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ON CLASSES OF PROJECTIONS IN A VON-NEUMANN ALGEBRA

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Abstracts.

Classes of projections in a von-Neumann algebra are studied, and thereby fairly general conditions for unitary implementation (of isomorphisms) are obtained. By introducing a relation between classes of projections we also get a unified proof and generalizations of some results in the spatial theory for von-Neumann algebras.

Introduction.

Conditions, assuring that an algebraic isomorphism between von-Neumann algebras be spatial (unitarily implemented), appear in a rather non-uniform way in the litterature (cf. [3], [4], [6]). In this article we shall study classes of projections in a von-Neumann algebra from a quite general point of view and thereby obtain a unitary implementation theorem for a fairly large class of von-Neumann algebras, the so-called GD (generalized discrete) algebras. As the name indicates, this is a generalization of the "classical" concept of a discrete (type I) von-Neumann algebra. In fact, any von-Neumann algebra whose commutant does not have any II_1-part is GD. A von-Neumann algebra with II_1 commutant may, or may not be GD.

Our basic building blocks will be the so-called primitive classes of projections (as an example, the class of abelian projections is primitive). We also introduce a relation between classes of projections and show how this may be used to give a unified proof of some spatial results for von-Neumann algebras.

1. Definitions, terminology and notation.

$\mathscr{A}$ and $\mathscr{B}$ will denote von-Neumann algebras over Hilbert spaces $\mathscr{H}$ and $\mathscr{K}$ respectively. All isomorphisms are *-isomorphisms. The letters E,F will denote projections and P,Q central projections. Central carrier

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of an element A is denoted by C_A. If $x \in \mathscr{H}$, then $[\mathscr{A}x]$ denotes the closure of the linear space

$$\{Ax\,;\ A \in \mathscr{A}\}$$

(or the orthogonal projection on this space). By a *partition* of E we mean an orthogonal family $\{E_i\}$ of projections with sum E. The family $\{E_i\}$ is said to be *homogeneous* if the elements are pairwise equivalent and *completely* disjoint if $C_{E_i}C_{E_j}=0$ for $i \neq j$. If $\{E_i\}_{i\in J}$ is homogeneous and card $J=n$, we say $E=\sum_{i\in J}E_i$ is an *n-multiple* of any of the summands E_i. An arbitrary n-multiple of a projection F is denoted by $n\cdot F$.

DEFINITION 1.1. Let $\mathscr{P}$ be a property of von-Neumann algebras. A projection E in $\mathscr{A}$ is said to have the property $\mathscr{P}$ (relatively $\mathscr{A}$) if the reduced algebra $\mathscr{A}_E$ has the property $\mathscr{P}$.

The symbol $\mathscr{P}$ will also be used to denote the class of projections having the property $\mathscr{P}$. Of course, we only consider properties which are preserved under unitary equivalence. Further we shall confine ourselves to properties which are "proper" in the sense that they persist under restrictions to central projections (i.e., if $E \in \mathscr{P}$ and P is central, then $PE \in \mathscr{P}$).

If n is a cardinal, we denote by $n\cdot\mathscr{P}$ the class of projections which may be written as n-multiples of elements from $\mathscr{P}$. The projection E is said to be *semi-$\mathscr{P}$* if every nonzero subprojection of E majorizes a nonzero $\mathscr{P}$-projection. (Note that if E is semi-$\mathscr{P}$, E may be written as a sum of $\mathscr{P}$-projections, by Zorn's lemma.) The projection E is said to be $\sigma-\mathscr{P}$ if it may be written as a completely disjoint sum of $\mathscr{P}$-projections. If $\{P_\alpha\}$ is a central partition of the unit such that $P_\alpha E \in \mathscr{P}$, we say $\{P_\alpha\}$ is a $\mathscr{P}$-partition for E.

