MATH. SCAND. 31 (1972), 248–256

SOME REDUCTION FORMULAS FOR HYPERGEOMETRIC FUNCTIONS

PER W. KARLSSON

1. Introduction.

In certain cases, hypergeometric functions can be expressed in terms of Gamma functions and elementary functions. Classical examples of such reduction formulas are the theorems of Gauss and Kummer, by which the Gaussian hypergeometric function ${}_2F_1(a,b;c;z)$ is expressed for $z=1$ and for $z=-1$, $c=1+a-b$, respectively.

Recently, other formulas of this kind for ${}_2F_1$ and for Appell's F_1 have been derived by Spiegel [3] and by Lavoie & Trottier [2] using rational substitutions of the third degree (in trigonometrical disguise) in the Eulerian integral representation of these functions.

In the present paper, the same method is used to obtain further results of this kind. Some definitions and results from the theory of hypergeometric functions of several variables are briefly mentioned in section 2. (Cf. e.g., [1, §§ 37–38].)

2. Preliminaries.

Gauss's function ${}_2F_1$ and Appell's function F_1 are both special cases of Lauricella's hypergeometric function F_D defined by the power series

$$(2.1)\qquad F_D\left[\begin{matrix}a;\, b_1,\ldots,b_n; \\ c;\end{matrix}\; z_1,\ldots,z_n\right]$$

$$= \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} \frac{(a)_{m_1+\ldots+m_n}(b_1)_{m_1}\ldots(b_n)_{m_n}}{(c)_{m_1+\ldots+m_n}} \frac{z_1^{m_1}\ldots z_n^{m_n}}{m_1!\ldots m_n!},$$

where $(a)_m \equiv \Gamma(a+m)/\Gamma(a)$; for convergence, $|z_m|<1$, $m=1,2,\ldots,n$.

The F_D function is represented by a single integral of Euler's type,

$$(2.2)\qquad F_D\left[\begin{matrix}a;\, b_1,\ldots,b_n; \\ c;\end{matrix}\; z_1,\ldots,z_n\right]$$

$$= B(a,c-a)^{-1} \int_0^1 t^{a-1}(1-t)^{c-a-1} \prod_{m=1}^{n} \{(1-z_m t)^{-b_m}\}\, dt\,,$$

Received December 19, 1971

where B denotes Euler's Beta function. The integral converges for $\operatorname{Re} c > \operatorname{Re} a > 0$; however, these restrictions are of no importance because of the principle of analytical continuation. Furthermore, it is required that $|\arg(1-z_m)| < \pi$, $m = 1,2,\ldots,n$.

There are $n+1$ independent linear transformations of the F_D function:

$$F_D\begin{bmatrix} a; b_1 \ldots, b_n; & z_1, \ldots, z_n \\ c; & \end{bmatrix} \tag{2.3}$$

$$= (1-z_1)^{-a}\, F_D\begin{bmatrix} a; c-\sum_m b_m, b_2, \ldots, b_n; & \dfrac{-z_1}{1-z_1}, \dfrac{z_2-z_1}{1-z_1}, \ldots, \dfrac{z_n-z_1}{1-z_1} \\ c; & \end{bmatrix},$$

etc., and

$$F_D\begin{bmatrix} a; b_1, \ldots, b_n; & z_1, \ldots, z_n \\ c; & \end{bmatrix} \tag{2.4}$$

$$= \prod_{m=1}^{n}\{(1-z_m)^{-b_m}\}\, F_D\begin{bmatrix} c-a; b_1, \ldots, b_n; & \dfrac{z_1}{z_1-1}, \ldots, \dfrac{z_n}{z_n-1} \\ c; & \end{bmatrix}.$$

The number of z-variables in the F_D-function is diminished by one if (i) one of the b-parameters equals zero, (ii) one of the z-variables equals zero, (iii) one of the z-variables equals unity, or (iv) two z-variables are equal. Reduction formulas for these four cases are readily derived.

3. The method.

In the integral (2.2) the variable of integration is changed by means of a (regular) rational substitution which leaves the limits of integration unaltered or reversed. By such a substitution the form of the integrand is unaffected, although in general the number of different factors is increased; the number of free parameters (in the exponents) cannot increase. It follows that the integral, with the substitution performed, in general represents a special F_D with a greater number of variables. This number is found to be somewhat greater than kn, where k denotes the degree of the substitution.

