MATH. SCAND. 31 (1972), 231–236

ON THE POSSIBILITY OF FINDING CERTAIN CRITERIA FOR THE IRRATIONALITY OF A NUMBER DEFINED AS A LIMIT OF A SEQUENCE OF RATIONAL NUMBERS

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1.

In 1910 I (Viggo Brun) put forth the following theorem in an article entitled "Ein Sats über Irrationalität" (See [1]).

If the sequence

$$\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, \ldots \frac{a_n}{b_n}, \frac{a_{n+1}}{b_{n+1}} \ldots$$

is composed of strictly increasing positive rational numbers which are converging towards c, while the sequence

$$\frac{a_2-a_1}{b_2-b_1}, \frac{a_3-a_2}{b_3-b_2}, \ldots \frac{a_{n+1}-a_n}{b_{n+1}-b_n}, \frac{a_{n+2}-a_{n+1}}{b_{n+2}-b_{n+1}} \ldots$$

is composed of strictly decreasing numbers, then c is irrational. Here $a_1, a_2, \ldots a_n, \ldots$ and $b_1, b_2, \ldots b_n, \ldots$ are supposed to be positive integers such that $b_{n+1} > b_n$.

The geometrical considerations that led me to this theorem, but which are not mentioned in [1], are the following:

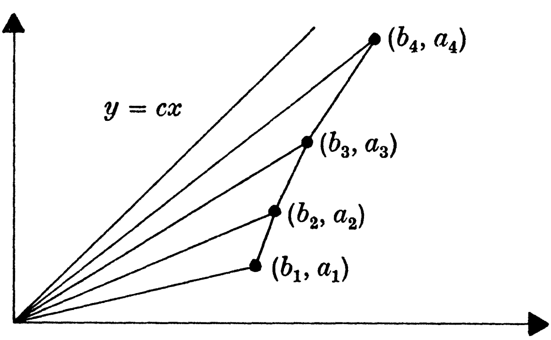

Fig. 1.

Received April 30, 1972.

Let x and y be coordinates in the plane. The points in the plane with integral coordinates will be called lattice points. Let c be the limit of the given sequence. Then the lattice points (b_i, a_i) form a configuration as shown in figure 1.

The characteristic feature of this configuration is that the polygon formed by the lattice points (b_i, a_i) shows its convex side towards the limit line $y = cx$. If c is rational, the line $y = cx$ will contain infinitely many equidistant lattice points. On each side of the line there will exist a stripe which does not contain any lattice points. This is impossible when the configuration of latticepoints is as in fig. 1. For a non-geometrical proof see [1]. The above theorem is simple but unfortunately not very useful since the picture very often will look as in fig. 2.

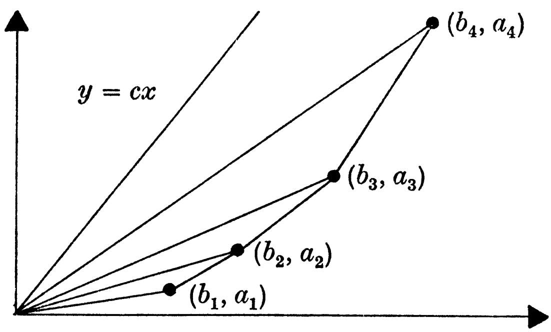

Fig. 2.

Here the polygon shows its concave side towards the limit line. Since the situation in fig. 2 occurs "much more frequently" than the situation in fig. 1, the value of my theorem is very limited.

In 1963 Al Froda published a generalization of the theorem in this journal [2].

I myself have made many efforts to find useful generalizations. My attempts have been to determine lattice points in the triangles

$$\langle (0,0), (b_n, a_n), (b_{n+1}, a_{n+1}) \rangle$$

which lie closer to the origin than (b_n, a_n) and (b_{n+1}, a_{n+1}) do. Possibly I will give some results in this direction in a future article.

During my work I made some temporary hypotheses. The first one was the following:

HYPOTHESIS 1. Let

$$c = u_1 + u_2 + \ldots + u_n + \ldots$$

be the sum of a convergent series, where the u_i's are positive rational numbers. If

$$\lim_{n\to\infty} u_{n+1}/u_n = 0$$

then c is irrational.

This condition at least excludes all geometric series. I mentioned this temporary hypothesis to my young friend Finn Faye Knudsen. He recognized very quickly that the hypothesis was wrong and gave a counterexample.