The following terminology will be used in connection with classes:

DEFINITION 1.2. Let $\mathscr{P}$ and $\mathscr{Q}$ be classes (properties). We say that $\mathscr{P}$ is

i) *dominated* by $\mathscr{Q}$, and write $\mathscr{P} \ll \mathscr{Q}$, if $E \in \mathscr{P}$, $F \in \mathscr{Q}$ and $C_E \leqq C_F$ implies $E \prec F$. We say $\mathscr{P}$ and $\mathscr{Q}$ are *related* if either $\mathscr{P} \ll \mathscr{Q}$ or $\mathscr{Q} \ll \mathscr{P}$.
ii) *primitive* if $E, F \in \mathscr{P}$ and $C_E = C_F$ implies $E \sim F$.
iii) *almost primitive* if $\aleph_0\cdot\mathscr{P}$ is primitive.
iv) *hereditary* if $E \in \mathscr{P}$ and $F \leqq E$ implies $F \in \mathscr{P}$.
v) *invariant* (resp. *σ-invariant*) if $\mathscr{P}$ persists under orthogonal (resp. completely disjoint) sums; the meaning of finitely (resp. countably) invariant should be clear.

vi) *homogeneously unique* (resp. *almost homogeneously unique*) if $E,F \in \mathscr{P}$ and $n\cdot E = m\cdot F$ (resp.: and $n,m \geqq \aleph_0$) implies $n=m$.
vii) *symmetric* if $[\mathscr{A}x] \in \mathscr{P}$ implies $[\mathscr{A}'x] \in \mathscr{P}$.

REMARKS. If $\mathscr{P}$ and $\mathscr{Q}$ are related, then obviously $\mathscr{P}\cap\mathscr{Q}$ is primitve. Further, $\mathscr{P}$ is primitive if and only if $\mathscr{P} \ll \mathscr{P}$. Indeed, suppose $E,F \in \mathscr{P}$ with $C_E \leqq C_F$. Then $C_E = C_E C_F = C_{C_E F}$ and so $E \sim C_E F \leqq F$.

2. General conditions for unitary implementation.

We shall make repeated use of the following structure theorem for isomorphisms, due to Dixmier ([1; 5.1.3.] and [2; 4, th. 3, corollaire]).

THEOREM 2.1. *Let $\varphi\colon \mathscr{A} \to \mathscr{B}$ be an isomorphism. Then there exists a von-Neumann algebra $\mathscr{D}$ and projections $E',F' \in \mathscr{D}'$ with $C_{E'} = C_{F'} = I$ such that*

$$\mathscr{A} = \mathscr{D}_{E'}, \quad \mathscr{B} = \mathscr{D}_{F'},$$

and φ may be identified with the mapping $TE' \to TF'$, for $T \in \mathscr{D}$. Also, φ is spatial if and only if $E' \sim F'$.

From the definition of primitivity we then get:

COROLLARY 1. *Let $\mathscr{P}$ be a primitive property and suppose $\mathscr{A}'$ and $\mathscr{B}'$ belong to the class $\mathscr{P}$. Then every isomorphism $\varphi\colon \mathscr{A} \to \mathscr{B}$ is spatial.*

If $\varphi\colon \mathscr{A} \to \mathscr{B}$ is an isomorphism and $E' \in \mathscr{A}'$ and $F' \in \mathscr{B}'$ are such that $\varphi(C_{E'}) = C_{F'}$, then also the mapping $\varphi^{E',F'}\colon AE' \to AF'$ from $\mathscr{A}_{E'}$ to $\mathscr{B}_{F'}$ is an isomorphism [6; p. 331]. From theorem 1 we then get:

COROLLARY 2. *Let $\varphi\colon \mathscr{A} \to \mathscr{B}$ be an isomorphism. If there exist partitions $\{E'_i\}$ and $\{F'_i\}$ of the units in $\mathscr{A}'$ and $\mathscr{B}'$ respectively such that $\varphi(C_{E'_i}) = C_{F'_i}$ and such that $\varphi^{E'_i,F'_i}$ is spatial for all i, then φ is spatial.*

PROOF. Let $\mathscr{D}$, E' and F' be as in theorem. We have $E' = \sum E'_i$ and $F' = \sum F'_i$, and $\varphi^{E'_i,F'_i}$ is given by $TE'_i \to TF'_i$, for $T \in \mathscr{D}$, from $\mathscr{D}_{E'_i}$ to $\mathscr{D}_{F'_i}$. Since $\varphi^{E'_i,F'_i}$ is spatial, we have $E'_i \sim F'_i$ and so

$$E' = \sum E'_i \sim \sum F'_i = F'.$$

3. The unitary implementation theorem for GD (generalized discrete) algebras.

In this section we shall study von-Neumann algebras whose commutants may be decomposed into primitive constituents, the so-called generalized discrete algebras. We give a precise definition of this concept:

DEFINITION 3.1. Let $\mathscr{A}$ be a von-Neumann algebra and let $\mathscr{P}$ be a primitive, homogeneously unique (resp. almost primitive, almost homogeneously unique) property. Suppose that for each cardinal n (resp. for each cardinal $n \geqq \aleph_0$) there exists a maximal central projection P_n such that $\mathscr{A}'_{P_n}$ belongs to the class $n \cdot \mathscr{P}$ and suppose l.u.b. $\{P_n\} = I$. In either case we say $\mathscr{A}$ is generalized discrete (abbreviated GD) with respect to $\mathscr{P}$. The family $\{P_n\}$ is said to be a characteristic family for $\mathscr{A}$ (with respect to $\mathscr{P}$).

REMARK. If the family $\{P_n\}$ exists, it is unique and the P_n's are orthogonal; this follows from the homogeneous uniqueness of $\mathscr{P}$ and the maximality of the P_n's.

In the next proposition we discuss some properties of the classoperations $\mathscr{P} \to \sigma - \mathscr{P}$ and $\mathscr{P} \to n \cdot \mathscr{P}$ and the relation $\ll$, introduced in section 1.

PROPOSITION 3.1. *Let $\mathscr{P}$ and $\mathscr{Q}$ be classes of projections and let n be a cardinal. Then*

i) $\mathscr{P} \ll \mathscr{Q} \Leftrightarrow \sigma - \mathscr{P} \ll \sigma - \mathscr{Q} \Leftrightarrow \sigma - \mathscr{P} \ll \mathscr{Q} \Leftrightarrow \mathscr{P} \ll \sigma - \mathscr{Q}$.
In particular, if $\mathscr{P}$ is primitive, so is $\sigma - \mathscr{P}$.

ii) $n \cdot (\sigma - \mathscr{P}) = \sigma - (n \cdot \mathscr{P})$.
In particular, if $\mathscr{P}$ is almost primitive, so is $\sigma - \mathscr{P}$.

iii) $\mathscr{P} \ll \mathscr{Q} \Rightarrow n \cdot \mathscr{P} \ll n \cdot \mathscr{Q}$.
In particular, if $\mathscr{P}$ is primitive, so is $n \cdot \mathscr{P}$; and if $\mathscr{P}$ is dominated by the property "properly infinite", then $\mathscr{P}$ is almost primitive.

iv) *If $\mathscr{P}$ is homogeneously unique (resp. almost homogeneously unique) so is $\sigma - \mathscr{P}$.*

PROOF. i) We prove $\mathscr{P} \ll \mathscr{Q} \Rightarrow \sigma - \mathscr{P} \ll \sigma - \mathscr{Q}$. Let $E \in \sigma - \mathscr{P}$, $F \in \sigma - \mathscr{Q}$ with $C_E \leqq C_F$ and let $\{P_\alpha\}$ (resp. $\{\mathscr{Q}_\beta\}$) be a $\mathscr{P}$-partition (resp. $\mathscr{Q}$-partition) for E (resp. for F). If $R_{\alpha\beta} = P_\alpha Q_\beta$, then $\{R_{\alpha\beta}\}$ is a $\mathscr{P}$-partition for E and a $\mathscr{Q}$-partition for F. We have

$$C_{R_{\alpha\beta}E} = R_{\alpha\beta}C_E \leqq R_{\alpha\beta}C_F = C_{R_{\alpha\beta}F}$$

and so $R_{\alpha\beta}E \prec R_{\alpha\beta}F$, since $\mathscr{P} \ll \mathscr{Q}$. But then

$$E = \sum R_{\alpha\beta}E \prec \sum R_{\alpha\beta}F = F$$

and so $\sigma - \mathscr{P} \ll \sigma - \mathscr{Q}$.

