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GORENSTEIN MODULES AND RELATED MODULES

HANS-BJØRN FOXBY

The finitely generated projective modules over a commutative noetherian ring A with identity and the Gorenstein modules G over A (defined in [16] and below) share the following property:

$$\text{(PG)}\ \operatorname{Hom}(G,G) \textit{ is projective and } \operatorname{Ext}^i(G,G) = 0 \textit{ for } i>0 .$$

If G is a finitely generated non-zero A-module with the property (PG) (short: G is a PG-module), and if M is any A-module of finite weak (respectively, injective) dimension, then there is a natural isomorphism

$$\varphi_M \colon H \otimes M \to \operatorname{Hom}(G, G \otimes M)$$

(respectively

$$\psi_M \colon G \otimes \operatorname{Hom}(G,M) \to \operatorname{Hom}(H,M)) ,$$

where $H = \operatorname{Hom}(G,G)$ (H is projective by assumption). This is established in the first section. In the second section these results are applied to Gorenstein modules. Here the main result is:

Suppose that G is a Gorenstein module and that A is of finite Krull-dimension. Then for any module M with $\operatorname{Supp} M \subseteq \operatorname{Supp} G$ *we have*:

$$\operatorname{weak\,dim} M < \infty \textit{ if and only if } \operatorname{inj\,dim} G \otimes M < \infty \textit{ and}$$
$$\operatorname{inj\,dim} M < \infty \textit{ if and only if } \operatorname{weak\,dim} \operatorname{Hom}(G, M) < \infty .$$

This result is well-known (and easy to prove) in the case $G = A$, but it is also an extension and generalization of a result due to Sharp (Theorem (2.9) of [17]).

In section 3 the natural homomorphism

$$\sigma_M \colon M \otimes \operatorname{Hom}(G, G) \to \operatorname{Hom}(\operatorname{Hom}(M, G), G)$$

is considered, when G is a PG-module. This leads to generalizations of some results due to Auslander and Bridger [1] and Fossum [7]. Finally, the converse to Sharp's theorem on existence of a rank 1 Gorenstein module (see Theorem (2.1.i) of [17]) is proved in the last section.

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NOTATION. The ring A will always be commutative and noetherian, while G always denote a *finitely generated* (f.g.) non-zero A-module. The endomorphism ring $\mathrm{Hom}_A(G,G)$ of G will be denoted by H. Arbitrary A-modules will be denoted by M and N, and the notation will be the same as that used in [8] with the following exceptions:

$$\begin{aligned} [M,N]_A &= \mathrm{Hom}_A(M,N) \text{ (or just } [M,N]). \\ \mathrm{pd}\,_A M &= \text{the projective dimension of } M, \\ \mathrm{wd}_A M &= \text{the weak dimension of } M, \text{ and} \\ \mathrm{id}\,_A M &= \text{the injective dimension of } M. \end{aligned}$$

1. On PG-modules.

There are two natural homomorphisms:

$$\varphi_M\colon H \otimes M \to [G, G \otimes M] \quad \text{and} \quad \psi_M\colon G \otimes [G, M] \to [H, M]\,,$$

defined by $\varphi_M(h \otimes m)(g) = h(g) \otimes m$ and $\psi_M(g \otimes f)(h) = fh(g)$ (cf. [11, § 1] and [6,VI,§5]).

Let Φ_G denote the class of A-modules M such that φ_M is an isomorphism and $\mathrm{Tor}_i(G,M)=0$ and $\mathrm{Ext}^i(G,G\otimes M)=0$ for all $i>0$. Let Ψ_G denote the class of A-modules M such that ψ_M is an isomorphism and $\mathrm{Ext}^i(G,M)=0$ and $\mathrm{Tor}_i(G,[G,M])=0$ for all $i>0$.

The homomorphisms φ_M and ψ_M and the classes Φ_G and Ψ_G will be studied in this section together with the class of PG-modules.

