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TOPOLOGIES ON THE EXTREME POINTS OF COMPACT CONVEX SETS

ALAN GLEIT[1]

Abstract.

Suppose there is a topology on the extreme points of a metrizable compact convex set whose closed sets are the traces on the extreme points of a family $\mathscr{T}$ satisfying 1–5 below. If the topology satisfies a certain condition (C 2, below), then the properties of first countability, second countability and local compactness are equivalent. We give several examples of collections to which the theorems are applicable.

Introduction.

Let K be a compact convex subset of a locally convex topological vector space E. Let $\mathscr{T}$ be any collection of closed convex subsets of K. Let

$$R = \bigcap \{T_\alpha \mid \emptyset \neq T_\alpha \in \mathscr{T}\} .$$

It is called the *radical* of the collection $\mathscr{T}$. Suppose $\mathscr{T}$ satisfies the following conditions:

1. $\emptyset, K \in \mathscr{T}$.
2. $T_1, T_2 \in \mathscr{T} \Rightarrow \mathrm{co}(T_1, T_2) \in \mathscr{T}$.
3. $T_\alpha \in \mathscr{T}$ each $\alpha \in A \Rightarrow \bigcap T_\alpha \in \mathscr{T}$.
4. $\mathrm{Ext}(T) = \mathrm{Ext}(K) \cap T$ for each $T \in \mathscr{T}$, $T \neq R$.
5. $\mathrm{Ext}(R) = \mathrm{Ext}(K) \cap R$ or $\emptyset = \mathrm{Ext}(K) \cap R$

In property 4, $\mathrm{Ext}(\cdot)$ are the extreme points of the given set. By property 3 the radical $R \in \mathscr{T}$. Faces obviously satisfy properties 4 and 5. Given the collection $\mathscr{T}$ as above, we define the τ-topology on $\mathrm{Ext}(K)$ by the following scheme:

$$F \subseteq \mathrm{Ext}(K) \text{ is } \tau\text{-closed} \Leftrightarrow F = \mathrm{Ext}(K) \cap T \text{ for some } T \in \mathscr{T} .$$

Hence, all τ-closed sets are of the form $F = \mathrm{Ext}(T)$ for some $T \in \mathscr{T}$, but the converse need not hold. This paper studies the τ-topology. We prove results similar to those in [5], [7] and [10] with very similar meth-

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ods. We include proofs for completeness and give references to their origins.

In Section 1 we begin our study. With the help of an auxiliary condition in Section 2 we show that for the τ-topology the concepts of first countability, second countability, local compactness, and local sequential compactness are all equivalent. In Section 3 we give several examples of collections $\mathscr{T}$ to which the theorems are applicable.

I should like to thank the referee for his many valuable comments. In particular, the notion of the radical of $\mathscr{T}$ is due to him.

0. Notations.

For any net, Greek subscripts, e.g. α, β, denote arbitrary index sets while Latin subscripts, e.g. i, j, n, denote the natural numbers as an index set, i.e. $\{x_n\}$ is a sequence.

For any set $A \subseteqq K$, $\bar{A}$ or A^- will be the closure of A in the topology of E.

For convergent nets, $p_\alpha \to q$ will denote convergence in the topology of E while $p_\alpha \to_\tau q$ will denote convergence in the τ-topology.

For any set $A \subseteqq K$, $\mathrm{Ext}(A)$ will denote the extreme points of A.

We denote by Z the set $\mathrm{Ext}(K)^-$.

For a function f, we denote by $f|A$ the restriction of f to A.

1. Preliminary study and several maps.

We let K, E, and $\mathscr{T}$ be as in the Introduction. For $q \in K$, we take $T(q)$ to be the minimal element of $\mathscr{T}$ which contains q. (It exists by properties 1 and 3 of $\mathscr{T}$.) Following [5], [7] and [10] (the notation is that of [7]) we define

$$\Phi(q) = \mathrm{Ext}(K) \cap T(q) \qquad \text{for each } q \in K\,,$$
$$\psi(p) = \{q \in Z \mid p \in \Phi(q)\} \quad \text{for each } p \in \mathrm{Ext}(K)\,.$$

We note that $\Phi(q)$ is τ-closed for each $q \in K$. Clearly $\Phi(q) \neq \emptyset$ for $q \notin R$.

