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A REDUCIBILITY ARISING FROM THE BOONE GROUPS

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If B is a set of numbers and e is a number, let φ_e^B denote the eth function partial recursive in B under some fixed formalism. If $\varphi_e^B(n)$ is defined, let $Q_e^B(n)$ denote the (finite) set of numbers u such that the membership or non-membership of u in B is used to compute $\varphi_e^B(n)$. The purpose of this paper is to study special cases of Turing reducibility given by reductions where $Q_e^B(n)$ is recursively bounded as a function of n in some sense. Specifically, for sets A and B, we say that A is *bounded-search reducible* to B ($A \leq_{\mathrm{bs}} B$) if there is a number e and a recursive function f such that $A(n) = \varphi_e^B(n)$ and $|Q_e^B(n)| \leqq f(n)$ for all n.[2] (We are identifying sets with their characteristic functions and writing $|Q|$ for the cardinality of Q.) This reducibility is of interest because the arguments of [1] show that for every recursively enumerable (r.e.) set A there is a finitely presented group G_A such that A and the word problem of G_A are mutually bounded-search reducible when the latter is coded as a set of integers.[3]

It is natural to inquire whether bounded-search reducibility coincides with either truth-table reducibility ($\leq_{\mathrm{tt}}$) or Turing reducibility ($\leq_{\mathrm{T}}$) on the r.e. sets. A negative answer for the case of truth-table reducibility follows from work of Lachlan on a reducibility introduced by Friedberg and Rogers [4, p. 124] called *weak truth-table reducibility* ($\leq_{\mathrm{w}}$).[4] To define $A \leq_{\mathrm{w}} B$ in the terminology of this paper simply replace "$|Q_e^B(n)|$" by "$\max Q_e^B(n)$" (i.e. the greatest element of

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[2] This terminology was suggested by T. G. Mclaughlin. The question of the relationship of this reducibility to other reducibilities was raised by W. W. Boone.

[3] In footnote 4 of [1] it is claimed that the arguments there in fact yield truth-table reducibility. As pointed out in [2], this claim is not justified since the given arguments seem to suffice only for bounded-search reducibility and not for truth-table reducibility. The existence of truth-table reductions is proved by different arguments in [3].

[4] The formal definition of $\leqq_w$ given by Friedberg and Rogers applies only to r.e. sets. Our definition (which is equivalent to Robinson's [6]) may be applied to arbitrary sets.

$Q_e{}^B(n) \cup \{0\}$) in the definition of $A \leq_{bs} B$. Clearly $\leq_{\mathrm{bs}}$ is weaker than (i.e. implied by) $\leq_{\mathrm{w}}$, and Lachlan showed [5, p. 26] that $\leq_{\mathrm{w}}$ is strictly weaker than $\leq_{\mathrm{tt}}$ on the r.e. sets. In view of Lachlan's result, the question now becomes whether $\leq_{\mathrm{bs}}$ coincides with either $\leq_{\mathrm{w}}$ or $\leq_{\mathrm{T}}$ on the r.e. sets. We shall obtain negative answers to both questions by showing that $\leq_{\mathrm{bs}}$ is not transitive on the r.e. sets, whereas it is clear that $\leq_{\mathrm{w}}$ and $\leq_{\mathrm{T}}$ are transitive.

In order to be able to assert the failure of transitivity for $\leq_{\mathrm{bs}}$ in a strong form, it is convenient to introduce yet another reducibility. We write $A \leq_2 B$ if $A \leq_{\mathrm{bs}} B$ and, moreover, the bounding function f for $|Q_e^B(n)|$ may be taken to be constant with value 2.

In notation, we shall generally follow [7]. In particular let φ_i denote the ith partial recursive function, and let $\langle\cdot,\cdot\rangle$ be a fixed recursive pairing function for the natural numbers. The characteristic function of a set A is 1 on A and 0 on the complement of A. Lower case italic letters, such as a, e, n, are reserved as variables for natural numbers.

THEOREM. *There are* r.e. *sets* A, B, C *such that* $A \leq_2 B$, $B \leq_{\mathrm{tt}} C$, *but* $A \not\leq_{\mathrm{bs}} C$.