The other implications are either obvious or quite analogous to the one just proved.

ii) We prove $n\cdot(\sigma - \mathscr{P}) \subseteqq \sigma - (n\cdot\mathscr{P})$. Let $E \in n\cdot(\sigma - \mathscr{P})$. Then $E = \sum E_i$ where $E_i \sim E_j$ and $E_i \in \sigma - \mathscr{P}$. Let $\{P_\alpha\}$ be a common $\mathscr{P}$-partition for all the E_i's (this is possible since the E_i's are equivalent) and set $F_\alpha = \sum_i P_\alpha E_i$. Then $F_\alpha \in n\cdot\mathscr{P}$ and the F_α's are completely disjoint. But $E = \sum_\alpha F_\alpha$ and so $E \in \sigma - (n\cdot\mathscr{P})$, that is

$$n\cdot(\sigma - \mathscr{P}) \subseteqq \sigma - (n\cdot\mathscr{P})\ .$$

The proof of the converse inclusion is quite analogous.

iii) Suppose $\mathscr{P} \ll \mathscr{Q}$ and let $E = n\cdot E_0$, $F = n\cdot F_0$ with $C_E \leqq C_F$, where $E_0 \in \mathscr{P}$, $F_0 \in \mathscr{Q}$. Then

$$C_{E_0} = C_E \leqq C_F = C_{F_0}$$

and so $E_0 \prec F_0$. It follows that

$$E = n\cdot E_0 \prec n\cdot F_0 = F\ .$$

Now let $\mathscr{Q}$ denote the property "properly infinite" and suppose $\mathscr{P} \ll \mathscr{Q}$. Then $\mathscr{Q} = \aleph_0\cdot\mathscr{Q}$ [2; p. 298] and so

$$\aleph_0\cdot\mathscr{P} \ll \aleph_0\cdot\mathscr{Q} = \mathscr{Q}\ .$$

But $\aleph_0\cdot\mathscr{P} \subseteqq \mathscr{Q}$ and it follows that $\aleph_0\cdot\mathscr{P} = (\aleph_0\cdot\mathscr{P}) \cap \mathscr{Q}$ is primitive.

iv) Suppose $\mathscr{P}$ is homogeneously unique (resp. almost homogeneously unique) and let $\{E_i\}_{i\in J}$ and $\{F_k\}_{k\in K}$ be homogeneous families from $\sigma - \mathscr{P}$ with

$$\sum\nolimits_{i\in J} E_i = \sum\nolimits_{k\in K} F_k$$

(resp.: and such that card $J \geqq \aleph_0$, card$K \geqq \aleph_0$). Let $\{P_\alpha\}$ and $\{Q_\beta\}$ be $\mathscr{P}$-partitions for the E_i's and the F_k's respectively. If $R_{\alpha\beta} = P_\alpha Q_\beta$, then $\{R_{\alpha\beta}\}$ is a $\mathscr{P}$-partition for the E_i's as well as for the F_k's. Since $\sum R_{\alpha\beta} = I$, there is a nonzero element R_0 in the family $\{R_{\alpha\beta}\}$. We have

$$\sum\nolimits_{i\in J} R_0E_i = \sum\nolimits_{k\in K} R_0F_k$$

and so card$J$ = card$K$ since $R_0E_i, R_0F_k \in \mathscr{P}$ and the sums are homogeneous.

We now state the unitary implementation theorem for GD algebras.

THEOREM 3.1. *Let $\mathscr{A}$ and $\mathscr{B}$ be* GD *algebras with respect to the primitive (resp. almost primitive) property $\mathscr{P}$, with characteristic families $\{P_n\}$ and $\{Q_n\}$ respectively. Then, if $\varphi: \mathscr{A} \to \mathscr{B}$ is an isomorphism such that $\varphi(P_n) = Q_n$ for all n, φ is spatial.*

PROOF. i) Suppose $\mathscr{P}$ is primitive. Then, for any cardinal n, also $n \cdot \mathscr{P}$ is primitive (proposition 3.1, iii)). By theorem 2.1, corollary 1, each φ_{P_n} is spatial, and by corollary 2, φ itself is spatial.

ii) Suppose $\mathscr{P}$ is almost-primitive. For any cardinal $n \geqq \aleph_0$ we obviously have $n \cdot \mathscr{P} \subseteqq n \cdot (\aleph_0 \cdot \mathscr{P})$, and we are back in the primitive case. The theorem follows.

To obtain conditions for generalized discreteness, we shall need the following lemma, due to Dixmier.