Lemma 1.1. *Let $\{p_\alpha\}$ be a net in* $\mathrm{Ext}(K)$. *If $p_\alpha \to q$ and $\Phi(q) \neq \emptyset$, then $p_\alpha \to_\tau p$ for each $p \in \Phi(q)$.*

Proof. [6, Lemma 2.3]. Suppose $p \in \Phi(q)$ but $\{p_\alpha\}$ does not converge to p in the τ-topology. Then there is a subnet $\{p_\beta\}$ and a τ-closed set F such that $p_\beta \in F$ while $p \notin F$. Let $T \in \mathscr{T}$ satisfy $F = \mathrm{Ext}(K) \cap T$. As T is closed, $q \in T$ and so $T(q) \subseteqq T$. Hence, $p \in T(q) \subseteqq T$ and so $p \in \mathrm{Ext}(K) \cap T = F$, which is a contradiction.

Following [5] and [10] (the notation is that of 5), for each $p \in \operatorname{Ext}(K)$, we let

$$\Delta(p) = \bigcap \{\bar{N} \mid N \in \mathcal{N}(p)\}$$

where $\mathcal{N}(p)$ is the collection of τ-neighborhoods of p. An alternate description of $\Delta(p)$ is provided by the following:

LEMMA 1.2. *For each* $p \in \operatorname{Ext}(K)$,

$$\Delta(p) = \{q \mid \text{There is a net } \{p_\alpha\} \subseteqq \operatorname{Ext}(K) \text{ such that } p_\alpha \to_\tau p \text{ while } p_\alpha \to q\} .$$

If the τ-topology is first countable at p and the given topology on K is first countable, one can use sequences rather than nets.

PROOF. [10, p. 212]. Let $p_\alpha \to_\tau p$ and $p_\alpha \to q$. Let $N \in \mathcal{N}(p)$. Then $p_\alpha \in N$ eventually. Thus $q \in \bar{N}$ and so $q \in \Delta(p)$. Conversely, let $q \in \Delta(p)$. Let $\mathcal{U}(q)$ be a neighborhood base at q. Order $\mathcal{U}(q) \times \mathcal{N}(p)$ by inclusion. For each $U \in \mathcal{U}(q)$ and $N \in \mathcal{N}(p)$, choose $p_{(U,N)} \in U \cap N$. Then $p_{(U,N)} \to_\tau p$ and $p_{(U,N)} \to q$.

COROLLARY 1.3. *Let* $p \in \operatorname{Ext}(K)$. *Then* $\Psi(p) \subseteqq \Delta(p)$.

PROOF. [10, p. 212]. Let $q \in \Psi(p)$. Then $p \in \Phi(q)$. Let $\{p_\alpha\}$ be a net in $\operatorname{Ext}(K)$ converging to q. This always exists for $q \notin \operatorname{Ext}(K)$ and for $q \in \operatorname{Ext}(K)$ we may take the constant net. Then $p_\alpha \to_\tau p$ by Lemma 1.1 and so $q \in \Delta(p)$ by Lemma 1.2.

COROLLARY 1.4. *Let* $G \subseteqq K$ *have interior which includes* $\Delta(p)$ *for some* $p \in \operatorname{Ext}(K)$. *Then* $G \cap \operatorname{Ext}(K)$ *is a τ-neighborhood of* p.

PROOF. [5, Corollary 7.4]. If not, then there is a net $\{p_\alpha\} \subseteqq \operatorname{Ext}(K) - G$ which τ-converges to p. Going to a subnet, we may assume that there is a $q \in K$ and that $p_\alpha \to q$. Then $q \in \Delta(p) \subseteqq G$ by Lemma 1.2. Hence $p_\alpha \in G$ eventually, which is a contradiction.

LEMMA 1.5. *Suppose f is an upper semi-continuous real valued convex function on K. Suppose that* $f(R) = 0$. *Then, for each* $c > 0$,

$$D = \{p \in \operatorname{Ext}(K) \mid f(p) \geqq c\}$$

is τ-compact. If K is metrizable, then D is also τ-sequentially compact.

PROOF. [7, proof of Theorem 3.3], [5 Proposition 4.8]. Let $\{p_\alpha\}$ be a net in D. Going to a subnet, we may assume $p_\alpha \to q$. Since f is upper semi-continuous, $f(q) \geqq c$. As $f(R)=0$, we have that $q \notin R$ and so $\Phi(q) \neq \emptyset$. Since $f | T(q)$ is convex and upper semi-continuous, it assumes its maximum value at some point of $\operatorname{Ext} T(q)=\Phi(q)$ [3, page 7]. Hence, for some $p \in \Phi(q)$, $f(p) \geqq f(q) \geqq c$ and so $p \in D$. From Lemma 1.1, $p_\alpha \rightarrow_\tau p$.