These substitutions generally lead to very complicated expressions. However, there are some special cases which may well be characterized as "sensible" or "interesting". Some of these cases, with k equal to 3 or 2 and with $n=1$, are studied in the following sections. (When $k=1$, the linear transformations (2.3) and (2.4) are obtained, cf. e.g., [1, § 38].)

4. First cubic rational substitution.

In (2.2), n is taken equal to unity, and a change of variable in the integral is effected by the substitution

$$t = (1-y)\tau^3/(1-y\tau), \quad y \neq 1 . \tag{4.1}$$

As a result we obtain

$$B(a,c-a)\,{}_2F_1\begin{bmatrix} a,b; & z \\ c; & \end{bmatrix} \tag{4.2}$$

$$= 3(1-y)^a \int_0^1 \tau^{3a-1}(1-\tau)^{c-a-1}(1-y\tau)^{b-c}(1-\tfrac{2}{3}y\tau)(1-p_1\tau)^{c-a-1} \times$$
$$\times (1-p_2\tau)^{c-a-1}(1-r_1\tau)^{-b}(1-r_2\tau)^{-b}(1-r_3\tau)^{-b}d\tau ,$$

where

$$\left.\begin{matrix} p_1 \\ p_2 \end{matrix}\right\} = \tfrac{1}{2}[y-1 \pm \{(y-1)(y+3)\}^{\frac{1}{2}}] , \tag{4.3}$$

while r_1, r_2, r_3 are defined by

$$1 - zt \equiv (1-r_1\tau)(1-r_2\tau)(1-r_3\tau)/(1-y\tau) . \tag{4.4}$$

The integral in (4.2) represents an F_D with seven variables $\{y, \frac{2}{3}y, p_1, p_2, r_1, r_2, r_3\}$ but only three free parameters. More interesting formulas can be expected if the number of variables is diminished by imposing additional conditions. We first demand that two of the r-variables be equal. This leads to the conditions

$$z = \frac{4y^3}{27(y-1)}, \tag{4.5a}$$

that is

$$r_1 = r_2 = \tfrac{2}{3}y, \quad r_3 = -\tfrac{1}{3}y ,$$

or

$$z = 0, \quad \text{that is,} \quad r_1 = r_2 = 0, \quad r_3 = y . \tag{4.5b}$$

Utilizing the conditions (4.5a) in (4.2) we find the reduction formula

$$F_D\begin{bmatrix} 3a; c-b, 2b-1, b, 1+a-c, 1+a-c; & y, \frac{2}{3}y, -\frac{1}{3}y, p_1, p_2 \\ 2a+c; & \end{bmatrix} \tag{4.6}$$

$$= (1-y)^{-a}\frac{\Gamma(a)\Gamma(2a+c)}{3\Gamma(3a)\Gamma(c)}\,{}_2F_1\begin{bmatrix} a,b; & \dfrac{4y^3}{27(y-1)} \\ c; & \end{bmatrix},$$

provided that the y-plane is cut along the real axis from $-\infty$ to -3 and from 1 to $+\infty$. In particular, the variables p_1, p_2 both disappear if we take $c = a+1$; then, for y in the cut plane,

$$(4.7)\qquad F_D\begin{bmatrix}3a;\ 1+a-b,\ 2b-1,\ b; & y, \tfrac{2}{3}y, -\tfrac{1}{3}y\\ 3a+1; & \end{bmatrix}$$

$$= (1-y)^{-a}{}_2F_1\begin{bmatrix}a, b; & \dfrac{4y^3}{27(y-1)}\\ a+1; & \end{bmatrix}.$$

The ${}_2F_1$ in (4.7) is a power function multiplied by an incomplete Beta function.

Next, with $z=0$ in (4.2) we obtain another reduction formula

$$(4.8)\qquad F_D\begin{bmatrix}3a;\ c,\ -1,\ 1+a-c,\ 1+a-c; & y, \tfrac{2}{3}y, p_1, p_2\\ 2a+c; & \end{bmatrix}$$

$$= (1-y)^{-a}\,\frac{\Gamma(a)\Gamma(2a+c)}{3\Gamma(3a)\Gamma(c)};$$

the y-plane must be cut as described above.