LEMMA 3.1. *Let $\{E_i\}_{i \in J}$ be a homogeneous family in $\mathscr{A}$. Then there is a central projection Q in $\mathscr{A}$ and a homogeneous family $\{F_k\}_{k \in K}$ such that*

i) $J \subseteqq K$,
ii) $F_i \sim E_i Q, \quad i \in J$,
iii) *if we put $F_0 = Q - \sum_{k \in K} F_k$, then $F_0 \prec F_k$ (strictly).*

Furthermore, if $\operatorname{card} K \geqq \aleph_0$, we may suppose $Q = \sum_{k \in K} F_k$. [2; III,1, Th. 1, corollaire 2].

As an intermediate result we now get:

LEMMA 3.2. *Let $\mathscr{P}$ be a hereditary property and let $\mathscr{A}$ be a semi-$\mathscr{P}$ von-Neumann algebra. Suppose one of the following two conditions is fulfilled:*

i) *$\mathscr{P}$ is primitive.*
ii) *$\mathscr{P}$ is finitely invariant.*

Then there is a central partition $\{P_\alpha\}$ of the unit in $\mathscr{A}$ and a corresponding family $\{n_\alpha\}$ of cardinals such that $\mathscr{A}_{P_\alpha}$ belongs to the class $n_\alpha \cdot \mathscr{P}$.

PROOF. i) Suppose $\mathscr{P}$ is primitive. Let E be a $\mathscr{P}$-projection and let Q, F_0 and $\{F_k\}_{k \in K}$ be as in lemma 3.1, constructed with respect to the one-element family $\{E\}$. Since $F_0 \prec F_k$, we have $F_0 \in \mathscr{P}$ by heredity of $\mathscr{P}$, and since F_0 is not equivalent to F_k, we have $C_{F_0} > C_{F_k}$ (strictly), by primitivity of $\mathscr{P}$. Set $P = C_{F_k} - C_{F_0}$. Then $PF_0 = 0$ and so

$$PQ = P = P(F_0 + \textstyle\sum_{k \in K} F_k) = \sum_{k \in K} PF_k .$$

$\{PF_k\}_{k \in K}$ is a homogeneous family of $\mathscr{P}$-projections and so $\mathscr{A}_P$ belongs to the class $n\cdot\mathscr{P}$ where $n=\operatorname{card}K$. We may now repeat the argument for $\mathscr{A}_{I-P}$ (which is semi-$\mathscr{P}$), and the lemma follows by transfinite induction.

ii) Suppose $\mathscr{P}$ is finitely invariant and let E, Q, F_0 and $\{F_k\}_{k\in K}$ be as above. If $\operatorname{card}K<\aleph_0$, then $\mathscr{A}_Q$ belongs to the class $\mathscr{P}$. If $\operatorname{card}K\geqq\aleph_0$, we may suppose $Q=\sum_{k\in K}F_k$ and so $\mathscr{A}_Q$ belongs to $n\cdot\mathscr{P}$, where $n=\operatorname{card}K$. The proof is now completed as in part i).

COROLLARY. *A finite projection is σ-countably decomposable.*

PROOF. Let $\mathscr{C}$ denote the property "countably decomposable". Then any von-Neumann algebra $\mathscr{A}$ is semi-$\mathscr{C}$, since every non-zero projection in $\mathscr{A}$ majorizes a nonzero cyclic projection. Also, the property $\mathscr{C}$ is obviously finitely invariant and hereditary. Now, if $\mathscr{A}$ is finite, then all the n_α's in lemma 3.1 must be finite. The corollary follows.

The following lemma clarifies the relationship between primitivity and homogeneous uniqueness. We omit the proof, since it is identical with the proof of a corresponding lemma in Dixmier [2; p. 239], concerning abelian projections.

LEMMA 3.3. *A primitive subclass of the class of finite projections is homogeneously unique.*

In particular, the property "having a generating and separating vector" is homogeneously unique when restricted to finite von-Neumann algebras.

