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ANTI-LATTICES AND PRIME SETS

CHU CHO-HO

It is well-known that a compact convex is a Bauer simplex if and only if the continuous functions on its extreme boundary for which the Dirichlet problem can be solved form a lattice (cf. [15, Théorème 6.2]). The other extreme case for simplexes was obtained by Effros and Kazdan [11] who introduced a geometrical notion of prime simplexes and proved that a simplex is prime if and only if the continuous functions on its extreme boundary for which the Dirichlet problem can be solved form an anti-lattice, in other words, if one can solve the Dirichlet problem for the boundary data f and g, then one can never solve it for $\max(f,g)$ unless $f \leqq g$ or $g \leqq f$.

In this paper, we extend the definition of prime simplexes to arbitrary compact convex sets and obtain the same characterization of prime sets in terms of the Dirichlet problems. We prove that every compact convex set with dense extreme points is prime. Nevertheless, the converse is not true in general and we shall quote from [11] an example of prime simplex in which the extreme points are not dense. In the case of C^*-algebras without non-zero GCR ideals, we do have, *inter alia*, that the state spaces are prime if and only if the pure states are dense. We shall state the duality of anti-lattices and prime sets in the settings of function algebras and C^*-algebras. A necessary and sufficient condition for the annihilator of a closed two-sided ideal in a C^*-algebra with identity to be a prime face is given.

We also generalize the results of Alfsen, Effros and Størmer concerning the closed faces of the state spaces of C^*-algebras. In fact, we prove that a closed face of the state space of a C^*-algebra with identity is the annihilator of a one-sided ideal if and only if it is semi-exposed, and that it is the annihilator of a closed two-sided ideal if and only if it is semi-exposed and the self-adjoint part of its annihilator is positively generated.

1. Preliminaries.

We shall always denote by K a compact convex set in a locally convex (Hausdorff) space and by ∂K, the *extreme boundary*, i.e., the set of extreme

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points, of K. The Banach space of continuous affine functions on K is designated by $A(K)$. Let

$$A(K)^+ = \{f \in A(K) : f \geqq 0\}$$

be the closed cone in $A(K)$ and let

$$Q(K) = \{\max(a_1, \ldots, a_n) : a_i \in A(K),\ i=1,\ldots,n\}.$$

Suppose F is a subset of K. The space of all restrictions to F of the continuous affine functions on K is denoted by $A(F; K)$. A convex subset F of K is called a *face* if $\lambda y + (1-\lambda)z \in F$ with $0<\lambda<1$ entails that $y, z \in F$.

When $F \subseteqq K$ and $I \subseteqq A(K)$, we write

$$F_\perp = \{f \in A(K) : f(x)=0,\ \forall x \in F\}$$

and

$$I^\perp = \{x \in K : f(x)=0,\ \forall f \in I\}.$$

A subset F of K is said to be *exposed* if there exists a function g in $F_\perp$ with $g \geqq 0$ and $g^{-1}(0)=F$, and F is said to be *semi-exposed* if for each $y \in K \setminus F$, there exists a function g in $F_\perp$ with $g \geqq 0$ and $g(y)>0$. It is clear that a set is semi-exposed if and only if it is the intersection of family of exposed sets. Evidently, every semi-exposed set is a face. When K is metrizable, the notions of semi-exposed faces and exposed faces coalesce (cf. [12]).

A closed face F of K is said to be *Archimedean* if $F=I^\perp$, where I is an order ideal of $A(K)$ satisfying:

(1) $A(K)/I$ is Archimedean in the quotient ordering,

(2) I is positively generated, viz., $I=(I \cap A(K)^+)-(I \cap A(K)^+)$. Every Archimedean face is semi-exposed (cf. [2, II, 5.17]).

LEMMA 1.1 (Alfsen [1]). *A closed face F of K is Archimedean if and only if for any $g \in Q(K)$ with $g|F \leqq a \in A(F; K)$, there exists c in $A(K)$ such that $g \leqq c$* and *$c|F=a$.*

LEMMA 1.2 (Ellis [12]). *Let F be a subset of K. Then F is semi-exposed if and only if $F=(F_\perp)^\perp$ and, given $f \in F_\perp$ and $\varepsilon>0$, there exists $g \in F_\perp$ with $g \geqq 0$ and $f \leqq g+\varepsilon$.*

COROLLARY 1.3. *If F is a semi-exposed face of K, then $F=(F_\perp \cap A(K)^+)^\perp$.*

PROOF. Plainly, $F \subseteqq (F_\perp \cap A(K)^+)^\perp$. Let $x \in (F_\perp \cap A(K)^+)^\perp$ and let $f \in F_\perp$. For each $\varepsilon>0$, there are functions g and h in $F_\perp \cap A(K)^+$ such that $-g-\varepsilon \leqq f \leqq h+\varepsilon$. It follows that $-\varepsilon \leqq f(x) \leqq \varepsilon$ since $g(x)=h(x)=0$. As ε was arbitrary, $f(x)=0$. As $f \in F_{\perp}$ was arbitrary, $x \in (F_{\perp})^{\perp}=F$. The proof is complete.