COUNTEREXAMPLE 1. Put

$$u_n = \frac{n}{(n+1)!}.$$

Here the partial sums are

$$s_n = \frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\ldots+\frac{n}{(n+1)!} = 1-\frac{1}{(n+1)!}.$$

Therefore

$$c = \lim_{n\to\infty} s_n = 1$$

whereas

$$\lim_{n\to\infty} u_{n+1}/u_n = 0\,.$$

After a while I mentioned another possible hypothesis for irrationality (which excluded counterexample 1).

HYPOTHESIS 2. Let again

$$c = u_1+u_2+u_3+\ldots+u_n+\ldots$$

be the sum of a convergent series whose terms are positive rational numbers such that

(1) $$\lim_{n\to\infty} u_{n+1}/u_n = 0\,,$$

(2) $$u_n(u_{n+2}+u_{n+3})-u_{n+1}^2 > 0\,.$$

Then c is irrational.

When I showed this hypothesis to my friend, he proved that also these conditions are insufficient and he gave a

COUNTEREXAMPLE 2. Put

$$u_n = \frac{n2^n}{(n+2)!} = \frac{2^n}{(n+1)!}-\frac{2^{n+1}}{(n+2)!}.$$

The rest of this paper, which is written by Finn Faye Knudsen, will be devoted to a general theorem which shows that criteria of "this type" can never lead to the goal. Even though this is a negative result, it gives in my opinion a valuable contribution to the study of this fundamental problem.

2.

Let k be a natrual number, and let $E \subset \mathsf{R}^k$ be the subset of real k-dimensional space defined by $x_1 > 0$, $x_2 > 0, \ldots, x_k > 0$.

Definition 1. By a *criterion* we shall mean a finite number of real-valued functions, $F_1, F_2, \ldots, F_m$ and G, defined on E.

Let $u_1 + u_2 + u_3 + \ldots + u_n + \ldots$ be a series with real positive terms.

Definition 2. We shall say that the series $u_1 + u_2 + u_3 + \ldots + u_n + \ldots$ *satisfies the criterion* $(F_1, F_2, \ldots, F_m, G)$ if there exists a natural number N with the property that for all $n \geqq N$ we have:

(1) $F_i(u_n, u_{n+1}, u_{n+2}, \ldots, u_{n+k-1}) > 0, \quad 1 \leqq i \leqq m,$
(2) $\lim_{n \to \infty} G(u_n, u_{n+1}, \ldots, u_{n+k-1}) = 0.$

Definition 3. We shall say that a criterion $(F_1, F_2, \ldots, F_m, G)$ is an *irrationality criterion* if the following is true.

Whenever $u_1 + u_2 + u_3 + \ldots + u_n + \ldots$ is a convergent series whose terms are positive *rational* numbers and satisfies the criterion $(F_1, F_2, \ldots, G)$, its sum c is an irrational number.

Definition 4. A criterion $(F_1, F_2, \ldots, F_m, G)$ will be said to be of *continuous type* if all the functions $F_1, F_2, \ldots, F_m$ and G are continuous functions.

We shall now prove that *there exists no irrationality criterion of continuous type* or more precisely:

Theorem. *Let $(F_1, F_2, \ldots, F_m, G)$ be a criterion of continuous type and suppose that a series $u_1 + u_2 + u_3 + \ldots + u_n + \ldots$ with rational positive terms satisfies the criterion. Also suppose $c = \sum u_n < \infty$. Then we can find another convergent series of positive rational terms $v_1 + v_2 + \ldots + v_n + \ldots$ which also satisfies the criterion and whose sum is rational.*

The method of proof will be to change the terms in the original series u_i by sufficiently small positive rational numbers δ_i. Then define $v_i = u_i + \delta_i$. We will divide the proof into several parts.

Lemma 1. *Let F be a real-valued continuous function on E and let $u_1, u_2, \ldots, u_n, \ldots$ be a sequence of positive real numbers such that*

(1) $\sum u_i < \infty$,
(2) $F(u_n, u_{n+1}, \ldots, u_{n+k-1}) > 0$ *for all* $n > N$.