We now give a sufficient condition for generalized discreteness:

THEOREM 3.2. *Let $\mathscr{A}$ be a von-Neumann algebra (resp. such that $\mathscr{A}'$ is properly infinite) and suppose $\mathscr{A}'$ is semi-$\mathscr{P}$ where $\mathscr{P}$ is*

i) *primitive, homogeneously unique and hereditary (resp. i') almost primitive, almost homogeneously unique, finitely invariant and hereditary).*

Then $\mathscr{A}$ is GD *with respect to $\sigma-\mathscr{P}$.*

PROOF. i) Suppose the unprimed conditions are fulfilled. Then, by proposition 3.1, it follows that $\sigma-\mathscr{P}$ is primitive and homogeneously unique. By lemma 3.2 there is a central partition $\{P_\alpha\}$ of the unit such that $\mathscr{A}'_{P_\alpha}$ belongs to $n_\alpha\cdot\mathscr{P}$ for some cardinal n_α. Set

$$P_n=\sum\{P_\alpha;\ n_\alpha=n\}\,.$$

Then $\mathscr{A}'_{P_n}$ belongs to $\sigma-(n\cdot\mathscr{P})=n\cdot(\sigma-\mathscr{P})$ and P_n is maximal with respect to this property (by homogeneous uniqueness of $\sigma-\mathscr{P}$). It follows that $\{P_n\}$ is a characteristic family for $\mathscr{A}$, with respect to $\sigma-\mathscr{P}$.

i') Suppose $\mathscr{A}'$ is properly infinite and the primed conditions are fulfilled. Then $\sigma-\mathscr{P}$ is almost primitive and almost homogeneously unique (proposition 3.1). As in part i) we obtain families $\{P_\alpha\}$ and $\{n_\alpha\}$ such that $\mathscr{A}'_{P_\alpha}$ belongs to $n_\alpha\cdot\mathscr{P}$. If n_α is finite, then the elements of the homogeneous partition in $\mathscr{A}'_{P_\alpha}$ are properly infinite (Indeed, let $\mathscr{B}$ be a properly infinite von-Neumann algebra and suppose $E_1+E_2=I$, $E_1\sim E_2$. If E_1 were not properly infinite, there would exist a nonzero projection P in the center of $\mathscr{B}$ such that PE_1, and consequently PE_2, was finite. But then also $PE_1+PE_2=P$ would be finite, contradicting the proper infiniteness of $\mathscr{B}$). Since a properly infinite projection is equivalent to an $\aleph_0$-multiple of itself [2; p. 298], we have that $\mathscr{A}_{P_\alpha}$ belongs to $\aleph_0\cdot\mathscr{P}$ for finite n_α's. Altogether, we may suppose that all the n_α's are greater than or equal to $\aleph_0$. The proof is now completed as in part i).

Many theorems in the spatial theory for von-Neumann algebras now follow as easy corollaries from the above theorem.

COROLLARY 1. *A type* I *von-Neumann algebra is* GD *with respect to abelian projections.*

PROOF. If $\mathscr{A}$ is type I, so is $\mathscr{A}'$. In our language this means that $\mathscr{A}'$ is semi-abelian. Let $\mathcal{A}$ denote the property "abelian"; then $\mathcal{A}$ is primitive [2; p. 239] and hereditary. Since every abelian projection is finite, $\mathcal{A}$ is homogeneously unique (lemma 3.3). Since $\sigma-\mathcal{A}=\mathcal{A}$, the corollary follows.

COROLLARY 2. *A von-Neumann algebra with properly infinite commutant is* GD *with respect to σ-countably decomposable projections.*

PROOF. Let $\mathscr{C}$ denote the property "countably decomposable". Then, as noted before, any von-Neumann algebra is semi-$\mathscr{C}$. Furthermore, the property $\mathscr{C}$ is almost primitive ([2; p. 299] and our proposition 3.1, iii), almost homogeneously unique ([2; p. 224, lemma 6]) finitely invariant and hereditary. The corollary follows.

COROLLARY 3. *A semi-finite von-Neumann algebra with properly infinite commutant is* GD *with respect to finite projections.*

PROOF. If $\mathscr{A}$ is semi-finite, so is $\mathscr{A}'$. Let $\mathscr{F}$ denote the property "finite". Then $\mathscr{F}$ is finitely invariant and hereditary. Since every finite projection is σ-countably decomposable (lemma 3.2, corollary) $\mathscr{F}$ is also almost primitive and almost homogeneously unique (cf. the proof of the preceding corollary). The corollary follows.

