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A POSSIBLE CHARACTERIZATION OF GENERIC STRUCTURES

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Recently A. Robinson has introduced into model theory two kinds of forcing, which he calls finite forcing and infinite forcing. Details of these forcing notions can be found in [3], [2] for finite forcing and [4], [5] for infinite forcing. Several people have noticed that infinite forcing can be "explained" using standard model theoretic techniques (see for instance [1]). In this note I make several remarks which may eventually help to similarly explain finite forcing.

1. The main result.

Let L be any first order language and let T be any L-theory. Let $\mathscr{M}$ be the class of L-structures which are substructures of models of T (so that $\mathscr{M}$ is elementary being the class of models of the universal part of T). Finite forcing is used to construct a subclass $\mathscr{F} \subseteq \mathscr{M}$ called the class of T-generic structures. We will consider how $\mathscr{F}$ can be described without using forcing.

Remember that two theories T_1, T_2 are mutually model consistent if each model of the one is embeddable in a model of the other, equivalenty if T_1, T_2 have the same universal part. Remember also that a model $\mathfrak{A}$ of a theory T' is a completing model if for each model $\mathfrak{B}$ of T′,

$$\mathfrak{A} \subseteq \mathfrak{B} \Rightarrow \mathfrak{A} \prec \mathfrak{B}\,.$$

We can now state our main theorem.

THEOREM 1. *For any L-theory T there is at most one class $\mathscr{F}$ of L-structures such that*

(F1) *T and $Th(\mathscr{F})$ are mutually model consistent,*
(F2) *$\mathscr{F}$ is the class of completing models of $Th(\mathscr{F})$.*

If such a class $\mathscr{F}$ exists then $\mathscr{F}$ is the class of T-generic structures and $Th(\mathscr{F}) = T^f$.

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To prove theorem 1 we need some notation.

For each integer $n \geqq 0$ let $\forall_n$ be the set of formulas in prenex normal form whose prenex consists of n blocks of quantifiers, the first being universal, the second being existential, the third being universal, etc. For any two structures $\mathfrak{A}, \mathfrak{B}$ let $\mathfrak{A} \prec_n \mathfrak{B}$ mean that $\mathfrak{A} \subseteqq \mathfrak{B}$ and for each formula $\varphi \in \forall_n$ and $\mathfrak{A}$-assignment a,

$$\mathfrak{A} \models \varphi[a] \Rightarrow \mathfrak{B} \models \varphi[a] .$$

From now on we suppose that $\mathscr{F}$ satifies (F1, 2) and we put $T^* = Th(\mathscr{F})$.

For each integer $n \geqq 0$ let $\mathscr{F}_n$ be the subclass of $\mathscr{M}$ given by

$$\mathfrak{A} \in \mathscr{F}_n \Leftrightarrow (\forall \mathfrak{B} \models T^*)\ [\mathfrak{A} \subseteqq \mathfrak{B} \Rightarrow \mathfrak{A} \prec_n \mathfrak{B}]$$

and let $T_n = Th(\mathscr{F}_n)$. We see that $\mathscr{F}_0 = \mathscr{M}$ and we have a descending chain

$$(h) \qquad \mathscr{F}_0 \supseteqq \mathscr{F}_1 \supseteqq \mathscr{F}_2 \supseteqq \cdots \supseteqq \mathscr{F} .$$

(The inclusion $\mathscr{F} \subseteqq \mathscr{F}_n$ follows from (F2)).

First we prove some simple facts about this chain.

Lemma 2. *For each* $n \geqq 0$, $T^* \cap \forall_{n+1} \subseteqq T_n$.

Proof. Consider any sentence $\sigma \in T^* \cap \forall_{n+1}$, and any structure $\mathfrak{A} \in \mathscr{F}_n$. We show that $\mathfrak{A} \models \sigma$.