A face F of K is said to be *split* if there exists a face G of K, disjoint from F, such that each x in K has a unique decomposition $x=\lambda y+(1-\lambda)z$, where $y \in F$, $z \in G$ and $0 \leqq \lambda \leqq 1$. The collection of all sets $F \cap \partial K$, where F is a closed split face of K, satisfies the axioms of closed sets for a topology, weaker than the induced topology on ∂K, which is called the *facial topology* of ∂K. Moreover, ∂K is compact in the facial topology. Alfsen and Andersen [3] have shown that each function $a: \partial K \to R$ continuous in the facial topology can be extended uniquely to a continuous affine function on K. Thus the facially continuous functions on ∂K can be identified with a subspace Z of $A(K)$, termed the *centre* of $A(K)$. The centre Z consists of all functions f in $A(K)$ such that for every $g \in A(K)$, there exists $h \in A(K)$ satisfying $h(x)=f(x)g(x)$ for each x in ∂K (cf. [4]). Every closed split face is Archimedean. For the details of Archimedean and split faces, see [1], [3] and [23].

A compact convex set K is a simplex if and only if the space $A(K)$ has the so-called *Riesz interpolation property*, i.e., for any $f,g,h,k \in A(K)$, if $f,g \leqq h,k$, then there is a function a in $A(K)$ such that $f,g \leqq a \leqq h,k$. A compact simplex K is called a *Bauer simplex* if ∂K is closed.

The *Dirichlet problem* for a family of continuous functions on ∂K is to determine which functions can be extended to continuous affine functions on K. It turns out that the solvability of the Dirichlet problem characterizes the Bauer simplexes among the compact convex sets, more explicitly, a compact convex set is a Bauer simplex if and only if the Dirichlet problem for each continuous function on ∂K is solvable (cf. [15, Théorème 6.2]). Another characterization of Bauer simplexes is due to Alfsen and Andersen [3] who proved that a compact convex set K is a Bauer simplex if and only if the facial topology and the relative topology coincide on ∂K. Thus the centre of $A(K)$ is $A(K)$ itself whenever K is a Bauer simplex.

Let E be a normed linear space partially ordered by a cone. A positive linear functional f on E is called a *state* if $\|f\|=1$. The *state space* $S(E)$ is the set of all states, together with the relative weak topology $w(E',E)$. The extreme points of $S(E)$ are called the *pure states*.

2. Anti-lattices and prime sets.

Let K be a compact convex set in a locally convex space. Then K is said to be *prime* if the following condition is satisfied:

If $K = \mathrm{co}(F \cup G)$ for any two semi-exposed faces F and G, then either $K = F$ or $K = G$.

Plainly, the discs and n-gons ($n \geqq 5$) in the plane are prime.

A partially ordered linear space E is called an *anti-lattice* if for any two elements x and y in E, the lattice infimum $x \wedge y$ exists in E implies that either $x \wedge y = x$ or $x \wedge y = y$. Because of the equality $x \vee y = -(-x \wedge -y)$, we see that if $x \vee y$ exists in an anti-lattice, then necessarily $x \vee y = x$ or $x \vee y = y$. Thus an antilattice is a partially ordered linear space in which only the trivial lattice infima and suprema exist and the terminology for it is justified. Anti-lattices exist in profusion. For instance, the linear spaces of all real harmonic functions defined on the Euclidean spaces E^n ($n \geqq 2$) with the usual pointwise ordering are anti-lattices. This is a direct consequence of a result of Picard (cf. [16, Theorem 1.11]) which states that a real harmonic function on E^n is neither bounded above nor bounded below unless it is a constant. Another example of anti-lattice is the space of real polynomials on $[0, 1]$ with the usual pointwise ordering, which has the Riesz interpolation property and which is not a lattice. For a thorough discussion of anti-lattices, we refer to [13].

We shall call $A(K)$ a *quasi-anti-lattice* if the following condition is satisfied:

For any f and g in $A(K)$, $f \wedge g$ exists in $A(K)$ and $(f \wedge g)(x) = f(x) \wedge g(x)$ for each x in ∂K imply that either $f \leqq g$ or $g \leqq f$.