DEFINITION. G is called a PG-*module* if $H=[G,G]$ is projective and $\mathrm{Ext}^i(G,G)=0$ for $i>0$ (still G is f.g. and non-zero). If G is a PG-module we will define the rank-map r_G: $\mathrm{Spec}\,A \to \mathsf{R}$ by $r_G(\mathfrak{p}) = (\mathrm{rank}\,H_\mathfrak{p})^{\frac{1}{2}}$ (cf. [4, § 5, n° 2, Theorème 1. c]).

Now we will give some examples of PG-modules.

EXAMPLE 1. If G is projective then $G \in \Phi_G \cap \Psi_G$, it is a PG-module, and r_G is the usual rank-map (by [4, § 5, n° 3], [6,VI, § 5, Proposition 5.2], and [11, § 1]). In fact, if A is a Gorenstein ring of dimension 1 and G is a PG-module, then G is projective, by Theorem (3.1) of [18].

EXAMPLE 2. If G is a Gorenstein module (that is $\mathrm{id}_{A_\mathfrak{p}} G_\mathfrak{p} = \mathrm{depth}_{A_\mathfrak{p}} G_\mathfrak{p}$ $(<\infty)$ for all $\mathfrak{p} \in \mathrm{Supp}\,G$, cf. (3.7) of [16]), then G is a PG-module and $r_G(\mathfrak{p}) = \mu^{\mathrm{ht}\mathfrak{p}}(\mathfrak{p},G)$ for all $\mathfrak{p}$ (by [12, § 2] and [8, Proposition 3.1 (a')]).

Remark that G is a PG-A-module iff $G_\mathfrak{p}$ is a PG-$A_\mathfrak{p}$-module for all $\mathfrak{p} \in \mathrm{Supp}\,G$ (or just the maximal ones).

EXAMPLE 3. If $A=A_1 \oplus A_2$ (direct product of rings) and M_i is a f.g. non-zero A_i-module, then $M=M_1 \oplus M_2$ is a PG-A-module if and only if M_i is a PG-A_i-module for $i=1,2$. This gives examples of PG-modules which are neither projective nor Gorenstein.

EXAMPLE 4. Let A be a Cohen-Macaulay local ring with maximal ideal $\mathfrak{m}$ and let G be a PG-module of rank 1 (i.e. $r_G(\mathfrak{m})=1$). Then it is easy to see that $\mu^s(A)=\beta_0(G)\mu^s(G)$ where $s=\operatorname{depth} A$ and $\beta_0(G)$ is the minimal number of generators of G. ($\operatorname{Ext}^s(A/\mathfrak{m},A)=\operatorname{Ext}^s(G/\mathfrak{m}G,G)$ by induction on s using Proposition 1.1 below.) This shows that if $\mu^s(A)$ is a prime number then G is either free or Gorenstein (cf. the remark after Theorem 4.1 in section 4). Furthermore, if A is Gorenstein then $G=A$.

Further basic properties of PG-modules are catalogued in the following:

PROPOSITION 1.1. *Let G be a* PG-*module.*

(i) $r_G{}^2$: $\operatorname{Spec} A \to \mathbf{Z}$ *is continuous* (*i.e. constant on the connected components of* Spec A).

(ii) $A=A' \oplus \operatorname{Ann} G$, *where* $A'=\operatorname{Ann}\operatorname{Ann} G$, *that is* $A'=A/\operatorname{Ann} G$. *Furthermore* $A_{\mathfrak{p}}=A'_{\mathfrak{p}'}$, *for all* $\mathfrak{p} \in \operatorname{Supp} G$, *when* $\mathfrak{p}'=\mathfrak{p}/\operatorname{Ann} G \in \operatorname{Spec} A'$.

(iii) G *is a* PG-A'-*module.*

(iv) $\operatorname{Ann} G=\operatorname{Ann} H=\operatorname{Ann} A'$ *and* $\operatorname{Ann}_{A'} G=0$.

(v) $\operatorname{Ass} G=\operatorname{Ass} H=\operatorname{Ass} A'=\operatorname{Ass} A \cap \operatorname{Supp} G$.

(vi) $zG=zH=z_A A' \subseteq zA$ (*Here* $zM=z_A M$ *denotes the set of zero-divisors on the A-module M.*)

(vii) G/aG *is a* PG-$A/(a)$-*module for all G-regular* $a \in A$.