Let $p \in \operatorname{Ext}(K)-R$. We say that the τ-topology is first countable at p if p has a countable τ-neighborhood base. We say that the τ-topology is locally (sequentially) compact at p if p has a τ-neighborhood base of (sequentially) compact sets. Using the above analysis we get the following result.

PROPOSITION 1.6. *Let $p \in \operatorname{Ext}(K)$ and suppose K is metrizable. If $\Delta(p)=\psi(p)$, then the following hold:*

(1) *The τ-topology is first countable at p.*
(2) *The τ-topology is locally compact at p.*
(3) *The τ-topology is locally sequentially compact at p.*

In fact, if only the relative topology of E is first countable at p, conclusion (1) *still holds.*

PROOF. Let $N \in \mathscr{N}(p)$. Find $T \in \mathscr{T}$ satisfying $\operatorname{Ext}(T)=\operatorname{Ext}(K)-N$. If $q \in T$, then $\Phi(q) \subseteqq T(q) \subseteqq T$ and so $p \notin \Phi(q)$. Hence $q \notin \Psi(p)=\Delta(p)$ and so $T \cap \Delta(p)=\emptyset$.

To show conclusion (1) [5, Theorem 7.6], we let $\{G_n\}$ be a basis for the relative topology of E at p which is closed under finite unions. Note that $\Delta(p)$ is the intersection of compact sets and so is compact. Thus, a trivial compactness argument yields the existence of G_n satisfying $\Delta(p) \subseteqq G_n$ and $G_n \cap T=\emptyset$. Hence, $G_n \cap \operatorname{Ext}(K)$ is a τ-neighborhood of p by Corollary 1.4 within N.

For conclusions (2) and (3) [7, Theorem 3.3] we have to specify the metric we shall be using for K. Let $A(K)$ be the space of all affine continuous functions on K. It is a closed subspace of the separable space $C(K)$. Let $\{\xi_n \mid n=0,1,\ldots\}$ be dense in the unit ball of $A(K)$. For $x, y \in K$, define

$$\operatorname{dist}(x,y) = \sum 2^{-n}|\xi_n(x)-\xi_n(y)| .$$

Using local convexity of E and the fact that each ξ_n is continuous, one easily shows that $\operatorname{dist}(\cdot,\cdot)$ is a metric on K generating the same topology as the given one.

From above, $\Delta(p) \cap T=\emptyset$. Since both are compact,

$$\operatorname{dist}(\Delta(p),T) = \delta > 0 .$$

Let

$$F = \{z \in Z \mid \operatorname{dist}(z,T) \geqq \delta/2\} .$$

As interior$(F) \supseteqq \Delta(p)$, $\operatorname{Ext}(K) \cap F$ is a τ-neighborhood of p by Corollary 1.4. Since $\operatorname{Ext}(K) \cap F \subseteqq N$, it suffices to show that $\operatorname{Ext}(K) \cap F$ is both τ-compact and τ-sequentially compact.

Consider the function defined on K by:

$$f(x) = \operatorname{dist}(x,T) .$$

It is obviously continuous. Since $R \subseteqq T$, $f(R)=0$. By Lemma 1.5, we will be done if we can show that f is convex. So, let $x,y \in K$ and $0 \leqq \lambda \leqq 1$. Choose $t_x, t_y \in T$ satisfying

$$f(x) = \operatorname{dist}(x,t_x) \quad \text{and} \quad f(y) = \operatorname{dist}(y,t_y) .$$

These exist since T is compact. Then

$$\begin{aligned} f(\lambda x+(1-\lambda)y) &= \operatorname{dist}(\lambda x+(1-\lambda)y, T) \\ &\leqq \operatorname{dist}(\lambda x+(1-\lambda)y,\, \lambda t_x+(1-\lambda)t_y) \end{aligned}$$

since $\lambda t_x+(1-\lambda)t_y \in T$ by convexity. Since ξ_n is affine, one easily gets that

$$\begin{aligned} \operatorname{dist}(\lambda x+(1-\lambda)y,\, \lambda t_x+(1-\lambda)t_y) \leqq \lambda \operatorname{dist}(x,t_x) + \\ + (1-\lambda) \operatorname{dist}(y,t_y) . \end{aligned}$$

Hence,

$$f(\lambda x+(1-\lambda)y) \leqq \lambda f(x)+(1-\lambda)f(y)$$

and so f is convex.