Now $\mathscr{F}_n \subseteqq \mathscr{M}$, and so (F1) gives us $\mathfrak{A} \subseteqq \mathfrak{B}$ for some model $\mathfrak{B}$ of T^*. In particular $\mathfrak{B} \models \sigma$. But, from the definition of $\mathscr{F}_n$, $\mathfrak{A} \prec_n \mathfrak{B}$, and so $\mathfrak{A} \models \sigma$, as required.

Corollary 3. *For each model* $\mathfrak{B}$ *of* T_n *there is some model* $\mathfrak{C}$ *of* T^* *such that* $\mathfrak{B} \prec_n \mathfrak{C}$.

Theorem 4. *For each* $n \geqq 0$ *the following are equivalent.*

(i) $\mathfrak{A} \in \mathscr{F}_{n+1}$.

(ii) *There is some model* $\mathfrak{B}$ *of* T_n *such that* $\mathfrak{A} \subseteqq \mathfrak{B}$, *and for each model* $\mathfrak{B}$ *of* T_n,

$$\mathfrak{A} \subseteqq \mathfrak{B} \Rightarrow \mathfrak{A} \prec_{n+1} \mathfrak{B} .$$

Proof. (i) $\Rightarrow$ (ii). Suppose $\mathfrak{A} \in \mathscr{F}_{n+1}$. The existence of $\mathfrak{B}$ such that $\mathfrak{A} \subseteqq \mathfrak{B} \models T_n$ follows from (F1) (or corollary 3). Also, for any such $\mathfrak{B}$, corollary 3 shows that $\mathfrak{B} \prec_n \mathfrak{C}$ for some model $\mathfrak{C}$ of T^*. But $\mathfrak{A} \in \mathscr{F}_{n+1}$ and so $\mathfrak{A} \prec_{n+1} \mathfrak{C}$. This gives $\mathfrak{A} \prec_{n+1} \mathfrak{B}$, as required.

(ii) $\Rightarrow$ (i) follows immediately from the definition of $\mathscr{F}_{n+1}$ since $T_n \subseteqq T^*$.

THEOREM 5. $\mathscr{F} = \bigcap_{n<\omega} \mathscr{F}_n$.

PROOF. We have already noted that

$$\mathscr{F} \subseteqq \bigcap_{n<\omega} \mathscr{F}_n$$

so it is sufficient to show the reverse inclusion.

Suppose $\mathfrak{A} \in \mathscr{F}_n$ for all $n \geqq 0$, we must show that $\mathfrak{A}$ is a completing model of T^*. Consider any $\mathfrak{A} \subseteqq \mathfrak{B} \models T^*$. Since $\mathfrak{A} \in \mathscr{F}_n$ we have $\mathfrak{A} \prec_n \mathfrak{B}$, and this holds for all $n \geqq 0$, hence $\mathfrak{A} \prec \mathfrak{B}$, as required,

PROOF OF THEOREM 1. Suppose such a class $\mathscr{F}$ exists, and consider the hierachy (h).

We have $\mathscr{F}_0 = \mathscr{M}$ and so $\mathscr{F}_0$ is uniquely determined. Moreover theorem 4 shows that each $\mathscr{F}_{n+1}$ is uniquely determined in terms of $\mathscr{F}_n$, and so each $\mathscr{F}_n$ is uniquely determined. Finally theorem 5 shows that $\mathscr{F}$ is uniquely determined.

We must now show that $\mathscr{F}$ is the class of T-generic structures and $T^* = T^f$.

First from (F_2) and [2, theorem 4.9] we see that T^* is forcing complete, i.e.

$$T^* = T^{*f}.$$

Also from [2, theorem 2.19] we have

$$T^{*f} = (T^* \cap \forall_1)^f, \qquad T^f = (T \cap \forall_1)^f.$$

However (F1) shows that

$$T^* \cap \forall_1 = T \cap \forall_1$$

so that

$$T^* = T^f.$$

Finally (F2) and [2, theorem 3.4] show that $\mathscr{F}$ is the class of T-generic structures.