(viii) $a_1,\ldots,a_p \in A$ *is an A'-regular sequence if and only if it is G-regular.*

(ix) $\operatorname{depth}_A G=\operatorname{depth}_A A'$.

(*x*) $\operatorname{grade}_G M=\operatorname{grade}_A M$ *for all* f.g. *A-modules with* $\operatorname{Supp} M \subseteq \operatorname{Supp} G$.

PROOF. For (i) and (ii) confer [4, § 4 n° 4]. However we can give a short prove of (ii). Remark first that $\operatorname{Ann} G=\operatorname{Ann} H$ and hence

$$(\operatorname{Ann} G)_{\mathfrak{p}} = \operatorname{Ann}_{A_{\mathfrak{p}}} G_{\mathfrak{p}} = \operatorname{Ann}_{A_{\mathfrak{p}}} H_{\mathfrak{p}} = 0$$

for all $\mathfrak{p} \in \operatorname{Supp} G=V(\operatorname{Ann} G)$, since $H_{\mathfrak{p}}$ is $A_{\mathfrak{p}}$-free. Next, $\operatorname{Supp} A'=V(\operatorname{Ann} A') \subseteq V(\operatorname{Ann} G)$, and we have proved $\mathfrak{p} \in V(\operatorname{Ann} G)$ if and only if $\mathfrak{p} \notin \operatorname{Supp} \operatorname{Ann} G=V(A')$, and $\mathfrak{p} \notin V(A')$ if and only if $\mathfrak{p} \in \operatorname{Supp} A'$, and hence

$$(A'+\operatorname{Ann} G)_{\mathfrak{p}}=A_{\mathfrak{p}} \quad \text{and} \quad (A' \cap \operatorname{Ann} G)_{\mathfrak{p}} = A'_{\mathfrak{p}} \cap (\operatorname{Ann} G)_{\mathfrak{p}} = 0$$

for all prime ideals $\mathfrak{p}$. Therefore $A=A' \oplus \operatorname{Ann} G$. The assignment $\mathfrak{p} \mapsto \mathfrak{p}'=$

$\mathfrak{p}/\mathrm{Ann}\, G$ defines a homeomorphism: $\mathrm{Supp}\, G \to \mathrm{Spec}\, A'$ and $A_\mathfrak{p} = A'_{\mathfrak{p}'}$, and hence $G_\mathfrak{p} = G_{\mathfrak{p}'}$ is a PG-$A'_{\mathfrak{p}'}$-module, i.e. (ii) and (iii) are established.

(iv) is now obvious.

(v) and (vi) $\mathrm{Ass}\, G = \mathrm{Ass}\, H \subseteq \mathrm{Ass}\, A \cap \mathrm{Supp}$ G, and hence $zG = zH \subseteq zA$ (cf. [5, § 1, n° 4, Proposition 8 and 10]). Assume now $\mathfrak{p} \in \mathrm{Ass}\, A \cap \mathrm{Supp}\, G$. Then $z_{A_\mathfrak{p}} G_\mathfrak{p} = z_{A_\mathfrak{p}} H_\mathfrak{p} = z_{A_\mathfrak{p}} A_\mathfrak{p} = \mathfrak{p}_\mathfrak{p}$, since $H_\mathfrak{p}$ is $A_\mathfrak{p}$-free, hence $\mathfrak{p}_\mathfrak{p} \in \mathrm{Ass}_{A_\mathfrak{p}} G_\mathfrak{p} \cap \mathrm{Ass}_{A_\mathfrak{p}} A'_{\mathfrak{p}'}$, and therefore $\mathfrak{p} \in \mathrm{Ass}_A G \cap \mathrm{Ass}_A A'$ (by [5, § 1, n° 2]).

(vii). We may assume A is local (with $a \in \mathfrak{m}$ = the maximal ideal). Then a is A-regular too (since $zG = zH = zA$ when A is local). By use of $[G, -]_A$ on the exact sequence

$$0 \to G \xrightarrow{a\cdot} G \to \bar{G} \to 0 ,$$

(here $\bar{N} = N/aN$), we get that $[\bar{G}, \bar{G}]_{\bar{A}} = [G, \bar{G}]_A = \bar{H}$ is $\bar{A}$-projective (free), and that $\mathrm{Ext}_{\bar{A}}^i(\bar{G}, \bar{G}) = \mathrm{Ext}_A^i(G, \bar{G}) = 0$ for $i > 0$ (cf. [14, § 0.1]).

