MATH. SCAND. 31 (1972), 220–224

ON THE REALIZATION OF CERTAIN MODULES OVER THE STEENROD ALGEBRA

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1.

Let $\mathcal{A}$ denote the (mod p) Steenrod algebra, p any prime and let Q_j be the unique non-zero primitive in degree $2p^j-1$. In the present note we show that for any string of integers $0 \leqq j_0 < j_1 < \ldots < j_q$ there exists a spectrum whose cohomology (as an $\mathcal{A}$-module) is isomorphic to $\mathcal{A}/\mathcal{A}(Q_{j_0},\ldots,Q_{j_q})$. Here $\mathcal{A}(Q_{j_0},\ldots,Q_{j_q})$ denotes the left ideal generated by $Q_{j_0},\ldots,Q_{j_q}$. Previously, Margolis [5] has realized the modules $\mathcal{A}/\mathcal{A}(Q_i)$ and $\mathcal{A}/\mathcal{A}(Q_0,Q_i)$ by other (and quite complicated) methods.

Let MU denote the Thom spectrum of the universal unitary bundle over BU. It is well-known that

$$\pi_*(MU) = \mathbf{Z}[x_1,x_2,\ldots] ,$$

where $\deg x_i = 2i$.

For any string of integers $0 < n_1 < n_2 < \ldots < n_q$ there exists a spectrum $MU\langle n_1,\ldots,n_q\rangle$ such that

$$\pi_*(MU\langle n_1,\ldots,n_q\rangle) = \mathbf{Z}[x_{n_1},\ldots,x_{n_q}]$$

and a map

$$\nu\colon MU \to MU\langle n_1,\ldots,n_q\rangle .$$

The Thom spectrum MU is a ring spectrum and $MU\langle n_1,\ldots,n_q\rangle$ is a "module" spectrum over MU, that is there is a map

$$MU \wedge MU\langle n_1,\ldots,n_q\rangle \to MU\langle n_1,\ldots,n_q\rangle .$$

Further, $\nu\colon MU \to MU\langle n_1,\ldots,n_q\rangle$ is a "module" map. The induced map ν_* on homotopy groups is just

$$\begin{aligned} \nu_*(x_i) &= 0 \quad \text{if } i \notin \{n_1,\ldots,n_q\} \\ &= x_i \quad \text{if } i \in \{n_1,\ldots,n_q\} \end{aligned}$$

The construction of $MU\langle n_1,\ldots,n_q\rangle$ is based on ideas of Sullivan and was carried out in detail in [2]. The spectrum $MU\langle n_1,\ldots,n_q\rangle$ appears

Received December 21, 1970; in revised form February 20, 1972.

as representing object for a bordism theory of manifolds with certain cone singularities.

We shall calculate $H^*(MU\langle n_1,\ldots,n_q\rangle; \mathsf{Z}_p)$ as a module over the Steenrod algebra in case all the n_i's are of the form p^j-1. The calculation is based on a simple application of the Atiyah–Hirzebruch spectral sequence for a generalized homology theory. If X and Y are spectra (with zero homotopy in negative degrees) then the spectral sequence, denoted $E^r{}_{*,*}(X,Y)$ is a first quadrant, homology type, spectral sequence with

$$E^2{}_{*,*} = H_*(X;\pi_*(Y))$$
$$E^\infty{}_{*,*} = E^0 Y_*(X)\,,$$

where $Y_*(X)=\pi_*(Y\wedge X)\cong X_*(Y)$. We shall only use the spectral sequence in the case where X is the Eilenberg–MacLane spectrum and we write $E^r{}_{*,*}(Y)$ instead of $E^r{}_{*,*}(K(\mathsf{Z}_p),Y)$. If Y is torsion free then

$$E^2{}_{*,*}(Y) = H_*(K(\mathsf{Z}_p);\mathsf{Z})\otimes\pi_*(Y)$$

and the spectral sequence converges to $Y_*(K(\mathsf{Z}_p))\cong H_*(Y;\mathsf{Z}_p)$.