This completes the proof of theorem 1.

2. Further remarks.

Some properties of T-generic structures can be derived from theorem 1 and the hierachy (h). For instance we will prove the following theorem, (c.f. [2, theorem 3.7]).

THEOREM 6. *For any two structures* $\mathfrak{A}, \mathfrak{B}$,

$$\mathfrak{A} \prec_1 \mathfrak{B} \in \mathscr{F} \Rightarrow \mathfrak{A} \in \mathscr{F}.$$

This theorem follows from the following two lemmas.

LEMMA 7. *For each integer $n \geqq 0$, and any two structures $\mathfrak{A}, \mathfrak{B}$,*

$$\mathfrak{A} \prec_{n+1} \mathfrak{B} \in \mathscr{F} \Rightarrow \mathfrak{A} \prec_{n+2} \mathfrak{B} .$$

PROOF. Suppose that $\mathfrak{A} \prec_{n+1} \mathfrak{B} \in \mathscr{F}$, so that

$$(*) \qquad \mathfrak{A} \prec \mathfrak{C}, \quad \mathfrak{B} \prec_n \mathfrak{C}$$

for some suitable $\mathfrak{C}$. In particular we have

$$\mathfrak{C} \equiv \mathfrak{A} \models T^* \cap \forall_{n+1}$$

so that $\mathfrak{C} \prec_n \mathfrak{D}$ for some model $\mathfrak{D}$ of T^*. But $\mathfrak{B} \in \mathscr{F}$ and so

$$\mathfrak{B} \prec \mathfrak{D}, \quad \mathfrak{C} \prec_n \mathfrak{D}$$

which gives $\mathfrak{B} \prec_{n+1} \mathfrak{C}$. Thus, from (*), we get $\mathfrak{A} \prec_{n+2} \mathfrak{B}$, as required.

LEMMA 8. *For any two structures $\mathfrak{A}, \mathfrak{B}$,*

$$\mathfrak{A} \prec \mathfrak{B} \in \mathscr{F} \Rightarrow \mathfrak{A} \in \mathscr{F} .$$

PROOF. Suppose that $\mathfrak{A} \prec \mathfrak{B} \in \mathscr{F}$ and $\mathfrak{A} \subseteq \mathfrak{C} \models T^*$. Thus we have a commuting diagram

$$\begin{array}{ccc} \mathfrak{A} & \prec & \mathfrak{B} \\ \cap\!\!\!\mid\mid & & \cap\!\!\!\mid\mid \\ \mathfrak{C} & \overset{f}{\rightarrowtail} & \mathfrak{D} \end{array}$$

where f is an elementary embedding. In particular $\mathfrak{D} \models T^*$ and so (since $\mathfrak{B} \in \mathscr{F}$), $\mathfrak{B} \prec \mathfrak{D}$. This gives $\mathfrak{A} \prec \mathfrak{C}$, as required.

3. Open problems.

Theorem 1 says nothing about the existence of class $\mathscr{F}$. However it is known that for countable L the class of T-generic structures exists and satisfies (F1, 2), see [2, theorems 3.3, 3.9, 3.4, and 4.1]. Thus for countable L we have both existence and uniquencess. It has been noticed by Shelah, [6], and independently by Macintyre that for uncountable L there are theories T for which no T-generic structures exist. For these theories no class $\mathscr{F}$ exists.

Thus we have the following problem.

(A) Under what conditions does the class $\mathscr{F}$ exist?

Theorem 1 says the class $\mathscr{F}$ (if it exists) is the class of T-generic structures, however the converse of this is not known. Thus we can ask the following.

(B) Under what conditions does the class of T-generic structures give us a class $\mathscr{F}$.

There are many open problems concerning the behaviour of the heirachy (h), (even for countable L). For instance we have the following.

(C) Under what conditions is (h) finite?

(D) What are the possible patterns of equality between member of (h)?

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