PROOF. The sets A, B, C are obtained by a standard finite-injury priority argument. To insure that $A \leq_2 B$ we define a partial recursive function ψ such that

$$\text{(1)} \qquad \mathrm{domain}(\psi) = B$$

$$\text{(2)} \qquad (\forall n)[n \in A \Leftrightarrow n \in B \text{ and } \psi(n) \in B].$$

For each n let $L_n = \{\langle n,i\rangle : i \leq n\}$. To insure that $B \leq_{\mathrm{tt}} C$ we arrange that for all n

$$\text{(3)} \qquad n \in B \Leftrightarrow L_n \cap C \neq \emptyset.$$

To insure that $A \not\leq_{\mathrm{bs}} C$ we must satisfy for each pair (e,i) a corresponding *requirement* denoted $R_{\langle e,i\rangle}$. The requirement $R_{\langle e,i\rangle}$ asserts that there is a number n $(=n(e,i))$ such that if $\varphi_e{}^C(n)$ and $\varphi_i(n)$ are defined and $|Q_e{}^C(n)| \leq \varphi_i(n)$, then $A(n) \neq \varphi_e{}^C(n)$. The number $n(e,i)$ will be obtained as the limit of a sequence of recursive approximations. Rogers' terminology [7] of "moving markers" is helpful in describing the process of approximation and so for each e,i we introduce a "marker" $\Lambda_{e,i}$ such that at (the end of) each stage $s > \langle e,i\rangle$, $\Lambda_{e,i}$ is associated with a unique number (denoted $n(e,i,s)$ and called the *position* of $\Lambda_{e,i}$) which may be thought of as the present candidate for $n(e,i)$. The position of $\Lambda_{e,i}$ will be allowed to change, but only finitely many times. The last position of $A_{e,i}$ is then the desired $n(e,i)$. Each number will be the position of at most one marker over the entire construction, that is

$$(4) \qquad n(e,i,s) = n(a,b,t) \text{ only if } \langle e,i\rangle = \langle a,b\rangle .$$

We shall also have $-$tags which may be associated with numbers during the construction. A number is given a $-$tag if it is desirable to keep it out of C to satisfy some $R_{\langle e,i\rangle}$. (Actually in visualizing the construction it is convenient to imagine two separate copies of the natural numbers, the "A-list" and the "C-list", with the positions of the markers $A_{e,i}$ lying in the A-list and the $-$tags lying in the C-list.) The priorities of the construction are such that a number given a $-$tag for the sake of requirement $R_{\langle e,i\rangle}$ can be enumerated in C only for the sake of requirements $R_{\langle a,b\rangle}$ with $\langle a,b\rangle < \langle e,i\rangle$. A number k in the A-list is called *free* (at some time during the construction) if k has not been the position of any marker and has not been enumerated in A or B, no element of L_k has been enumerated in C or given a $-$tag, and k has not been placed in the domain or range of ψ. Let A^s, C^s denote the finite sets of numbers enumerated in A, C respectively by the end of stage s.

We now give the construction.

Stage 0. Do nothing, i.e. no numbers are to be enumerated in A, B or C; no markers or $-$tags are to be positioned, and ψ is to be left completely undefined.

Stage $s+1$. Associate the previously unused marker $A_{a,b}$, where $\langle a,b\rangle = s$, with the least free number.

Now ask whether there is a number $\langle e,i\rangle < s$ such that $R_{\langle e,i\rangle}$ *requires attention*, that is $\varphi_e^{C^s}(n)$ and $\varphi_i(n)$ are each defined in at most s steps (where n abbreviates $n(e,i,s)$) and $|Q_e^{C^s}(n)| \leqq \varphi_i(n)$, $n \notin A^s$, and $\varphi_e^{C^s}(n) = 0$. If no such $\langle e,i\rangle$ exists, proceed directly to stage $s+2$ without taking further action. Otherwise, take $\langle e,i\rangle$ minimal such that $R_{\langle e,i\rangle}$ requires attention, let $n = n(e,i,s)$, and *attack* $R_{\langle e,i\rangle}$ as follows:

Case 1. $\psi(n)$ is not yet defined. Define $\psi(n)$ to be the least free number k such that $k \geqq \varphi_i(n)$. Enumerate n in B (to preserve (1)) and $\langle n,0\rangle$ in C (to preserve (3)).

Case 2. $\psi(n)$ is already defined. When the construction is complete, it will be clear that $\psi(n) \geqq \varphi_i(n)$. Assuming this for the moment, it follows that

$$|L_{\psi(n)}| = \psi(n)+1 > \varphi_i(n) \geqq |Q|$$

where $Q = Q_e^{C^s}(n)$. Hence $L_{\psi(n)} - Q \neq \emptyset$, and we let c be the least element of $L_{\psi(n)} - Q$. For the sake of $R_{\langle e,i\rangle}$, enumerate n in A and associate a $-$tag with each element of $Q - C^s$. Also enumerate $\psi(n)$ in B (to preserve (2)) and c in C (to preserve (3)). In addition define $\psi\psi(n)=\psi(n)$ (to preserve (1)) and finally enumerate $\psi(n)$ in A (to preserve (2)).