Then there is a sequence $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n, \ldots$ of positive real numbers such that

(3) $\sum \varepsilon_i < \infty$,
(4) *for all sequences $\delta_1, \delta_2, \ldots, \delta_n, \ldots$ with $0 \leqq \delta_i \leqq \varepsilon_i$ we have*

$$F(u_n + \delta_n, u_{n+1} + \delta_{n+1}, \ldots, u_{n+k-1} + \delta_{n+k-1}) > 0 \quad \textit{for } n > N .$$

Proof. We can choose the first N ε's quite arbitrarily. For all $n > N$ we can choose a neighbourhood V_n of the point $(u_n, u_{n+1}, \ldots, u_{n+k-1})$ in E such that

$$F(v_n, v_{n+1}, \ldots, v_{n+k-1}) > 0$$

for all points $(v_n, \ldots, v_{n+k-1})$ in V_n. The set V_n certainly contains a closed box of type:

$$\{(\delta_n, \ldots, \delta_{n+k-1})\ ;\ 0 \leqq |u_i - \delta_i| \leqq \varepsilon_i{}^n\} .$$

If we put

$$\varepsilon'_n = \min(\varepsilon_n{}^{n-k+1}, \varepsilon_n{}^{n-k+2}, \ldots, \varepsilon_n{}^n)$$

and define $\varepsilon_n = 2^{-n} \min(1, \varepsilon'_n)$, the sequence $\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots, \varepsilon_n, \ldots$ has the required property.

Lemma 2. *Let G be a real-valued continuous function on E and let $u_1, u_2, \ldots, u_n, \ldots$ be a sequence of positive real numbers such that*

(1) $\sum u_i < \infty$
(2) $\lim_{n\to\infty} G(u_n, u_{n+1}, \ldots, u_{n+k-1}) = 0$.

Then there is a sequence of positive real numbers $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n, \ldots$ such that

(3) $\sum \varepsilon_i < \infty$
(4) *for all sequences $\delta_1, \delta_2, \ldots, \delta_n, \ldots$ with $0 \leqq \delta_i \leqq \varepsilon_i$ we have*

$$\lim_{n\to\infty} G(u_n + \delta_n, \ldots, u_{n+k-1} + \delta_{n+k-1}) = 0 .$$

Proof. Let V_k be a neighbourhood of the point $(u_n,\ldots,u_{n+k-1})$ in E such that

$$|G(u_n,\ldots,u_{n+k-1})-G(v_n,\ldots,v_{n+k-1})| < 2^{-n} .$$

Again V_n contains a box

$$\{(v_n,\ldots,v_{n+k-1})\ ;\ 0 \leqq |u_i-v_i| \leqq \varepsilon_i{}^n\} ,$$

and we see that the sequence $\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n,\ldots$ has the required property if we choose

$$\varepsilon_n = 2^{-n}[\min(1,\varepsilon_n{}^{n-k+1},\varepsilon_n{}^{n-k+2},\ldots,\varepsilon_n{}^n)] .$$

Lemma 3. *Let $\varepsilon_1,\varepsilon_2,\varepsilon_3,\ldots,\varepsilon_n,\ldots$ be a sequence of positive real numbers such that $\sum\varepsilon_i=\varepsilon<\infty$. Then given any number β, with $0<\beta<\varepsilon$, we can find a sequence $\delta_1,\delta_2,\ldots,\delta_n,\ldots$ of positive rational numbers such that*

(1) $0 \leqq \delta_i < \varepsilon_i$,

(2) $\sum\delta_i=\beta$.

Proof. Let $\varepsilon'_i=\varepsilon_i\varepsilon/\beta$ and let s_n be the partial sum,

$$s_n = \textstyle\sum_{i=1}^n \varepsilon'_i .$$

Inductively we define a sequence $k_0,k_1,\ldots,k_n,\ldots$ of rational numbers as follows:

$$k_0 = 0$$
$$k_n = \text{some rational number in the interval}$$
$$\langle \max(k_{n-1},s_{n+1}-\varepsilon_{n+1}),s_n\rangle .$$

If we define $\delta_n=k_n-k_{n-1}$ for $n=1,2,\ldots$ we have (1) and (2).

This completes the proof of the theorem.

REFERENCES

1. V. Brun, *Ein Satz über Irrationalität,* Archiv for Mathematik og Naturvidenskab (Kristiania) 31 (1910), 6 pp.
2. A. Froda, *Critères paramétriques d'irrationalité,* Math. Scand. 12 (1963), 199–208.

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