THEOREM 2.1. *Let K be a compact convex set in a locally convex space. Then the following statements are equivalent:*

(1) *K is prime.*

(2) *$A(K)$ is a quasi-anti-lattice.*

(3) *If $f, g \in C(\partial K)$ are any boundary data for which the Dirichlet problem in solvable, then it is not solvable for the data* $\max(f, g)$ *unless $f \leqq g$ or $g \leqq f$.*

(4) *If $f, g \in C(\partial K)$ are any boundary data for which the Dirichlet problem is solvable, then it is not solvable for the data* $\min(f, g)$ *unless $f \leqq g$ or $g \leqq f$.*

PROOF. The equivalence of (2), (3) and (4) can be easily verified.

(1) $\Rightarrow$ (2). Suppose the lattice infimum $f \wedge g$ of f and g in $A(K)$ exists and suppose $(f \wedge g)(x) = f(x) \wedge g(x)$ for each x in ∂K. Let

$$F = \{x \in K : f(x) = (f \wedge g)(x)\} \quad \text{and} \quad G = \{x \in K : g(x) = (f \wedge g)(x)\}.$$

Then F and G are exposed faces and $\partial K = \partial F \cup \partial G$. It follows that $K = \mathrm{co}(F \cup G)$ and hence $K = F$ or $K = G$, in other words, either $f \leqq g$ or $g \leqq f$.

(2) $\Rightarrow$ (1). We assume *ad absurdum* that there are two proper semi-exposed faces F and G of K such that $K = \mathrm{co}(F \cup G)$. Then $\partial K = \partial F \cup \partial G$ and we can find non-negative f and g in $A(K)$ such that $f(F) = g(G) = 0$ with $f \nleqq g$ and $g \nleqq f$. Evidently, $f \wedge g$ exists and is equal to 0. Moreover, for each x in ∂K, $(f \wedge g)(x) = 0 = f(x) \wedge g(x)$ since $\partial K = \partial F \cup \partial G$ and $f(F) = g(G) = 0$. Therefore $f \leqq g$ or $g \leqq f$ which contradicts the definitions of f and g. So we conclude that K is prime.

In order to seek an equivalent condition for $A(K)$ being an anti-lattice, we introduce the following definition.

A compact convex set K is said to be *ℓ-stable* if the following condition is satisfied:

For any f and g in $A(K)$, if the lattice infimum $f \wedge g$ exists in $A(K)$, then $(f \wedge g)(x) = f(x) \wedge g(x)$ for each x in ∂K.

The following theorem is immediate from Theorem 2.1.

THEOREM 2.2. *Let K be a compact convex set in a locally convex space. Then the following statements are equivalent*:

(1) *$A(K)$ is an anti-lattice.*

(2) *K is prime and ℓ-stable.*

Taylor [24] has shown that simplexes are ℓ-stable and hence the preceding theorem generalizes a result of Effros and Kazdan [11, Theorem 2.7] asserting the duality of prime simplexes and anti-lattices. We now extend Taylor's result to those compact convex sets with Archimedean extreme points.

LEMMA 2.3. *Let K be a compact convex set and let $x \in \partial K$ be Archimedean. Suppose the lattice supremum $f \vee g$ exists for f and g in $A(K)$. Then $(f \vee g)(x) = f(x) \vee g(x)$.*

PROOF. Certainly, $f(x) \vee g(x) \leqq (f \vee g)(x)$. Suppose $f(x) \vee g(x) < \alpha < (f \vee g)(x)$ for some real number α. By Lemma 1.1, there exists an h in $A(K)$ such that $\max(f, g) \leqq h$ and $h(x) = \alpha$. It follows that $f \vee g \leqq h$ and in particular, $\alpha < (f \vee g)(x) \leqq h(x) = \alpha$ which is absurd. Therefore $f(x) \vee g(x) = (f \vee g)(x)$.

Asimow [5] has proved that each extreme point of the state space of a function algebra is split and from Lemma 2.3, this state space is ℓ-stable. Thus we have a good many examples of ℓ-stable sets which are not simplexes. We shall give an example of a prime state space of a function algebra in section 3.

