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UNIFORM APPROXIMATION ON MANIFOLDS

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1. Introduction.

Suppose that M is a real C^1-manifold of dimension m, and that Φ is a family of complex-valued C^1-functions on M. Then *the exceptional set*, $E(\Phi)$, is the set

$$\{x \in M \ ; \ df_1 \wedge \ldots \wedge df_m(x) = 0, \ \forall (f_1, \ldots, f_m) \in \Phi^m\} \ .$$

We fix a compact subset X of M, and we shall often write E instead of $E(\Phi) \cap X$.

Let $A \subset C(X)$ denote the closed Banach-algebra generated by the restriction to X of the elements of Φ. Assume that A separates points in X and that $M_A = X$, where M_A is the maximal ideal space of A. It is an open problem, see [1, pp. 348–349], if A includes all continuous functions on X which vanish identically on E. Michael Freeman proved this in [2] under the additional hypothesis that both M and the functions in Φ are real-analytic. In this work we will solve the problem if M and the functions in Φ are of class C^r, for some sufficiently large real r.

Our result will be proved via the following corollary of theorem 3.1: If Σ is a C^r-manifold in $\mathbf{C}^n$ without complex tangents (see [4] for the precise meaning of the last term) and $K = \sigma(f_1, \ldots, f_n)$ is the spectrum of some members of A, then all continuous functions on K which vanish on $K - \Sigma$ operate on A.

The proof will follow by adaptation of a technique developed in the work of Hörmander and Wermer [4].

2. Fundamental constructions.

Assume that $r \geqq 1$ and that Σ is a closed, real C^r-sub-manifold, without complex tangents, of an open set Ω in $\mathbf{C}^n$. Let N_1 and N_2 be some open sets in $\mathbf{C}^n, \bar{N}_2 \subset N_1$.

The Euclidean distance between the point x and the set A will be denoted $d(x, A)$.

LEMMA 2.1. *Suppose that $u \in C^r(\Omega \cup N_1)$ is holomorphic in N_1. Then there exists a $v \in C^r(\Omega \cup N_2)$ with $v = u$ on $\Sigma \cup N_2$ and such that*:

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For every compact $F \subset \Omega \cup N_2$ and every $\eta > 0$ we can find a $\delta > 0$ with the property:

If $z \in F$ and $d(z,\Sigma) < \delta$, then $|\bar{\partial} v(z)| \leqq \eta\, d(z,\Sigma)^{r-1}$.

PROOF. This lemma is a restatement of lemma 4.3 in [4].

The next result is similar to theorem 3.1 in [4]. However, since the proof is a bit different, we will carry it out in some detail.

Consider Σ, Ω, N_1 and N_2 as above with $r=1$. Suppose A is a commutative Banach algebra with unit, and let $f_1,\ldots,f_n$ be elements of A. Define K to be the joint spectrum $\sigma(f_1,\ldots,f_n)$.

LEMMA 2.2. *Assume $K - N_2 \subset \Sigma$. Then there exist real numbers $\varepsilon_0 > 0$ and $t \in \langle 0,1\rangle$, elements $f_{n+1},\ldots,f_m \in A$, a compact set $F \subset \Sigma$, and, for every $\varepsilon \in \langle 0,\varepsilon_0\rangle$, a domain of holomorphy $\omega_\varepsilon \in \mathbf{C}^m$ such that*

(i) $\sigma(f_1,\ldots,f_m) \subset \omega_\varepsilon \subset \mathbf{C}^n \times \{|(z_{n+1},\ldots,z_m)| < 1/t\}$,

(ii) *if $z \in \mathbf{C}^m$ and*

$$d\big((z_1,\ldots,z_n,\varepsilon z_{n+1},\ldots,\varepsilon z_m),\sigma(f_1,\ldots,f_n,\varepsilon f_{n+1},\ldots,\varepsilon f_m)\big) < t\varepsilon ,$$

then $z \in \omega_\varepsilon$,

(iii) *if $z \in \omega_\varepsilon - (N_1 \times \mathbf{C}^{m-n})$, then $d\big((z_1,\ldots,z_n),F\big) < \varepsilon/t$.*

PROOF. Let N_3 be an open set such that $\bar{N}_2 \subset N_3 \subset \bar{N}_3 \subset N_1$. By applying the proof of theorem 3.1 in [4], we get:

An open set V such that $K \cap (\bar{N}_3 - N_2) \subset V \Subset N_1$, real numbers $\varepsilon_0 > 0$ and $t_1 \in \langle 0,1\rangle$, a compact set $F \subset \Sigma$, and, for every $\varepsilon \in \langle 0,\varepsilon_0\rangle$, a domain of holomorphy v_ε such that

(i) $\big(\sigma(f_1,\ldots,f_n) - N_2\big) \cup V \subset v_\varepsilon$,

(ii) if $z \in \mathbf{C}^n$ and

$$d\big((z_1,\ldots,z_n),\sigma(f_1,\ldots,f_n) - N_2\big) < t_1\varepsilon , \tag{2.1}$$

then $z \in v_\varepsilon$,

(iii) if $z \in v_\varepsilon - N_1$, then $d\big((z_1,\ldots,z_n),F\big) < \varepsilon/t_1$.