In the proof of conclusion (1) we could have taken $\{G_n\}$ to be a basis for all of K. Hence, we have the following.

THEOREM 1.7. *Suppose K is metrizable and $\Delta(p)=\Psi(p)$ for each $p \in \operatorname{Ext}(K)$. Then the following hold:*

1. *The τ-topology is first countable.*
2. *The τ-topology is second countable.*
3. *The τ-topology is locally compact.*
4. *The τ-topology is locally sequentially compact.*

To show that the converses hold, we will need to introduce additional hypotheses.

2. Auxiliary conditions and the main theorems.

Throughout this section we suppose that K is metrizable. The first auxiliary condition we introduce is:

(C 1) If $\{p_n\} \subseteqq \mathrm{Ext}(K)$ satisfy $p_n \to q$ and $p_n \to_\tau p$, then $\Phi(q) \neq \emptyset$ and $p \in \Phi(q)$.

LEMMA 2.1. *Suppose K satisfies* (C 1). *Let D be τ-sequentially compact. Then*

$$\bar{D} \subseteqq \Psi(D) .$$

PROOF. [5, Lemma 7.5]. Let $q \in \bar{D}$. Then there is a sequence $\{p_n\} \subseteqq D$ converging to q. Since D is τ-sequentially compact, going to a subsequence, there is a $p \in D$ such that $p_n \to_\tau p$. Then by (C 1), $p \in \Phi(q)$ and so $q \in \Psi(p) \subseteqq \Psi(D)$.

THEOREM 2.2. *Suppose K satisfies* (C 1). *Let $p \in \mathrm{Ext}(K)$. Then $\Psi(p) = \Delta(p)$ follows from each of the following:*

1. *The τ-topology is first countable at p.*
2. *The τ-topology is locally sequentially compact at p.*

PROOF. If the τ-topology is first countable at p, then $\psi(p) = \Delta(p)$ follows easily from Lemma 1.2, Corollary 1.3 and (C 1). For the second conclusion (see [5, Theorem 7.6]), if the τ-topology is locally sequentially compact, then

$$\begin{aligned} \Delta(p) &= \bigcap \{\bar{N} \mid N \in \mathcal{N}(p)\} \\ &= \bigcap \{\bar{N} \mid N \text{ is a } \tau\text{-sequentially compact neighborhood of } p\} \\ &\subseteqq \bigcap \{\Psi(N) \mid N \in \mathcal{N}(p)\} \end{aligned}$$

by Lemma 2.1. If $q \notin \Psi(p)$, then $p \notin \Phi(q)$. Let $N = \mathrm{Ext}(K) - \Phi(q) \in \mathcal{N}(p)$. Obviously $q \notin \Psi(N)$. Hence

$$\bigcap \{\Psi(N) \mid N \in \mathcal{N}(p)\} \subseteqq \Psi(p) .$$

The Theorem now follows from Corollary 1.3.

To show that the properties listed in the Introduction are all equivalent, we still have to consider the property of local τ-compactness. To do this, we need to introduce a second auxiliary condition:

(C 2) If $\{p_n\} \subseteqq \mathrm{Ext}(K)$ converges to q, then all the τ-cluster points of $\{p_n\}$ are in $\Phi(q)$.

The condition (C 2) is a partial converse to Lemma 1.1 and is clearly stronger than (C 1).

LEMMA 2.3. *Suppose* K *satisfies* (C 2). *Let* $F \subseteq \mathrm{Ext}(K)$. *Then* F *is* τ*-sequentially compact iff* F *is* τ*-compact.*

PROOF. [7, Proposition 3.1]. Suppose F is τ-sequentially compact. Let $\{p_\alpha\} \subseteq F$ be a net. Going to a subnet, we may assume there is a $q \in K$ such that $p_\alpha \to q$. Choose a sequence $\{p_n\}$ from the net $\{p_\alpha\}$ satisfying $p_n \to q$. Then $\{p_n\} \subseteq F$ is a sequence and so there is a subsequence $\{p_j\}$ and an element $p \in F$ satisfying $p_j \to_\tau p$. But then $p \in \Phi(q) \cap F$ by (C 1). Also, p is a limit point of $\{p_\alpha\}$ by Lemma 1.1 and so F is τ-compact.