COROLLARY 2.4. *Let K be a compact convex set in which every extreme point is Archimedean. Then K is ℓ-stable and the following statements are equivalent:*

(1) *K is prime.*

(2) *$A(K)$ is an anti-lattice.*

Effros and Kazdan [11] have shown that the state space arising from the solutions to Laplace's equation is prime if and only if its extreme boundary is not closed. Nevertheless, this is not the case in general. For example, the state space K of

$$A = \{f \in C[0,1] : f(n^{-1}) = (1-n^{-1})f(0)+n^{-1}f(1),\ n=2,3,\dots\}$$

is a simplex and the extreme boundary can be identified with

$$[0,1] \setminus \{n^{-1} :\ n \geqq 2\}$$

which is not closed (cf. [10, p. 388]) and yet K is not prime. In fact, A can be represented as the space $A(K)$ and A is not an anti-lattice for if we consider the function f in A defined by

$$\begin{aligned} f(x) &= 1 && \text{if } 0 \leqq x \leqq \tfrac{1}{2}, \\ &= 8|x-\tfrac{3}{4}|-1 && \text{if } \tfrac{1}{2} \leqq x \leqq 1, \end{aligned}$$

then $f \vee 0$ exists in A while $f \nleqq 0$ and $0 \nleqq f$.
However, if we consider a stronger condition, namely, the density condition, then it turns out to be a sufficient condition for being a prime set. This settles the question, communicated to me by D. A. Edwards in a conversation, whether a simplex with dense extreme points is prime.

PROPOSITION 2.5. *Let K be a compact convex set in which the extreme points are dense. Then K is prime.*

PROOF. We prove that $A(K)$ is a quasi-anti-lattice. Suppose $f \wedge g$ exists in $A(K)$ for f and g in $A(K)$ and suppose $(f \wedge g)(x) = f(x) \wedge g(x)$ for each x in ∂K. From the density of ∂K and the continuity of $f \wedge g$, we see that $f \wedge g$ is the pointwise minimum of f and g. It follows from the affineness of f and g that either $f \leqq g$ or $g \leqq f$. Therefore $A(K)$ is a quasi-anti-lattice and K is prime by Theorem 2.1.

REMARK. For examples of compact convex sets with dense extreme boundaries, see [20], [21] and [8, 11.2.4].

It is evident that the converse of the above result is not true in general, for instance, the discs are prime while their extreme boundaries are not dense. Even in the case of simplexes, the converse of Proposition 2.5 is false. The following example is due to Effros and Kazdan [11, section 5].

EXAMPLE. Let Ω be the open unit disc in the complex plane endowed with the usual topology $\mathcal{U}$. Define a *bundle of functions* $\mathcal{H}$ on Ω by

$$U \in \mathcal{U} \mapsto \mathcal{H}_U ,$$

where $\mathcal{H}_U$ is the family of harmonic functions in U. Then $(\Omega,\mathcal{H},\mathcal{U})$ is a *harmonic space* satisfying the *axiom of domination* (for definitions, see [6, section 1] and [7, section 1]). Let

$$C_n = \{r_n e^{i\theta} : r_n = 2^{-(n+1)} + 2^{-(n+2)},\ 0 \leqq \theta \leqq 4 \exp(-n^3)\}$$

for $n = 1, 2, \ldots$, and let

$$\omega = \Omega \setminus \left(\bigcup_{n=1}^{\infty} C_n \cup \{0\}\right) .$$

Let $A(\omega)$ be the functions continuous on $\bar{\omega}$ and harmonic in ω and let $SA(\omega)$ be the state space of $A(\omega)$. Then $SA(\omega)$ is a prime simplex since the Choquet boundary for $A(\omega)$ is not closed [11, Theorem 3.9] and it is clear that $\partial SA(\omega) \subseteqq \bar{\omega}$ is not dense in $\partial SA(\omega)$.

We now exhibit some simple properties of prime sets.

PROPOSITION 2.6. *Let K be a compact convex set. Then the statements below are related as follows:* (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Leftrightarrow$ (7).

(1) *K is prime.*
(2) *Any two nonempty facially open sets in ∂K intersect.*
(3) *Every nonempty facially open set is dense in ∂K.*
(4) *Every facially open set in ∂K is connected.*
(5) *The centre of $A(K)$ consists of constant functions.*
(6) *The facial topology for ∂K is connected.*
(7) *The centre Z of $A(K)$ does not admit nontrivial idempotents, that is, $f \in Z$ with $f = f^2$ only if $f = 0$ or $f = 1$.*

PROOF. Straightforward topological arguments.

REMARK. Since the centre of $A(K)$ is $A(K)$ itself whenever K is a Bauer simplex, we conclude that a Bauer simplex is never prime unless it degenerates to a point.

PROPOSITION 2.7. *Let K be a nonvoid compact convex set in the Euclidean space E^n. Then there are prime faces $F_1,\dots,F_m$ of K such that $K=\mathrm{co}(F_1,\dots,F_m)$.*

PROOF. We prove by induction on the dimension n. The case $n=1$ is trivial, since K is either a point or a closed interval. In general, if K is not prime, then $K=\mathrm{co}(F_1\cup F_2)$ for some proper exposed faces F_1 and F_2. Since $F_i=K\cap \mathrm{lin} F_i$ $(i=1,2)$, where $\mathrm{lin} F_i$ is the linear span of F_i, the dimensions of F_i $(i=1,2)$ are less than that of K and hence we can use the induction hypothesis to conclude the proof.