To proceed, we must now study $K \cap \bar{N}_2$. This was done in [4] by imposing a holomorphic convexity condition on this part of K. Instead, we will use the fact that K is the spectrum of elements from a Banach algebra; therefore the well known Arens–Calderón theorem applies here.

Obviously, $V_1 = (\mathbf{C}^n - \bar{N}_3) \cup V \cup N_2$ is an open cover of K. Consequently, we can find an open holomorphically convex set $U \Subset \mathbf{C}^m$ and elements $f_{n+1},\ldots,f_m \in A$ such that

$$\sigma(f_1,\ldots,f_m) \subset U \subset V_1 \times \mathbf{C}^{m-n} . \tag{2.2}$$

We can now define the required ω_ε for every $\varepsilon \in \langle 0, \varepsilon_0 \rangle$. Define

$$O_1 = (\mathbf{C}^n - \bar{N}_2) \times \mathbf{C}^{m-n} \quad \text{and} \quad O_2 = N_3 \times \mathbf{C}^{m-n}.$$

Since $O_1 \cup O_2 = \mathbf{C}^m$, it is enough to specify $\omega_\varepsilon \cap O_1$ and $\omega_\varepsilon \cap O_2$. We define

(i) $\omega_\varepsilon \cap O_1 = (v_\varepsilon \times \mathbf{C}^{m-n}) \cap U \cap O_1$,
(ii) $\omega_\varepsilon \cap O_2 = U \cap O_2$.

Now we must prove that (i) and (ii) agree on $O_1 \cap O_2$. It suffices to prove that $U \cap O_1 \cap O_2 \subset v_\varepsilon \times \mathbf{C}^{m-n}$.

Since

$$U \cap O_1 \cap O_2 \subset (V_1 \cap (N_3 - \bar{N}_2)) \times \mathbf{C}^{m-n}$$

by (2.2) and since $(V_1 \cap (N_3 - \bar{N}_2)) \subset V$ by the definition of V_1, we have

$$U \cap O_1 \cap O_2 \subset v_\varepsilon \times \mathbf{C}^{m-n}$$

by (i) in (2.1).

It is easy to check that (i), (ii), and (iii) of the lemma now hold, if we choose $t > 0$ small enough.

By applying a technique introduced by Nachbin in [5], we shall now determine a class of mappings of a manifold into $\mathbf{C}^n$. More precisely, let M be a k-dimensional real C^r-manifold, $r \geqq 1$. Suppose X is a compact subset of M. Assume further that $\Phi \subset C^r{}_{\mathbf{C}}(M)$ and separates points in X. Let E denote the exceptional set $E(\Phi) \cap X$.

Lemma 2.3. *For every compact set $X_0 \subset X - E$ we can find an open neighbourhood V of X_0, a finite number of functions, $f_1, \ldots, f_n \in \Phi$, and an open $\Omega \subset \mathbf{C}^n$ such that*

(i) *$(f_1, \ldots, f_n)(V)$ is a closed C^r-submanifold of Ω of dimension k and without complex tangents, and*

(ii) *$(f_1, \ldots, f_n)(X - V) \subset \mathbf{C}^n - \Omega$.*

Proof. Choose a finite number of functions $f_1, \ldots, f_n \in \Phi$ and an open neighbourhood V of X_0 with $E(\{f_1, \ldots, f_n\}) \cap V = \emptyset$. It follows from the inverse mapping theorem that the multiple function $(f_1, \ldots, f_n) : M \to \mathbf{C}^n$ is locally 1-1 on V. Obviously, the set

$$\{(x,y) \in (X_0 \times X_0) - \Delta \; ; \; f_i(x) = f_i(y), \; \forall i = 1, \ldots, n\}$$

is a compact subset of $X_0 \times X_0$, where Δ denotes the diagonal in $X_0 \times X_0$. Consequently, by adding some more functions if necessary, we may assume that $\{f_1, \ldots, f_n\}$ separate points in X_0. Shrinking V if necessary, we then get that $(f_1, \ldots, f_n)$ is 1-1 on $\bar{V}$ and that $\bar{V}$ is compact.

Since Φ separates points in X, we can also suppose, after further modifications, that $(f_1,\dots,f_n)(X-V)$ and $(f_1,\dots,f_n)(V)$ are disjoint.

The choice $\Omega = \mathbf{C}^n - (f_1,\dots,f_n)(X-V)$ finishes the proof.