Suppose that Y is connected ($\pi_0(Y)=\mathsf{Z}$, $\pi_i(Y)=0$ for $i<0$) and let $U\colon Y\to K(\mathsf{Z})$ be the map which induces the identity on π_0. The edge homomorphisms are then

$$h:\ \pi_*(Y)\to H_*(Y;\mathsf{Z}_p)$$
$$\chi\circ U_*:\ H_*(Y;\mathsf{Z}_p)\to H_*(K(\mathsf{Z});\mathsf{Z}_p)\cong H_*(K(\mathsf{Z}_p);\mathsf{Z})\,,$$

where h is the Hurewicz homomorphism and χ the isomorphism $H_*(K(\mathsf{Z});\mathsf{Z}_p)\cong H_*(K(\mathsf{Z}_p);\mathsf{Z})$.

If Y is a ring spectrum then $E^r{}_{*,*}(Y)$ is a spectral sequence of algebras over Z_p. If M is a "module" spectrum over the ring spectrum Y then $E^r{}_{*,*}(M)$ is a differential module over $E^r{}_{*,*}(Y)$ (that is

$$d_r(y\cdot m) = d_r y\cdot m+(-1)^{\deg y}y\cdot d_r m)\,.$$

2.

We first consider the case $Y=MU$. Then

$$E^2{}_{*,*}(MU) = \mathsf{Z}_p[\xi_1,\xi_2,\ldots]\otimes E\{\tau_1,\tau_2,\ldots\}\otimes\mathsf{Z}_p[x_1,x_2,\ldots]\,,$$

where $E\{\ \}$ denotes the exterior algebra and $\mathrm{bideg}(x_i)=(0,2i)$, $\mathrm{bideg}(\xi_i)=(2(p^i-1),0)$ and $\mathrm{bideg}(\tau_i)=(2p^i-1,0)$. (If $p=2$ then τ_i and ξ_i above should be interpreted as ξ_i and $\xi_i{}^2$, respectively.)

J. Cohen in [4] has examined the structure of $E^r{}_{*,*}(MU)$: The elements ξ_i all survives to E^∞ and there are elements $\bar\tau_j$ (where $\bar\tau_j\equiv\tau_j$ modulo decomposable elements) such that

$$d_{2p^j-1}(\bar{\tau}_j) = \bar{x}_k, \quad k = p^j - 1 ,$$

where $\bar{x}_k \equiv x_k$ modulo decomposable elements.

This is all easily implied by the remark that $\tau_j \in E^2{}_{*,0}$ cannot survive to E^∞ since $H_*(MU; \mathsf{Z}_p)$ is a polynomial algebra with one generator in each even dimension and the fact (Milnor [6]) that the composite map

$$\pi_*(MU) \to H_*(MU; \mathsf{Z}_p) \to QH_*(MU; \mathsf{Z}_p)$$

is zero in dimensions $2(p^j-1)$.

The structure of $E^r{}_{*,*}(MU)$ is now computed by a standard argument using the comparison theorem for spectral sequences

$$\begin{aligned} E^r{}_{*,*}(MU) = {} & \mathsf{Z}_p[\xi_1, \xi_2, \ldots] \otimes E\{\bar{\tau}_j \mid 2p^j - 1 \geqq r-1\} \\ & \otimes \mathsf{Z}_p[\{\bar{x}_k \mid k \neq p^j - 1, 2p^j - 1 \leqq r - 1\}] . \end{aligned}$$

3.

Next we consider $E^r{}_{*,*}(MU\langle n_1, \ldots, n_q\rangle)$ where all the n_i's are of the type $p^j - 1$, say $n_i = p^{j_i} - 1$. We have

$$\begin{aligned} E^2{}_{*,*}(MU\langle n_1, \ldots, n_q\rangle) = {} & \mathsf{Z}_p[\xi_1, \xi_2, \ldots] \otimes E\{\bar{\tau}_1, \bar{\tau}_2, \ldots\} \\ & \otimes \mathsf{Z}[x_{n_1}, \ldots, x_{n_q}] . \end{aligned}$$

The map $\nu: MU \to MU\langle n_1, \ldots, n_q\rangle$ induces an $E^r{}_{*,*}(MU)$-module map of spectral sequences

$$\nu_*: E^r{}_{*,*}(MU) \to E^r{}_{*,*}(MU\langle n_1, \ldots, n_q\rangle)$$

which on the E^2-level is just

$$\begin{aligned} \nu_*(\xi_i \otimes \bar{\tau}_j \otimes x_k) &= \xi_i \otimes \bar{\tau}_j \otimes x_k \quad \text{if } k \in \{n_1, \ldots, n_q\} \\ &= 0 \qquad\qquad\qquad \text{otherwise} . \end{aligned}$$