Now suppose K is a compact convex set and $0\in\partial K$. Let

$$A_0(K) = \{f\in A(K) : f(0)=0\}$$

and let $C_0(\partial K)$ be the continuous functions on ∂K vanishing at 0. We say K is *0-prime* if the following condition is satisfied:

If $K=\mathrm{co}(F\cup G)$ for any two semi-exposed faces F and G with $0\in F\cap G$, then either $K=F$ or $K=G$.

Analogously, we have the following results.

PROPOSITION 2.8. *Let K be a compact convex set with $0\in\partial K$. Then the following statements are equivalent:*

(1) *K is 0-prime.*

(2) *$A_0(K)$ is a quasi-anti-lattice.*

(3) *If $f,g\in C_0(\partial K)$ are any boundary data for which the Dirichlet problem is solvable, then it is not solvable for the data* $\max(f,g)$ *unless $f\leqq g$ or $g\leqq f$.*

PROPOSITION 2.9. *Let K be a compact convex set with $0\in\partial K$. Then $A_0(K)$ is an anti-lattice if and only if K is 0-prime and ℓ-stable.*

EXAMPLE. Let Ω be the open unit disc in the complex plane and let

$$H_0 = \{f\in C_{\mathbf{R}}(\bar\Omega) : f \text{ is harmonic in } \Omega \text{ and } f(0)=0\}\,.$$

Let K be the positive part of the closed unit ball in the dual space of H_0. Then K is w^*-compact and convex and, H_0 is isometrically order-isomorphic to $A_0(K)$ through the natural evaluation map. Using the mean value property and the Poisson representation of the harmonic functions, one can show that H_0 is an anti-lattice.

3. Function algebras.

Let X be a compact Hausdorff space and $C(X)$ the algebra of all complex-valued continuous functions on X. Recall that a function algebra on X is a closed subalgebra of $C(X)$ which separates the points of X and contains the constants.

Let A be a function algebra on X and let K be its state space. Then $A(K)$ coincides naturally with the uniform closure of $\mathrm{re}\,A$ since $\mathrm{re}\,A$ separates the points of K and contains the constants. We have stated Asimow's result [5] that every extreme point of K is split. Thus, applying Corollary 2.4, we obtain the following result.

PROPOSITION 3.1. *Let A be a function algebra and K the state space. Then $\overline{\mathrm{re}\,A}$ is an anti-lattice if and only if K is prime.*

EXAMPLE. Let $\mathscr{A}$ be the disc algebra and let

$$A = \{f \in \mathscr{A} : f(0) = f(1)\}.$$

Then A is a function algebra of which the state space K is prime. In fact, by identifying $\overline{\mathrm{re}\,A}$ with the space $A(K)$, we can show that $\overline{\mathrm{re}\,A}$ is an anti-lattice. Suppose there is an u in $\overline{\mathrm{re}\,A}$ such that $u \wedge 0$ exists in $\overline{\mathrm{re}\,A}$. Then $(u\wedge 0)(\varrho) = u(\varrho)\wedge 0$ for each ϱ in ∂K because K is ℓ-stable. Note that the Choquet boundary for A is

$$\{z \in \mathbf{C} : |z| = 1\} \setminus \{1\}$$

and can be identified with ∂K through the usual evaluation map (cf. [19, p. 54]). It follows from the continuity of $u \wedge 0$ that $(u\wedge 0)(1) = u(1)\wedge 0$. Hence we have $(u\wedge 0)(0) = (u\wedge 0)(1) = u(1)\wedge 0 = u(0)\wedge 0$. By the mean value property, this is possible only when $u \leqq 0$ or $0 \leqq u$. Therefore $\overline{\mathrm{re}\,A}$ is an anti-lattice.

It is straightforward to show that if $\overline{\mathrm{re}\,A}$ is an anti-lattice, then so is $\mathrm{re}\,A$. However, we have been unable to determine whether the converse is true or not.

4. C^*-algebras.

Let A be a C^*-algebra with identity and let A_{sa} be the real linear space of self-adjoint elements of A, partially ordered by the cone $A^+ = \{a^*a : a \in A\}$. Let $S(A)$ be the state space of A. Then A_{sa} is isometrically order-isomorphic to the space $A(S(A))$ via the map $a \mapsto \tilde{a}$, where $\tilde{a}$ is defined by $\tilde{a}(p) = p(a)$ for each p in $S(A)$.