A consideration of the equations $p_1=0$, $p_1=p_2$, $p_1=y$, etc., shows that a further decrease in the number of variables occurs when y is equal to -3 or 0. (Only trivial results are obtained with $c=a+1$ in (4.8) or, equivalently, $b=0$ in (4.7).)

With $y=-3$, we have $p_1=p_2=\frac{2}{3}y=-2$, and (4.8) becomes

$$(4.9\text{a})\qquad F_1\begin{bmatrix}3a;\ c,\ 1+2a-2c; & -3, -2\\ 2a+c; & \end{bmatrix} = 4^{-a}\,\frac{\Gamma(a)\Gamma(2a+c)}{3\Gamma(3a)\Gamma(c)}.$$

Application of the linear transformation (2.3) yields

$$(4.9\text{b})\qquad F_1\begin{bmatrix}3a;\ 2c-1,\ 1+2a-2c; & \tfrac{3}{4}, \tfrac{1}{4}\\ 2a+c; & \end{bmatrix} = 16^{a}\,\frac{\Gamma(a)\Gamma(2a+c)}{3\Gamma(3a)\Gamma(c)}$$

and

$$(4.9\text{c})\qquad F_1\begin{bmatrix}3a;\ c,\ 2c-1; & -\tfrac{1}{3}, \tfrac{2}{3}\\ 2a+c; & \end{bmatrix} = (\tfrac{27}{4})^{a}\,\frac{\Gamma(a)\Gamma(2a+c)}{3\Gamma(3a)\Gamma(c)},$$

while (2.4), when applied to these three results, yields

$$(4.9\text{d})\qquad F_1\begin{bmatrix}c-a;\ c,\ 1+2a-2c; & \tfrac{3}{4}, \tfrac{2}{3}\\ 2a+c; & \end{bmatrix} = (\tfrac{4}{9})^{c-a}\,\frac{\Gamma(a)\Gamma(2a+c)}{\Gamma(3a)\Gamma(c)},$$

$$(4.9\text{e})\qquad F_1\begin{bmatrix}c-a;\ 2c-1,\ 1+2a-2c; & -3, -\tfrac{1}{3}\\ 2a+c; & \end{bmatrix} = 9^{a-c}\,\frac{\Gamma(a)\Gamma(2a+c)}{\Gamma(3a)\Gamma(c)},$$

$$(4.9\text{f})\qquad F_1\begin{bmatrix}c-a;\ c,\ 2c-1; & \tfrac{1}{4}, -2\\ 2a+c; & \end{bmatrix} = (\tfrac{27}{4})^{a-c}\,\frac{\Gamma(a)\Gamma(2a+c)}{\Gamma(3a)\Gamma(c)}.$$

From these six reduction formulas for Appell's F_1, twelve reduction formulas for the Gaussian function ${}_2F_1$ are readily obtained by taking c equal to 0, $\frac{1}{2}$ or $a+\frac{1}{2}$.

Finally, let $y=0$. In this case, the variable z need not satisfy any additional condition. From (4.2) we then obtain the following formula, where $\varepsilon = \exp 2\pi i/3$ and z has been set equal to ζ^3;

$$(4.10)\quad F_D\left[\begin{matrix}3a;\,1+a-c,\,1+a-c,\,b,\,b,\,b;\\ 2a+c;\end{matrix}\ \varepsilon,\varepsilon^2,\zeta,\varepsilon\zeta,\varepsilon^2\zeta\right] = \frac{\Gamma(a)\,\Gamma(2a+c)}{3\Gamma(3a)\,\Gamma(c)}\,{}_2F_1\left[\begin{matrix}a,b;\\ c;\end{matrix}\ \zeta^3\right].$$

The ζ-plane must be cut along straight lines from 1, ε and ε^2 to ensure that ζ^3 is not a real number greater than unity.

By taking $b=0$ or $\zeta=0$ in (4.10) we obtain a result similar to (4.9a) and lending itself to similar linear transformations; for brevity, $\mu = 3^{-\frac{1}{2}}\exp i\pi/6$:

$$(4.11a)\quad F_1\left[\begin{matrix}3a;\,1+a-c,\,1+a-c;\\ 2a+c;\end{matrix}\ \varepsilon,\varepsilon^2\right] = \frac{\Gamma(a)\,\Gamma(2a+c)}{3\Gamma(3a)\,\Gamma(c)},$$