Conversely, if F is τ-compact, let $\{p_n\} \subseteq F$ be a sequence. Going to a subsequence, we may assume there is an element $q \in K$ such that $p_n \to q$. The net $\{p_n\}$ has a τ-cluster point $p \in F$. By (C 2), $p \in \Phi(q)$. So, Lemma 1.1 yields $p_n \to_\tau p$ and so F is τ-sequentially compact.

PROPOSITION 2.4. *Suppose* K *is metrizable satisfying* (C 2). *Then the following are equivalent:*

(1) $\mathrm{Ext}(K)$ *is* τ*-compact.*
(2) *For all* $q \in Z$, $\Phi(q) \neq \emptyset$.
(3) *Either* (a) *or* (b) *holds:*

(a) $\mathrm{Ext}(K) \cap R = \mathrm{Ext}(R)$,
(b) $Z \cap R = \emptyset$.

PROOF. (1) $\Rightarrow$ (2). Suppose $\mathrm{Ext}(K)$ is τ-compact. Let $q \in Z$ and find a sequence $\{p_n\} \subseteq \mathrm{Ext}(K)$ converging to q. Then $\{p_n\}$ has a τ-cluster point $p \in \mathrm{Ext}(K)$. By (C 2), $p \in \Phi(q)$ and so the latter set is non-empty.

Not(3) $\Rightarrow$ Not(2). By condition 5 of the defining properties of $\mathscr{T}$, we must have that $\mathrm{Ext}(K) \cap R = \emptyset$ and $Z \cap R \neq \emptyset$. Let $q \in Z \cap R$. Clearly $\Phi(q) = \emptyset$.

(3) $\Rightarrow$ (1). Let $\{p_\alpha\}$ be a net in $\mathrm{Ext}(K)$. By going to a subnet, we may assume that $p_\alpha \to q$ for some $q \in Z$. If $q \notin R$, then $\Phi(q) \neq \emptyset$. If $q \in R$, then (a) must hold and again $\Phi(q) = \mathrm{Ext}(K) \cap R \neq \emptyset$. So, in either case, we may find $p \in \Phi(q)$. But then $p_\alpha \to_\tau p$ by Lemma 1.1 and so $\mathrm{Ext}(K)$ is τ-compact.

Putting Proposition 1.6, Theorem 1.7, Theorem 2.2, and Lemma 2.3 together yields our main result.

THEOREM 2.5. *Suppose K is metrizable satisfying* (C 2). *Then the following are equivalent* (*for a fixed* $p \in \mathrm{Ext}(K)$):

1. *The τ-topology is first countable* (*at* p).
2. *The τ-topology is locally compact* (*at* p).
3. *The τ-topology is sequentially locally compact* (*at* p).

Further, the τ-topology is first countable for each $p \in \mathrm{Ext}(K)$ if and only if the τ-topology is second countable.

Finally, we may consider an alternate topology on K if the family $\mathscr{T}$ satisfies properties 1–4, $R \neq \emptyset$, and the first possibility for property 5, i.e.

$$\mathrm{Ext}(K) \cap R = \mathrm{Ext}(R) .$$

One may then define the τ_0-topology on $\mathrm{Ext}(K) - R$ by the following scheme:

$$F \subseteqq \mathrm{Ext}(K) - R \text{ is } \tau_0\text{-closed} \Leftrightarrow F = T \cap (\mathrm{Ext}(K) - R) \text{ for some } T \in \mathscr{T} .$$

We then define Φ_0 and ψ_0 as follows:

$$\begin{aligned} \Phi_0(q) &= T(q) \cap (\mathrm{Ext}(K) - R) && \text{for all } q \in K - R , \\ \Phi_0(q) &= \emptyset && \text{for all } q \in R , \\ \psi_0(p) &= \{q \in Z \mid p \in \Phi_0(q)\} && \text{for all } p \in \mathrm{Ext}(K) . \end{aligned}$$

By making the obvious modifications, the results of section 1 all hold for the τ_0-topology. The condition (C 2) becomes:

(C 2)$_0$ If $\{p_n\} \subseteqq \mathrm{Ext}(K) - R$ converges to q, then all the τ_0-cluster points of $\{p_n\}$ are in $\Phi_0(q)$.

The results for section 2 hold if we make the obvious modifications. The final result is the following.