By the *annihilator* in $S(A)$ of a subset M of A we mean the set

$$M^\circ = \{p \in S(A) : p(a)=0, \ \forall a \in M\},$$

and if $F \subseteqq S(A)$, we let

$$F_0 = \{a \in A : p(a)=0, \ \forall p \in F\}.$$

Notice that $F_0 \cap A_{sa} = F_\perp$ if we identify A_{sa} with $A(S(A))$.

We first investigate the nature of semi-exposed faces in the state space of a C^*-algebra.

PROPOSITION 4.1. *Let A be a C^*-algebra with identity. Then the following statements are equivalent*:

(1) *F is a semi-exposed face in $S(A)$.*

(2) *$F = N^\circ$ for some left ideal N in A.*

PROOF. (1) $\Rightarrow$ (2). Since F is semi-exposed, we have from Corollary 1.3

$$F = (F_0 \cap A^+)^\circ = (F_0)^\circ.$$

Let $N_F = \{a \in A : a^*a \in F_0\}$. Then the inequalities

$$(a+b)^*(a+b) \leqq 2(a^*a+b^*b)$$

and

$$(ba)^*(ba) \leqq \|b\|^2 a^*a$$

show that N_F is a left ideal (cf. [18, Lemma 1.1]).

We prove that $F_0 \cap A^+ \subseteqq N_F \subseteqq F_0$. First, let $a \in F_0 \cap A^+$. Then there is a $\lambda > 0$ such that $a^2 \leqq \lambda a$. So $0 \leqq p(a^*a) = p(a^2) \leqq \lambda p(a) = 0$ for each p in F. Hence $a^*a \in F_0$ and $a \in N_F$. Now suppose $b \in N_F$. Then

$$|p(b)|^2 \leqq p(1)p(b^*b) = 0$$

for each p in F. So $b \in F_0$. It follows that $(F_0 \cap A^+)^\circ \supseteqq N_F{}^\circ \supseteqq (F_0)^\circ$ and hence $F = N_F{}^\circ$.

(2) $\Rightarrow$ (1). Let N be a left ideal in A and let $p \in S(A) \setminus N^\circ$. Then there is an element a in N such that $p(a) \neq 0$. So we have $p(a^*a) > 0$ for otherwise the Cauchy–Schwarz inequality would give $p(a)=0$. Since N is a left ideal, $a^*a \in N \cap A^+$ and so

$$(a^*a)^\sim \in A(S(A)), \quad (a^*a)^\sim(N^\circ) = 0, \quad (a^*a)^\sim(p) > 0.$$

This shows that N° is semi-exposed.

PROPOSITION 4.2. *Let A be a C^*-algebra with identity and let $F \subseteqq S(A)$ be a semi-exposed face such that $F_0 \cap A_{sa}$ is positively generated. Then F is the annihilator of a closed two-sided ideal.*

PROOF. We have $F_0 \cap A^+ \subseteq N_F \subseteq F_0$, where

$$N_F = \{a \in A : a^*a \in F_0\}$$

is a closed left ideal. From this and from the fact that $F_0 \cap A_{sa}$ is positively generated, we obtain $F_0 \cap A_{sa} \subseteq N_F$. It follows that $F_0 \subseteq N_F$ and so $F_0 = N_F$ is a left ideal. Similarly, $F_0 = N_F{}^*$ is a right ideal. Therefore $F = (F_0)^\circ$ is the annihilator of a closed two-sided ideal.

The following familiar result shows that the positively generating condition in Proposition 4.2 can not be dropped.

PROPOSITION 4.3. *Let N be a closed left ideal in a C^*-algebra with identity. Then the following statements are equivalent:*
(1) *N is a two-sided ideal.*
(2) *$(N^\circ)_0 \cap A_{sa}$ is positively generated.*
(3) *$N = (N^\circ)_0$.*
(4) *$N = N^*$.*

PROOF. The implications (3) ⇒ (4) ⇒ (1) are obvious.
(1) ⇒ (2) since N is a two-sided ideal, $N = (N^\circ)_0$ and $(N^\circ)_0 \cap A_{sa} = N \cap A_{sa}$ is positively generated (cf. [22, Theorem 2, Lemma 2.3]).
(2) ⇒ (3). Since N is a closed left ideal, we have $N^\circ = (N \cap A^+)^\circ$, because for any p in $(N \cap A^+)^\circ$ and for any a in N, we have $a^*a \in N \cap A^+$ and $|p(a)|^2 \leqq p(1)p(a^*a) = 0$. Moreover,

$$N \cap A^+ = ((N \cap A^+)^\circ)_0 \cap A^+$$

[9, Theorem 2.5]. It follows that

$$N \cap A^+ = (N^\circ)_0 \cap A^+ .$$

Now condition (2) yields $N = (N^\circ)_0$.

