, \quad \forall m \in L_n,\ \forall n \geqq 1 .$$

It is of course no loss of generality to assume that $K \subseteq K_n$ for all $n \geqq 1$. Hence by Lemma 7.5 in [18] we have that $C = \bigcup_{n=1}^\infty K_n$ is compact and $C \subseteq U$. We shall now show that C satisfies (4.5). If $m \in L_n$ for some $n \geqq 1$, then

$$\int_C f\,dm \geqq \int_{K_n} f\,dm \geqq \Phi(m) - \varepsilon .$$

If $m \in L \setminus \bigcup_1^\infty L_n$, then $m \in M^\circ{}_n$ for all $n \geqq 1$, and so

$$\int_{V_n} f\,dm > \Phi(m) - \varepsilon \quad \forall n \geqq 1 ,$$

since $M^\circ{}_n \subseteq M_n$. From this inequality and the fact that $V_n \downarrow K$ we find that

$$\int_C f\,dm \geqq \int_K f\,dm = \lim_{n\to\infty} \int_{V_n} f\,dm \geqq \Phi(m) - \varepsilon .$$

Hence the induction step is completed and Theorem 9 is proved.

Corollary 8. *Let X be a completely regular Hausdorff space, which is σ-complete at every compact set. If L is compact scattered subset of $(M^+(X), w^*)$, then L is uniformly tight.*

Compare with the result of LeCam [12].

Corollary 9. *Let X be a completely regular Hausdorff space, which is σ-complete at every compact set. If L is a compact, countable subset of $(M^+(X), w^*)$, then L is uniformly tight.*

Corollary 10. *Let X be a completely regular Hausdorff space, which is complete at every compact set. Let L be a compact subset of $(M^+(X), w^*)$. Then L is uniformly tight, if and only if the perfect kernel of L is uniformly tight.*

PROOF. Let P be the perfect kernel of L and suppose that P is uniformly tight, and let $\varepsilon > 0$ be given. Then we can find a compact subset K of X, such that $m(X \setminus K) < \frac{1}{2}\varepsilon$ for all $m \in P$. Since X is complete at K, we may assume that X has countable base at K. Let $\{V_n\}$ be a decreasing neighborhood base at K, such that $V_1 = X$, and let

$$M_n = \{m \in L \mid m(X \setminus V_n) < \tfrac{1}{2}\varepsilon\} .$$

Then M_n is an open neighborhood of P, so by normality of L there exists an open set M°_n in L, such that $M^\circ_1 = X$ and that

$$P \subseteq M^\circ_n \subseteq \mathrm{cl}(M^\circ_n) \subseteq M_n .$$

Let $L_n = \mathrm{cl}(M^\circ_n) \setminus M^\circ_{n+1}$. Then L_n is a compact subset of $L \setminus P$, in particular we find that L_n is compact and scattered. Since

$$\textstyle\int_X 1_{V_n}\, dm \geqq m(X) - \frac{1}{2}\varepsilon \quad \forall m \in L_n ,$$

we find from Theorem 9 that there exist compact sets $K_n \subseteq V_n$, such that

$$m(K_n) \geqq m(X) - \varepsilon \quad \forall m \in L_n .$$

It is no loss of generality assuming that $K \subseteq K_n$. Hence by Lemma 7.5 in [18] we have that $C = \bigcup_{n=1}^\infty K_n$ is compact. If $m \in L_n$ for some $n \geqq 1$, then

$$m(X \setminus C) \leqq m(X \setminus K_n) \leqq \varepsilon .$$

If $m \in L \setminus \bigcup_{n=1}^\infty L_n$, then since $M^\circ_1 = L$, we have that $m \in M^\circ_n$ for all $n \geqq 1$, and so

$$m(X \setminus V_n) < \tfrac{1}{2}\varepsilon \quad \forall n \geqq 1 .$$

Since $V_n \downarrow K$, we find that

$$m(X \setminus C) \leqq m(X \setminus K) = \lim_{n\to\infty} m(X \setminus V_n) \leqq \varepsilon .$$

That is L is uniformly tight, and so Corollary 9 is proved.