4. A note on generating vectors.

By theorem 3.2, corollaries, the only possible pure type non-GD algebras are the $\mathrm{II}_{x,1}$-algebras ($x=1$ or ∞). And indeed, a $\mathrm{II}_{x,1}$-algebra need not be GD since, for instance, a $\mathrm{II}_{\infty,1}$-factor with non-trivial fundamental group permits non-spatial automorphisms [5] (and so can't be GD, by theorem 3.1; on the other hand, a $\mathrm{II}_{1,1}$-algebra with a generating and separating vector is GD). In general then, when we deal with $\mathrm{II}_{x,1}$-algebras, we must look for other criteria for unitary implementation than those developed in the preceding paragraphs.

For finite-finite ("finite with finite commutant") algebras, and in particular for $\mathrm{II}_{1,1}$-algebras, one may formulate a criterion in terms of the coupling-operator ([2] and [3]).

For $\mathrm{II}_{\infty,1}$-algebras there is no canonical coupling-operator at hand. However, for algebras with generating vectors there is a condition for unitary implementation, due to Kadison, which says that an isomorphism between such algebras is spatial if it preserves maximal cyclicity [6; p. 349]. (For finite-finite algebras with generating vectors it is easy to see that an isomorphism $\varphi\colon \mathscr{A} \to \mathscr{B}$ preserves maximal cyclicity if it preserves the coupling-operator; indeed, in this case $C_{\mathscr{A}}^{-1}=[\mathscr{A}'x]^{\natural}$, $C_{\mathscr{B}}^{-1}=[\mathscr{B}'y]^{\natural}$ where $C_{\mathscr{A}}, C_{\mathscr{B}}$ are the coupling-operators and x, y are generating vectors. By assumption, $\varphi([\mathscr{A}'x]^{\natural})=[\mathscr{B}'y]^{\natural}$; but $\varphi([\mathscr{A}'x]^{\natural}=(\varphi([\mathscr{A}'x]))^{\natural}$, by uniqueness of the trace, and so $\varphi([\mathscr{A}'x])\sim[\mathscr{B}'y]$, by faithfulness of the trace.) We now contend that a $\mathrm{II}_{\infty,1}$-algebra with countably decomposable center has a generating vector. This will follow from the following more general result, which gives a condition for the existence of separating vectors in terms of the relation $\ll$. At the same time we also get a new and unified proof of two similar results in Dixmier ([2; p. 19] and [2; p. 302]). (Note that in view of lemma 3.2, corollary, and theorem 2.1, corollary 2, the restriction to algebras with countably decomposable centers is not a very severe one.)

PROPOSITION 4.1. *Suppose $\mathscr{A}$ belongs to the class $\mathscr{P}$ and $\mathscr{A}'$ belongs to the class $\mathscr{Q}$, where $\mathscr{P} \ll \mathscr{Q}$ and $\mathscr{Q}$ is symmetric. Then, if $\mathscr{A}$ is countably decomposable, $\mathscr{A}$ has a separating vector.*

PROOF. By [2; p. 18] we may assume that $\mathscr{A}$ has a generating vector x. Then, if $E=[\mathscr{A}'x]$, we have that $C_E=I$ and $E\in\mathscr{Q}$, and so, by hypothesis, $I\prec E$, that is $I\sim E$. The algebra $\mathscr{A}_E$ has a separating vector, and so the same must hold for $\mathscr{A}$ ($\mathscr{A}$ is spatially isomorphic to $\mathscr{A}_E$).

COROLLARY 1. *A countably decomposable abelian von-Neumann algebra has a separating vector.*

PROOF. The property "abelian" is dominated by any property [2; p. 239].

COROLLARY 2. *A countably decomposable von-Neumann algebra with properly infinite commutant has a separating vector.*

PROOF. The property "countably decomposable" is dominated by the property "properly infinite" [2; p. 292], and the latter is symmetric. [2; p. 231].

COROLLARY 3. *A properly infinite von-Neumann algebra with finite commutant and countably decomposable center has a generating vector.*

PROOF. If $\mathscr{A}$ satisfies the hypothesis of the corollary, $\mathscr{A}'$ is countably decomposable (lemma 3.2, corollary). Thus $\mathscr{A}'$ has a separating vector, that is $\mathscr{A}$ has a generating vector.

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