A face F of $S(A)$ is called *invariant* if $p_a \in F$ whenever $p \in F$, where

$$p_a(b) = p(a^*a)^{-1}p(a^*ba) \quad (p(a^*a) \neq 0) .$$

The equivalence of (1), (2), (4) and (5) of the following proposition was proved in [3, Proposition 7.1].

PROPOSITION 4.4. *Let A be a C^*-algebra with identity and let F be a closed face of $S(A)$. Then the following statements are equivalent:*
(1) *F is invariant.*
(2) *F is Archimedean.*
(3) *F is semi-exposed and $F_0 \cap A_{sa}$ is positively generated.*

(4) *F is the annihilator of a closed two-sided ideal.*
(5) *F is a split face.*

PROOF. (2) $\Rightarrow$ (3). Cf. [2, II.5.17].
(3) $\Rightarrow$ (4). Proposition 4.2.

We should remark here that the property of a closed face F of a compact convex set K being semi-exposed and $F_\perp \subseteqq A(K)$ being positively generated does not necessarily imply that F is Archimedean, although this is the case in the setting of C^*-algebras.

The following example is due to Ng Kung-Fu (cf. [12, Example b]). Let

$$K = \{(\alpha,\beta) \in \mathrm{R}^2 : (\alpha-1)^2+(\beta-1)^2 \leqq 1\} \cup \{(\alpha,\beta) \in \mathrm{R}^2 : 0 \leqq \alpha,\beta \leqq 1\}$$

and let

$$F = \{(\alpha,0) \in K : 0 \leqq \alpha \leqq 1\} .$$

Then $F=(F_\perp)^\perp$ and $F_\perp$ is positively generated while $A(K)/F_\perp$ is not Archimedean ordered.

We now set up a necessary and sufficient condition for the annihilator of a closed two-sided ideal to be a prime face.

THEOREM 4.5. *Let A be a C^*-algebra with identity and let I be a closed two-sided ideal. Then the following statements are equivalent:*
(1) *I° is a prime face.*
(2) *If $N \cap M \subseteqq I$ where N and M are left ideals such that $(N\cap M)^\circ = \mathrm{co}(N^\circ \cup M^\circ)$, then either $N \subseteqq I$ or $M \subseteqq I$.*

PROOF. (1) $\Rightarrow$ (2). Suppose $N \cap M \subseteqq I$, where N and M are left ideals satisfying $(N\cap M)^\circ = \mathrm{co}(N^\circ \cup M^\circ)$. Then

$$I^\circ = \mathrm{co}\big((N^\circ \cap I^\circ) \cup (M^\circ \cap I^\circ)\big)$$

with $N^\circ \cap I^\circ$ and $M^\circ \cap I^\circ$ being semi-exposed faces in I°. As I° is prime, we have either $I^\circ = N^\circ \cap I^\circ$ or $I^\circ = M^\circ \cap I^\circ$. It follows that either $N \subseteqq (N^\circ)_0 \subseteqq I$ or $M \subseteqq (M^\circ)_0 \subseteqq I$.

(2) $\Rightarrow$ (1). Suppose $I^\circ = \mathrm{co}(F \cup G)$, where F and G are semi-exposed faces of I°. Since I° is split, F and G are semi-exposed in $S(A)$. Furthermore, $I = (I^\circ)_0 = F_0 \cap G_0$. Let

$$N_F = \{a \in A : a^*a \in F_0\} \quad \text{and} \quad M_G = \{a \in A : a^*a \in G_0\}$$

be the aforecited left ideals. Then $F = N_F{}^\circ$, $G = M_G{}^\circ$ and

$$I \cap A^+ = F_0 \cap G_0 \cap A^+ \subseteqq N_F \cap M_G \subseteqq F_0 \cap G_0 = I .$$

Hence $I=N_F \cap M_G$ and

$$(N_F \cap M_G)^\circ = I^\circ = \mathrm{co}(F \cup G) = \mathrm{co}(N_F{}^\circ \cup M_G{}^\circ) .$$

Thus by assumption, either $N_F \subseteqq I$ or $M_G \subseteqq I$. This entails either $I^\circ \subseteqq N_F{}^\circ = F$ or $I^\circ \subseteqq M_G{}^\circ = G$. Therefore either $I^\circ = F$ or $I^\circ = G$ which concludes the proof.

Recall that a two-sided ideal I in a C^*-algebra is said to be *prime* if either $I_1 \subseteqq I$ or $I_2 \subseteqq I$ whenever $I_1 I_2 \subseteqq I$ for any two-sided ideals I_1 and I_2 in A. We call A a *prime C^*-algebra* if the ideal (0) is prime.