PROPOSITION 4. *Let X be a completely regular Hausdorff space and L a compact subset of $(M^+(X), w^*)$. Let Φ be an upper semi-continuous function from L into the real line and f a lower semi-continuous function from X into the positive real line* $[0, \infty)$, *such that*

$$\textstyle\int_X f\, dm \geqq \Phi(m) \quad \forall m \in L . \tag{4.6}$$

Then for every $\varepsilon > 0$ there exists a closed set $F \subseteq U = \{x \in X \mid f(x) > 0\}$ such that

$$\textstyle\int_F f\, dm \geqq \Phi(m) - \varepsilon \quad \forall m \in L . \tag{4.7}$$

If L is in addition uniformly tight, then F may be taken to be compact.

PROOF. Let us define $\mathscr{V}$ to be the class of open subsets V of X, such that $\mathrm{cl}(V) \subseteqq U$. For each $V \in \mathscr{V}$ we define

$$M_V = \{m \in L \mid \textstyle\int_V f\, dm > \Phi(m) - \varepsilon\} .$$

Then M_V is an open subset of L. If V' and V'' belongs to $\mathscr{V}$, then $V' \cup V''$ belongs to $\mathscr{V}$ and $M_{V'} \cup M_{V''} \subseteqq M_{V' \cup V''}$. That is $\{M_V \mid V \in \mathscr{V}\}$ is filtering upwards.

If $m \in L$, then by regularity of m, there exists a compact set $K \subseteqq U$, such that

$$\textstyle\int_K f\, dm > \Phi(m) - \varepsilon .$$

Since X is regular, there exists $V \in \mathscr{V}$, such that $K \subseteqq V$, that is $m \in M_V$, and so $M_V \uparrow L$, hence by compactness of L, there exists $V \in \mathscr{V}$, such that $L = M_V$. That is, $F = \mathrm{cl}(V)$ satisfies (4.7), and so the first part of Proposition 4 is proved. The last part of Proposition 4 is an easy consequence of the first part.

THEOREM 10. *Let X be a completely regular semi-Radonian space and let v be a regular finite Borel measure on $(M(X), w^*)$. Let M be a Borel subset of $(M(X), w^*)$ and $\varepsilon > 0$. Then there exists a w^*-closed, u^*-bounded, uniformly tight set $L \subseteqq M$, such that $|v|(M \setminus L) < \varepsilon$.*

PROOF. Let $K \subseteqq M$ be a u^*-bounded, w^*-compact set such that $|v|(M \setminus K) < \varepsilon$, which exists since $|v|$ is regular and σ-additive.

The proof of Theorem 2 shows, that the set function

$$s(A) = \textstyle\int_K |m|(A)\, |v|(dm)$$

exists for all $A \in \mathscr{B}(X)$, and that s is a finite regular measure on $(X, \mathscr{B}(X))$. Hence there exist compact sets C_n, such that

$$s(X \setminus C_n) \leqq \varepsilon 2^{-2n} \quad \forall n \geqq 1 .$$

Let

$$K_n = \{m \in K \mid |m|(X \setminus C_n) \leqq 2^{-n}\} .$$

Then K_n is w^*-closed by Proposition 1, and we have

$$\begin{aligned} |v|(K \setminus K_n) &= \textstyle\int_{K \setminus K_n} d|v| \leqq 2^n \int_K |m|(X \setminus C_n)\, |v|(dm) \\ &= 2^n s(X \setminus C_n) \leqq \varepsilon 2^{-n} . \end{aligned}$$

Let $L = \bigcap_{n=1}^{\infty} K_n$. Then L is u^*-bounded, w^*-closed and uniformly tight.

Furthermore $L \subseteq K \subseteq M$ and

$$|v|(K \setminus L) \leqq \textstyle\sum_{n=1}^{\infty} |v|(K \setminus K_n) < \varepsilon .$$

Hence $|v|(M \setminus L) < 2\varepsilon$ and so Theorem 10 is proved.

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