$$(4.11b)\quad F_1\left[\begin{matrix}3a;\,3c-2,\,1+a-c;\\ 2a+c;\end{matrix}\ \bar{\mu},\,-\varepsilon\right] = 3^{3a/2}e^{-i\pi a/2}\,\frac{\Gamma(a)\,\Gamma(2a+c)}{3\Gamma(3a)\,\Gamma(c)},$$

$$(4.11c)\quad F_1\left[\begin{matrix}3a;\,1+a-c,\,3c-2;\\ 2a+c;\end{matrix}\ -\varepsilon^2,\mu\right] = 3^{3a/2}e^{i\pi a/2}\,\frac{\Gamma(a)\,\Gamma(2a+c)}{3\Gamma(3a)\,\Gamma(c)},$$

$$(4.11d)\quad F_1\left[\begin{matrix}c-a;\,1+a-c,\,1+a-c;\\ 2a+c;\end{matrix}\ \mu,\bar{\mu}\right] = 3^{a-c}\,\frac{\Gamma(a)\,\Gamma(2a+c)}{\Gamma(3a)\,\Gamma(c)},$$

$$(4.11e)\quad F_1\left[\begin{matrix}c-a;\,3c-2,\,1+a-c;\\ 2a+c;\end{matrix}\ \varepsilon,\,-\varepsilon^2\right] = 3^{3(a-c)/2}e^{i\pi(c-a)/6}\,\frac{\Gamma(a)\,\Gamma(2a+c)}{\Gamma(3a)\,\Gamma(c)},$$

$$(4.11f)\quad F_1\left[\begin{matrix}c-a;\,1+a-c,\,3c-2;\\ 2a+c;\end{matrix}\ -\varepsilon,\,\varepsilon^2\right] = 3^{3(a-c)/2}e^{i\pi(a-c)/6}\,\frac{\Gamma(a)\,\Gamma(2a+c)}{\Gamma(3a)\,\Gamma(c)}.$$

With $c=\frac{2}{3}$, Watson's formulas [4] for ${}_2F_1$ are obtained; $c=a+1$ leads to trivialities.

5. Second cubic rational substitution.

Results similar to those obtained in the preceding section are found by application of the rational substitution

(5.1) $$t = \tau^2(1-x\tau)/(1-x), \quad x \neq 1 .$$

Insertion in (2.2) yields

(5.2) $$B(a,c-a)\,{}_2F_1\begin{bmatrix} a,b; & \\ c; & z \end{bmatrix}$$

$$= 2(1-x)^{-a}\int_0^1 \tau^{2a-1}(1-\tau)^{c-a-1}(1-x\tau)^{a-1}(1-\tfrac{3}{2}x\tau)(1-p_1\tau)^{c-a-1}\times$$
$$\times(1-p_2\tau)^{c-a-1}(1-r_1\tau)^{-b}(1-r_2\tau)^{-b}(1-r_3\tau)^{-b}\,d\tau ,$$

where now

(5.3) $$\left.\begin{matrix} p_1 \\ p_2 \end{matrix}\right\} = \tfrac{1}{2}[-1 \pm \{(1+3x)/(1-x)\}^{\frac{1}{2}}]$$

and

(5.4) $$1-zt \equiv (1-r_1\tau)(1-r_2\tau)(1-r_3\tau) .$$

Again, the integral represents an F_D with seven variables $\{x, \tfrac{3}{2}x, p_1, p_2, r_1, r_2, r_3\}$, but only three free parameters. Conditions for diminishing the number of r-variables are

(5.5a) $$z = \tfrac{27}{4}x^2(1-x) ,$$

that is,

$$r_1 = r_2 = \tfrac{3}{2}x, \quad r_3 = -3x ,$$

or

(5.5b) $$z = 0, \quad \text{that is,} \quad r_1 = r_2 = r_3 = 0 .$$

With (5.5a) inserted, (5.2) yields the reduction formula

(5.6) $$F_D\begin{bmatrix} 2a;\, 1-a,\, 2b-1,\, b,\, 1+a-c,\, 1+a-c; & \\ a+c; & x,\, \tfrac{3}{2}x,\, -3x,\, p_1,\, p_2 \end{bmatrix}$$

$$= (1-x)^a\,\frac{\Gamma(a)\Gamma(a+c)}{2\Gamma(2a)\Gamma(c)}\,{}_2F_1\begin{bmatrix} a,\, b; & \\ c; & \tfrac{27}{4}x^2(1-x) \end{bmatrix} .$$