3. Approximation theorems.

Let X be a compact Hausdorff space, and let $C(X)$ denote the Banach space under the supremum norm of continuous complex-valued functions. The notation $A \propto C(X)$ means that A is a closed linear subspace which is closed under pointwise multiplication, separates points and contains the constant functions.

If K is a compact subset of $\mathbf{C}^n$, then $A(K)$ is defined to be the class of continuous, complex-valued functions on K which can be uniformly approximated on K by functions holomorphic in a neighbourhood of K.

THEOREM 3.1. *Suppose $A \propto C(X)$, where X is a compact Hausdorff space. Let Σ be a closed k-dimensional submanifold of an open set $\Omega \subset \mathbf{C}^n$, without complex tangents, and of class C^r, $r = \frac{1}{2}k + 1$. Choose $f_1,\dots,f_n \in A$ and define $K = \sigma(f_1,\dots,f_n)$, $K_0 = \overline{K - \Sigma}$.*

If $u \in C(K)$ with $u_{|K_0} \in A(K_0)$, then $u \circ (f_1,\dots,f_n) \in A$.

REMARK. We can replace the condition $r = \frac{1}{2}k + 1$ with $r = \max\{\frac{1}{2}k, 1\}$, but the proof will then be more involved. More specifically, we need a stronger version of lemma 2.2.

PROOF OF THEOREM 3.1. Evidently, we may assume that $u \in C^r(\Omega \cup N^1)$ for some open neighbourhood N_1 of K_0, and also that u is holomorphic in N_1. We will further suppose that u has been modified as described in lemma 2.1. If N_2 is chosen as an open set with $K_0 \subset N_2 \Subset N_1$, we can apply lemma 2.2.

With notations as in lemma 2.2, we will define

$$\omega'_\varepsilon = \{(z_1,\dots,z_n,\varepsilon z_{n+1},\dots,\varepsilon z_m)\ ;\ (z_1,\dots,z_m) \in \omega_\varepsilon\}\ ,$$

and also

$$v_\varepsilon :\ \omega'_\varepsilon \to \mathbf{C} \text{ by } (z_1,\dots,z_m) \mapsto u(z_1,\dots,z_n)\ .$$

Condition (iii) in lemma 2.2 ensures that v_ε is well-defined in ω'_ε for all small enough $\varepsilon > 0$.

With this notation, lemma 2.2 says:

(i) $\sigma(f_1,\dots,f_n,\varepsilon f_{n+1},\dots,\varepsilon f_m) \subset \omega'_\varepsilon \subset \mathbf{C}^n \times \{|(z_{n+1},\dots,z_m)| < \varepsilon/t\}$;

(ii) if $z \in \mathbf{C}^m$ and $d((z_1,\ldots,z_m),\sigma(f_1,\ldots,f_n,\varepsilon f_{n+1},\ldots,\varepsilon f_m)) < t\varepsilon$, then $z \in \omega'_\varepsilon$,

(iii) if $z \in \omega'_\varepsilon - (N_1 \times \mathbf{C}^{m-n})$, then $d((z_1,\ldots,z_n),F) < \varepsilon/t$.

Then proceeding exactly as in the proof of theorem 4.1 in [4], we obtain a function v'_ε which is holomorphic in ω'_ε and which satisfies

$$\|v'_\varepsilon - v_\varepsilon\|_{\omega'_\varepsilon} \to 0 \quad \text{as} \quad \varepsilon \to 0 .$$

Then, since holomorphic functions operate on A,

$$v'_\varepsilon \circ (f_1,\ldots,f_n,\varepsilon f_{n+1},\ldots,\varepsilon f_m) \in A .$$

Since $v_\varepsilon \circ (f_1,\ldots,f_n,\varepsilon f_{n+1},\ldots,\varepsilon f_m) = u \circ (f_1,\ldots,f_n)$, the completeness of A implies $u \circ (f_1,\ldots,f_n) \in A$.

We are now able to prove a generalization of a result by Freeman [2].

THEOREM 3.2. *Let M be a k-dimensional real manifold of class C^r, $r = \frac{1}{2}k + 1$. Suppose that $\Phi \subset C^r(M)$ separates points on a compact subset X of M. Define $E = E(\Phi) \cap X$ and $A =$ the sup norm algebra in $C(X)$ generated by Φ. If $M_A = X$, then*

$$A \supset \{g \in C(X);\ g_{|E} \equiv 0\} .$$

PROOF. Choose any compact subset $X_0 \subset X - E$ and use lemma 2.3. It follows from theorem 3.1 that the family

$$\{g \in A \cap C_{\mathbf{R}}(X)\ ;\ g_{|E} \equiv 0 \text{ and } 0 \notin g(X_0)\}$$

is non-empty and separates points in X_0. The theorem now is a consequence of Stone–Weierstrass.

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