In either Case 1 or Case 2 the markers $\Lambda_{a,b}$ for $\langle a,b\rangle \leqq \langle e,i\rangle$ retain their positions while the markers $\Lambda_{a,b}$ for $\langle e,i\rangle < \langle a,b\rangle \leqq s$ are to be moved to distinct free integers. This completes the construction.

Observe first that if $\psi(n)$ is defined, where n is the position of any marker, then $\psi(n)$ became defined through Case 1. To prove this, assume that $\psi(k)$ was defined through Case 2, so k is in the range of ψ. Let $s+1$ be the stage where k first enters the range of ψ. Clearly Case 1 applies at stage $s+1$, and k cannot be the position of any marker through the end of stage $s+1$. After the end of stage $s+1$, k is not free and again it cannot become the position of any marker. Thus k is never a marker position, and the observation is proved. It follows from this and (4) that if $n=n(e,i,s)$ and $\psi(n)$ is defined, then $\psi(n) \geqq \varphi_i(n)$, as needed for Case 2. Also it follows that Case 2 can apply to a marker position n only after Case 1 has applied, and so n must have already been enumerated in B when Case 2 applies. From this and the parenthetical remarks in the construction it can easily be seen by induction that (1), (2), (3) hold stage-by-stage and so are valid. It remains only to show that $A \nleqq_{\mathrm{bs}} C$, or equivalently that the requirements $R_{\langle e,i\rangle}$ are satisfied. For this one must show in particular that each marker $\Lambda_{a,b}$ moves only finitely many times. Since $\Lambda_{a,b}$ moves only when some $R_{\langle e,i\rangle}$ for $\langle e,i\rangle < \langle a,b\rangle$ is attacked, it suffices to prove the following lemma to show that the markers come to rest.

LEMMA. *Each requirement $R_{\langle e,i\rangle}$ is attacked only finitely often.*

PROOF. Assume inductively that there is a stage $s_0 > \langle e,i\rangle$ such that no $R_{\langle a,b\rangle}$ with $\langle a,b\rangle < \langle e,i\rangle$ is attacked at any stage $s \geqq s_0$. By the remark before the Lemma, $\Lambda_{e,i}$ does not move after stage s_0. If $n=n(e,i,s_0)$, $R_{\langle e,i\rangle}$ can be attacked only when (if ever) $\psi(n)$ first becomes defined or n is first enumerated in A. This completes the proof of the Lemma.

To show that $R_{\langle e,i\rangle}$ is satisfied, let $n=n(e,i,s_0)$ where s_0 is in the proof of the Lemma. Assume that $\varphi_e^C(n)$ and $\varphi_i(n)$ are defined, and $|Q_e^C(n)| \leqq \varphi_i(n)$.

First assume that $n \notin A$. Then, if $\varphi_e^C(n)=A(n)=0$, $R_{\langle e,i\rangle}$ requires attention from some stage on, and so some $R_{\langle a,b\rangle}$, with $\langle a,b\rangle \leqq \langle e,i\rangle$ is attacked infinitely often in contradiction to the Lemma. Now assume $n \in A$. Suppose n is enumerated in A at stage s_1. From (4) and the fact that no marker position can be in the range of ψ (by the argument just after the construction) we see that $R_{\langle e,i\rangle}$ is attacked at stage s_1. Putting

$C_1 = C^{s_1}$ we claim that no element of $Q_e^{C_1}(n)$ $(=Q)$ is enumerated in C during or after stage s_1. For stage s_1 itself, this follows from the choice of C. At stages $s+1 > s_1$, if $R_{\langle a,b\rangle}$ is attacked then $\langle a\ b\rangle > \langle e,i\rangle$ by the proof of the Lemma. It follows that $L_{n(a,b,s)} \cap Q = \emptyset$ because $\Lambda_{a.b}$ first occupied $n(a,b,s)$ after s_1. Similarly, $L_{\psi(n(a,b,s))} \cap Q = \emptyset$ if $\psi(n(a,b,s))$ is defined. Hence no element of Q enters C at stage $s+1$ and the claim is proved. It follows that $\varphi_e{}^C(n) = \varphi_e{}^{C_1}(n) = 0$. On the other hand, $A(n) = 1$ since $n \in A$ and so $R_{\langle e,i\rangle}$ is satisfied. This completes the proof of the Theorem.

The fact that bounded search reducibility is not transitive shows that there is no corresponding degree concept. Presumably the transitive closure of the new reducibility also differs from Turing reducibility.

It might be of interest to study the bs-*complete* sets, that is r.e. sets B such that $A \leq_{\text{bs}} B$ for every r.e. set A. In particular one might consider whether the bs-complete sets coincide with either the w-complete or T-complete r.e. sets.

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