In particular, when $c=a+1$,

(5.7) $$F_D\begin{bmatrix} 2a;\, 1-a,\, 2b-1,\, b; & \\ 2a+1; & x,\, \tfrac{3}{2}x,\, -3x \end{bmatrix}$$

$$= (1-x)^a\,{}_2F_1\begin{bmatrix} a,\, b; & \\ a+1; & \tfrac{27}{4}x^2(1-x) \end{bmatrix} .$$

These two results are valid in that domain of the x-plane which contains the origin and is bounded by the inverse image of the real axis from 1 to $+\infty$ by the mapping $x \to 27x^2(1-x)/4$. This boundary consists of a cut from $-\infty$ to $-\frac{1}{3}$ and a curve through $\frac{2}{3}$, symmetric with respect to the real axis and with $\{x: \arg x = \pm\frac{1}{3}\pi\}$ as asymptotes.

When (5.5b) is inserted in (5.2), we obtain

$$(5.8)\qquad F_D\left[\begin{matrix}2a;\,1-a,\,-1,\,1+a-c,\,1+a-c;\\ a+c;\end{matrix}\; x, \tfrac{3}{2}x, p_1, p_2\right] = (1-x)^a\,\frac{\Gamma(a)\Gamma(a+c)}{2\Gamma(2a)\Gamma(c)},$$

provided that the x-plane is cut along the real axis from $\frac{2}{3}$ to $+\infty$.

A further decrease in the number of variables occurs when x is equal to $-\frac{1}{3}$ or $\frac{2}{3}$, and in the degenerate case $x=0$ (section 6).

When $x=-\frac{1}{3}$, we have $p_1=p_2=\frac{3}{2}x=-\frac{1}{2}$, and the following results, analogous to (4.9a)–(4.9f) are obtained from (5.8); they were given by Lavoie & Trottier [2] and are stated here for the sake of completeness:

$$(5.9a)\qquad F_1\left[\begin{matrix}2a;\,1-a,\,1+2a-2c;\\ a+c;\end{matrix}\; -\tfrac{1}{3}, -\tfrac{1}{2}\right] = (\tfrac{4}{3})^a\,\frac{\Gamma(a)\Gamma(a+c)}{2\Gamma(2a)\Gamma(c)},$$

$$(5.9b)\qquad F_1\left[\begin{matrix}2a;\,3c-2,\,1+2a-2c;\\ a+c;\end{matrix}\; \tfrac{1}{4}, -\tfrac{1}{8}\right] = (\tfrac{4}{3})^{3a}\,\frac{\Gamma(a)\Gamma(a+c)}{2\Gamma(2a)\Gamma(c)},$$

$$(5.9c)\qquad F_1\left[\begin{matrix}2a;\,1-a,\,3c-2;\\ a+c;\end{matrix}\; \tfrac{1}{9}, \tfrac{1}{3}\right] = 3^a\,\frac{\Gamma(a)\Gamma(a+c)}{2\Gamma(2a)\Gamma(c)},$$

$$(5.9d)\qquad F_1\left[\begin{matrix}c-a;\,1-a,\,1+2a-2c;\\ a+c;\end{matrix}\; \tfrac{1}{4}, \tfrac{1}{3}\right] = (\tfrac{2}{3})^{2(c-a)}\,\frac{\Gamma(a)\Gamma(a+c)}{\Gamma(2a)\Gamma(c)},$$

$$(5.9e)\qquad F_1\left[\begin{matrix}c-a;\,3c-2,\,1+2a-2c;\\ a+c;\end{matrix}\; -\tfrac{1}{3}, \tfrac{1}{9}\right] = 3^{a-c}\,\frac{\Gamma(a)\Gamma(a+c)}{\Gamma(2a)\Gamma(c)},$$

$$(5.9f)\qquad F_1\left[\begin{matrix}c-a;\,1-a,\,3c-2;\\ a+c;\end{matrix}\; -\tfrac{1}{8}, -\tfrac{1}{2}\right] = (\tfrac{2}{3})^{3(c-a)}\,\frac{\Gamma(a)\Gamma(a+c)}{\Gamma(2a)\Gamma(c)}.$$

For $x=\frac{2}{3}$, we find that $p_1=\frac{3}{2}x=1$, $p_2=-2$. Since unity appears, it is easier to insert in (5.2) than in (5.8). The results are analogous to those found in the other cases; only three formulas are stated, because (2.4) proves to yield nothing but trivial restatements:

(5.10a) $$F_1\begin{bmatrix}2a;\ 1-a,\ 1+a-c; & \tfrac{2}{3},\ -2\\ 2c; & \end{bmatrix} = 3^{-a}\frac{\Gamma(a)\Gamma(c-a)\Gamma(2c)}{2\Gamma(2a)\Gamma(2c-2a)\Gamma(c)},$$

(5.10b) $$F_1\begin{bmatrix}2a;\ 3c-2,\ 1+a-c; & -2,\ -8\\ 2c; & \end{bmatrix} = 3^{-3a}\frac{\Gamma(a)\Gamma(c-a)\Gamma(2c)}{2\Gamma(2a)\Gamma(2c-2a)\Gamma(c)},$$

(5.10c) $$F_1\begin{bmatrix}2a;\ 1-a,\ 3c-2; & \tfrac{8}{9},\tfrac{2}{3}\\ 2c; & \end{bmatrix} = 3^{a}\frac{\Gamma(a)\Gamma(c-a)\Gamma(2c)}{2\Gamma(2a)\Gamma(2c-2a)\Gamma(c)}.$$

Again, formulas for ${}_2F_1$ are obtained by assigning a special value to a or c. In particular, one of Spiegel's results [3, equation III] is obtained by taking $c=a+\tfrac{1}{2}$ in (5.9d).

6. Quadratic rational substitutions.

When $x=0$, the substitution (5.1) degenerates to a quadratic substitution. The result is

(6.1) $$F_D\begin{bmatrix}2a;\ 1+a-c,\ b,\ b; & -1,\ \zeta,\ -\zeta\\ a+c; & \end{bmatrix} = \frac{\Gamma(a)\Gamma(a+c)}{2\Gamma(2a)\Gamma(c)}\,{}_2F_1\begin{bmatrix}a,b; & \zeta^2\\ c; & \end{bmatrix},$$

where we have set $z=\zeta^2$; the ζ-plane must be cut along the real axis from $-\infty$ to -1 and from 1 to $+\infty$. For $b=0$ or $\zeta=0$, (6.1) reduces to Kummer's theorem.

One further quadratic substitution seems to be worth consideration, viz.,

(6.2) $$t = \tau(1-x\tau)/(1-x), \quad x\neq 1\,.$$

By methods similar to those used above it can be proved that (6.2) yields the reduction formula

(6.3) $$F_D\begin{bmatrix}a;\ 1-a,\ 1+a-c,\ 2b-1; & x,\ \dfrac{x}{1-x},\ 2x\\ c; & \end{bmatrix} = (1-x)^a\,{}_2F_1\begin{bmatrix}a,\ b; & 4x(1-x)\\ c; & \end{bmatrix}, \quad \operatorname{Re} x<\tfrac{1}{2}\,.$$

A perhaps more elegant formula is obtained by an application of the linear transformation (2.3) and introduction of $\xi\equiv x/(1-x)$:

$$(6.4) \qquad F_D\left[\begin{matrix} a; 2(c-b)-1, 2b-1, 1+a-c; \\ c; \end{matrix} \; -\xi, \xi, \xi^2\right]$$

$$= (1+\xi)^{-2a}{}_2F_1\left[\begin{matrix} a, b; \\ c; \end{matrix} \; \frac{4\xi}{(1+\xi)^2}\right], \quad |\xi| < 1 .$$

When $b = \frac{1}{2}$ and $c = 1$, (6.4) degenerates to a special case of one of the known quadratic transformations of ${}_2F_1$ [1, p. 7].

REFERENCES

1. P. Appell et J. Kampé de Fériet, *Fonctions hypergéométriques et hypersphériques, Polynomes d'Hermite*, Gauthier-Villars, Paris 1926.
2. J. L. Lavoie and G. Trottier, *On the sum of certain Appell series*, Ganita 20 (1969), 43–46.
3. M. R. Spiegel, *Some interesting special cases of the hypergeometric series*, Amer. Math. Monthly 69 (1962), 894–896.
4. G. N. Watson, *The cubic transformation of the hypergeometric series*, Quart. J. Math. 41 (1910), 70–79.

THE TECHNICAL UNIVERSITY, LYNGBY, DENMARK