THEOREM 2.6. *Suppose K is metrizable satisfying* (C 2)$_0$.*Then the following are equivalent* (*for a fixed* $p \in \mathrm{Ext}(K) - R$):

1. *The τ_0-topology is first countable* (*at* p).
2. *The τ_0-topology is locally compact* (*at* p).
3. *The τ_0-topology is sequentially locally compact* (*at* p).

Further, the τ_0-topology is first countable for each $p \in \mathrm{Ext}(K) - R$ if and only if the τ_0-topology is second countable. Last, $\mathrm{Ext}(K) - R$ *is τ_0-compact if and only if* $R \cap (\mathrm{Ext}(K) - R)^- = \emptyset$.

3. Applications.

All spaces in section 3 are real.

EXAMPLE 3.1. *Choquet simplexes* (see [10]).

We let K be a Choquet simplex in an lctvs$E$. We take $\mathscr{T}$ = set of all closed faces. Here $R=\emptyset$. Effros [4] showed that conditions 2 and 3 hold. The τ-topology on Ext(K) is called the structure topology. (C 2) holds for the structure topology of a metrizable K [10, Proposition 2.2]. Hence, Theorem 2.5 holds for a metrizable Choquet simplex [10, Theorem 2.4 and Proposition 2.7].

EXAMPLE 3.2. *Simplex spaces* (see [7]).

We let V be a simplex space, i.e. an ordered Banach space whose dual is an L-space. We take

$$K = P_1(V) = \{f \in V^* \mid \|f\| \leqq 1,\ f \geqq 0\}$$

considered in the weak* topology of V^*. We take $\mathscr{T}$ = set of all weak* closed faces which include zero. Here $R=\{0\}$. Effros [4] showed that conditions 2 and 3 hold. The τ_0-topology on Ext$(K)-\{0\}$ is called the structure topology or the hull-kernel topology of the maximal ideal space of V (see [4]). (C 2)$_0$ holds for separable simplex spaces [7, Corollary 1.5]. Hence, Theorem 2.6 holds for a separable simplex space [7, Theorem 3.3 and the remark following it].

EXAMPLE 3.3. *Conjugate L-spaces* (see [5]).

We let V be a Banach space whose dual is an L-space. We take K = unit ball of V^* considered in the weak* topology. We say that $p \in K$ dominates $q \in K$, written $q \prec p$, if $p=q+r$ with $\|p\|=\|q\|+\|r\|$.

A nonempty subset H of K is a biface if:

1. H is convex and symmetric.
2. If $x \neq 0$ is in H then $x/\|x\|$ is in H.
3. If $p \in H$ and $q \prec p$, then $q \in H$.

We take $\mathscr{T}$ = set of all weak* closed bifaces of K. Here $R=\{0\}$ which is not a subset of Ext(K), and so condition 5 holds. Effros [5] showed that conditions 2, 3, and 4 hold. The τ-topology on Ext(K) is called the structure topology. (C 2) holds for separable conjugate L-spaces [5, Lemma 7.1]. Hence, Theorem 2.5 holds for separable conjugate L-spaces [5, Theorem 7.6 and Theorem 7.7].

EXAMPLE 3.4. *L-ideals* (see [2]).

We take V to be a Banach space and K to be the unit ball of V^* with the weak* topology. A map e from V^* to V^* is an L-projection if it satisfies:

1. $e^2 = e$,
2. $\|p\| = \|ep\| + \|p - ep\|$ for all $p \in V^*$.

A subspace of V^* is an L-ideal if it is the range of an L-projection. We take $\mathscr{T}$ to be the collection of intersections of weak* closed L-ideals with K. We note that $R = \{0\}$ which is not a subset of $\text{Ext}(K)$ and so condition 5 holds. Alfsen and Effros showed that conditions 2 and 3 hold for $\mathscr{T}$ [2, Proposition 3.14 and (3.13)]. The τ-topology on $\text{Ext}(K)$ is called the structure topology. Hence, Proposition 1.6 and Theorem 1.7 hold for the structure topology if V is separable. If V^* is an L-space, then the structure topology described in Example 3.3 is the same topology as that described by the L-ideals [2, page 9.13]. Fairly strong conditions (to appear elsewhere), but which include Example 3.3. are sufficient to insure that (C 2) holds for this topology.

EXAMPLE 3.5. *Parallelizable faces* (see [9]).