If I and J are any closed two-sided ideals in A, then we have $IJ = I \cap J$ (cf. [8, 1.9.12]) and $(I \cap J)^\circ = \mathrm{co}(I^\circ \cup J^\circ)$ [22, Theorem 5].

COROLLARY 4.6. *If I is a closed two-sided ideal such that I° is prime, then I is a prime ideal.*

We have been unable to resolve whether the converse holds though we can prove it in some particular circumstance (cf. Theorem 4.9).

The following lemma is known and we give a proof for the sake of completeness.

LEMMA 4.7. *Let $\mathfrak{Z}$ be the algebraic centre of a C^*-algebra A with identity and let Z be the centre of $A(S(A))$. Then $\mathfrak{Z} = Z + iZ$.*

PROOF. Since $\mathfrak{Z} = (\mathfrak{Z} \cap A_{\mathrm{sa}}) + i(\mathfrak{Z} \cap A_{\mathrm{sa}})$, it suffices to show that $\mathfrak{Z} \cap A_{\mathrm{sa}} = Z$.

Notice that if a and b are any two positive elements commuting with each other, then ab is positive as well. Using Kadison's technique in the proof of [17, Lemma 3.2], one can show that if $z \in \mathfrak{Z} \cap A_{\mathrm{sa}}$, then $p(az) = p(a)p(z)$ for each pure state p and each element a in A. From this it follows that $\mathfrak{Z} \cap A_{\mathrm{sa}} \subseteqq Z$.

Now let $a \in Z$. Define a hull-kernel continuous real function f on the primitive ideal space $\mathrm{Prim}(A)$ by

$$f(\ker \pi_p) = p(a)$$

where π_p is the irreducible representation induced by the pure state p (cf. [8,2.5.4], [3, Theorem 7.6]). The Dauns–Hofmann Theorem (cf. [3, Theorem 7.6]) gives $a - f(\ker \pi_p) \in \ker \pi_p$ for each pure state p. Take $b \in A$. Then $ab - f(\ker \pi_p)b$ and $ba - f(\ker \pi_p)b$ belong to $\ker \pi_p$ for each pure state p. It follows that

$$ab - ba \in \bigcap_{p \in \partial S(A)} \ker \pi_p$$

and hence $ab = ba$. As b was arbitrary, a is central. The proof is complete.

PROPOSITION 4.8. *Suppose the state space of a C^*-algebra A with identity is prime. Then the centre of A consists of scalar multiples of the identity.*

PROOF. From Lemma 4.7 and Proposition 2.6.

A C^*-algebra is called an NGCR algebra if it does not contain any nonzero CCR ideal or what is equivalent, if it does not contain any nonzero GCR ideal (cf. [8, 4.2, 4.3]).

Tomiyama and Takesaki [25, Theorem 2] have proved that a C^*-algebra with identity is a prime NGCR algebra if and only if the pure states are dense in the state space.

THEOREM 4.9. *Let A be an* NGCR *algebra with identity. Then the following statements are equivalent:*

(1) *A is a prime algebra.*
(2) *$\partial S(A)$ is w^*-dense in $S(A)$.*
(3) *$S(A)$ is a prime set.*
(4) *$A_{sa}=A(S(A))$ is a quasi-anti-lattice.*

PROOF. (1) $\Rightarrow$ (2). Cf. [25, Theorem 2], [8, 11.2.4] and [14, p. 231–232].
(2) $\Rightarrow$ (3). Proposition 2.5.
(3) $\Rightarrow$ (1). Corollary 4.6.
(3) $\Leftrightarrow$ (4). Theorem 2.1.

We mention an example. Let H be a separable infinite dimensional Hilbert space and let $\mathscr{L}(H)$ (resp. $\mathscr{L}C(H)$) be the algebra of bounded operators (resp. compact operators) on H. Then $\mathscr{L}(H)/\mathscr{L}C(H)$ is a prime NGCR algebra with identity which is not separable and not isomorphic to a von Neumann algebra (cf. [8, 4.7.22]).

I am much indebted to Dr. A. J. Ellis for introducing me to Effros and Kazdan's paper from which the idea of prime sets stems, and for his valuable suggestions and comments.

ADDED IN PROOF. The converse of Corollary 4.6 has been proved by the author in a paper, *Prime faces in C^*-algebras* (submitted to Bull. London Math. Soc.). In fact, the following conditions are equivalent for a C^*-algebra A with identity: (1) A is a prime algebra. (2) $S(A)$ is a prime compact convex. (3) A_{sa} is an antilattice. Also, one can show that $S(A)$ is ℓ-stable.

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UNIVERSITY COLLEGE, SWANSEA, BRITAIN