Comparing the two spectral sequences via ν_* we see that the only non-zero transgressive differentials in $E^r{}_{*,*}(MU\langle n_1, \ldots, n_q\rangle)$ are d_{2n_1+1}, $d_{2n_2+1}, \ldots, d_{2n_q+1}$, and that

$$d_{2n_i+1}(\bar{\tau}_j) = \bar{x}_{n_i} \quad (n_i = p^j - 1) .$$

It follows that

$$\begin{aligned} E^\infty{}_{*,*}(MU\langle n_1, \ldots, n_q\rangle) = {} & \mathsf{Z}_p[\xi_1, \xi_2, \ldots] \\ & \otimes E\{\bar{\tau}_j \mid p^j - 1 \notin \{n_1, \ldots n_q\}, j > 0\} . \end{aligned}$$

The map

$$U_*: H_*(MU\langle n_1, \ldots, n_q\rangle; \mathsf{Z}_p) \to H_*(K(\mathsf{Z}); \mathsf{Z}_p)$$

is therefore an injection with image

$$\mathsf{Z}_p[\chi(\xi_1),\chi(\xi_2),\ldots]\otimes E\{\chi(\bar\tau_j)\mid p^j-1\notin\{n_1,\ldots,n_q\},j>0\}\,,$$

where χ is the canonical anti-automorphism of $\mathfrak{A}$.

The primitive element Q_j (of degree $2p^j-1$) in $\mathfrak{A}$ is dual to τ_j. Since $\chi(\bar\tau_j)\equiv\tau_j$ modulo decomposable elements,

$$U^*\colon H^*(K(\mathsf{Z});\mathsf{Z}_p)\to H^*(MU\langle n_1,\ldots,n_q\rangle;\mathsf{Z}_p)$$

maps $Q_{j_1},\ldots,Q_{j_q}$ $(n_i=p^{j_i}-1)$ to zero so that U^* factors to give an epimorphism

$$\nu^*\colon \mathfrak{A}/\mathfrak{A}(Q_0,Q_{j_1},\ldots,Q_{j_q})\to H^*(MU\langle n_1,\ldots,n_q\rangle;\mathsf{Z}_p)\,.$$

The two vector spaces have the same dimension and U^* is therefore an isomorphism. This proves

THEOREM A. *Let* $0<i_1<\ldots<i_q$ *and set* $n_i=p^{j_i}-1$. *There is an isomorphism of left* $\mathfrak{A}$*-modules*

$$\mathfrak{A}/\mathfrak{A}(Q_0,Q_{j_1},\ldots,Q_{j_q})\cong H^*(MU\langle n_1,\ldots,n_q\rangle;\mathsf{Z}_p)\,.$$

Let L be the Z_p-Moore spectrum, (L is the cofiber of a map of degree $p\colon S^0\to S^0$) and write $MU_p\langle n_1,\ldots,n_q\rangle$ for $MU\langle n_1,\ldots,n_q\rangle\wedge L$. Then

$$\pi_*(MU_p\langle n_1,\ldots,n_q\rangle) = \mathsf{Z}_p[x_{n_1},\ldots,x_{n_q}]$$

and again there is a MU-"module" map

$$\nu_p\colon MU\to MU_p\langle n_1,\ldots,n_q\rangle$$

which on homotopy is ν_* composed with reduction modulo p. Let

$$U_p\colon MU_p\langle n_1,\ldots,n_q\rangle\to K(\mathsf{Z}_p)$$

be a map which induces the identity on π_0. Arguing as before we get

THEOREM B. *There is a left* $\mathfrak{A}$*-module isomorphism*

$$\mathfrak{A}/\mathfrak{A}(Q_{i_1},\ldots,Q_{i_q})\cong H^*(MU_p\langle n_1,\ldots,n_q\rangle;\mathsf{Z}_p)\,.$$

REMARKS:

(a) The spectrum $MU_p\langle n_1,\ldots,n_q\rangle$ is again a representing object for a bordism theory of manifolds with singularities. In fact one just adds one more singularity to the singularities determining $MU\langle n_1,\ldots,n_q\rangle$, namely the manifold consisting of p distinct points.

(b) Let $0<m_1<\ldots<m_r$ be a string of integers not all of the form p^j-1. In order to compute even the additive structure of $H^*(MU\langle m_1,\ldots,m_r\rangle)$ one would need more precise information on the structure of $E^r{}_{*,*}(MU)$, than provided by Cohen [4].

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