Let K be a compact convex set in a locally convex topological vector space E. Let F be a closed face of K. We let F' be the union of all faces disjoint from F. If face (x) is the smallest face which includes $x \in K$, then

$$x \in F' \Leftrightarrow \text{face}(x) \cap F = \emptyset .$$

It is always true that $K = \text{co}(F \cup F')$ [1, Corollary 1.2]. Thus, for each $x \in K$, there are points $f \in F$, $f' \in F'$ and $0 \leqq \alpha \leqq 1$ satisfying

$$x = \alpha f + (1 - \alpha) f' . \tag{3.1}$$

The face F is said to be parallelizable if F' is a face and if for each $x \in K$, the coefficient α in the above decomposition is unique. We let $\mathscr{T}_0$ be the collection of all closed parallelizable faces in K. Clearly, $\mathscr{T}_0$ has properties 1, 4 and 5. Unfortunately, it need not have property 2 and only satisfies property 3 for downward filtered families [9, Proposition 8]. Thus, if $\mathscr{T} \subseteqq \mathscr{T}_0$ is such that $\emptyset, K \in \mathscr{T}$ and if $T_1, T_2 \in \mathscr{T}$ imply that both $T_1 \cap T_2$ and $\text{co}(T_1 \cup T_2)$ are in $\mathscr{T}$, then $\mathscr{T}$ satisfies all of our conditions. If it does, the τ-topology on $\text{Ext}(K)$ is called a facial topology. So, Proposition 1.6 and Theorem 1.7 hold for any facial topology of metrizable sets K. We do not know of any nice conditions which insure that (C 2) is satisfied.

EXAMPLE 3.6. *Split faces* (see [1], [8]).

Let K be a compact convex set in an l.c.t.v.s. E. Let F be a closed face of K. F is said to be a split face if F' is a face and the elements f, f', and α in the decomposition (3.1) are all unique. Hence, split faces are parallelizable faces but the converse is not true (take, for example, any closed edge of a rectangle). We take $\mathscr{T}$ to be the set of all closed split faces of K. Alfsen and Andersen [1] showed that $\mathscr{T}$ satisfies conditions 2 and 3 and the others are clear. The τ-topology on $\mathrm{Ext}(K)$ is called the facial topology. So Proposition 1.6 and Theorem 1.7 hold for the facial topology of metrizable sets K. Fairly strong conditions (to appear elsewhere) are sufficient for (C 2) to hold and, so, for Theorem 2.5 to hold. As a special case we cite the following:

THEOREM. *If the facial topology of a metrizable set K is T_1, then the properties of first countability, second countability, local compactness, and local sequential compactness for the facial topology are equivalent.*

Since every closed face of a simplex is split, this example already includes Example 3.1 (and via it, Example 3.2). This theorem covers more than just the previous examples as there is a metrizable compact convex set with a T_1 facial topology which is not a simplex [1, Theorem 6.4].

BIBLIOGRAPHY

1. E. Alfsen and T. Andersen, *Split faces of compact convex sets*, Proc. London Math. Soc. (3) 21 (1970), 415–442.
2. E. Alfsen and E. Effros, *Structure in real Banach spaces*, Ann. of Math. 96 (1972), 98–174.
3. H. Bauer, *Harmonische Räume und ihre Potentialtheorie*, Lecture notes in Mathematics 22, Springer-Verlag, Berlin · Heidelberg · New York, 1966.
4. E. Effros, *Structure in simplexes*, Acta. Math. 117 (1967), 103–121.
5. E. Effros, *On a class of real Banach spaces*, Israel J. Math. 9 (1971), 430–458.
6. E. Effros, and A. Gleit, *Structure in Simplexes* III, Trans. Amer. Math. Soc. 142 (1969), 355–380.
7. A. Gleit, *On the structure topology of simplex spaces*. Pacific J. Math. 34 (1970), 389–405.
8. F. Perdrizet, *Espaces de Banach ordonnés et idéaux*, J. Math. Pures Appl. 49 (1970), 61–98.
9. M. Rogalski, *Topologie faciale dans les convexes compacts*, C.R. Acad. Sci. Paris Sér. A. 270 (1970), 523–526.
10. P. Taylor, *The structure space of a Choquet simplex*, J. Functional Analysis 6 (1970), 208–217.

UNIVERSITY OF MASSACHUSETTS,
AMHERST, MASSACHUSETTS, U.S.A.