(viii). $a_1 \notin zG$ if and only if $a_1 \notin zA'$, by (vi), and $G/a_1 G \neq 0$ if and only if $A'/a_1 A' \neq 0$, since $\mathrm{Ann}_A G = \mathrm{Ann}_A A'$. The conclusion follows by an easy induction on p.

(ix) follows by (viii).

(x) follows by (ix) and Corollary 3.6 of [9].

Remark. It is easy to see that a replacement of "PG" by "Gorenstein" in the proposition only requires tiny modifications of the proof. This gives many of the results in [16]. (In particular (4.11) might be proved in a way similar to (vii).)

The next propositions deal with Φ_G and Ψ_G when G is a PG-module.

Proposition 1.2. *Let G be a* PG-*module.*

(φ) *If* wd $M < \infty$, *then* $M \in \Phi_G$.

(ψ) *If* id $M < \infty$, *then* $M \in \Psi_G$.

In particular, if G is Gorenstein, then $G \in \Psi_G$.

We start the proof with a lemma:

Lemma 1.3. *Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be an exact sequence of A-modules and let G be a* PG-*module. Then if two of the modules in the exact sequence belong to Φ_G then also the third belongs to Φ_G. The similar statement holds for Ψ_G.*

Proof. If $M_2, M_3 \in \Phi_G$ (respectively $M_1, M_3 \in \Phi_G$) then it is a straightforward application of the 5-lemma to see that also $M_1 \in \Phi_G$ (respectively $M_2 \in \Phi_G$).

Now assume $M_1, M_2 \in \Phi_G$. From the exact sequence we get $\mathrm{Tor}_i(G, M_3) = 0$ for $i > 1$ and an exact sequence

$$0 \to \mathrm{Tor}_1(G, M_3) \to G \otimes M_1 \to G \otimes M_2 ,$$

and hence we have a commutative diagram with exact rows

$$\begin{array}{ccccc} 0 \to [G, \mathrm{Tor}_1(G, M_3)] & \to & [G, G\otimes M_1] & \to & [G, G\otimes M_2] \\ & & \varphi_{M_1} \uparrow \cong & & \varphi_{M_2} \uparrow \cong \\ & 0 \to & H\otimes M_1 & \longrightarrow & H\otimes M_2 \end{array}$$

Therefore $[G, \mathrm{Tor}_1(G, M_3)] = 0$. This implies $\mathrm{Tor}_1(G, M_3) = 0$ since $\mathrm{Ann} G \subseteq \mathrm{Ann}\,\mathrm{Tor}_1(G, M_3)$, by the use of [4, § 4 n° 4, Proposition 20]. Now $\mathrm{Ext}^i(G, G\otimes M_3) = 0$ for $i > 0$ and we have a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 \to [G, G\otimes M_1] & \to & [G, G\otimes M_2] & \to & [G, G\otimes M_3] \to 0 \\ \varphi_{M_1} \uparrow \cong & & \varphi_{M_2} \uparrow \cong & & \varphi_{M_3} \uparrow \\ 0 \to H\otimes M_1 & \longrightarrow & H\otimes M_2 & \longrightarrow & H\otimes M_3 \to 0 . \end{array}$$

Thus φ_{M_3} is an isomorphism, by the 5-lemma, and hence $M_3 \in \Phi_G$.

The proof of the similar result for Ψ_G is quite the same, but here we use that if $G \otimes \mathrm{Ext}^1(G, M_1) = 0$ then $\mathrm{Ext}^1(G, M_1) = 0$ (if $N \neq 0$ and $\mathrm{Ann} N \supseteq \mathrm{Ann} G$, then $[G\otimes N, N] = [G, [N, N]] \neq 0$, therefore $G\otimes N \neq 0$).

The proof of the proposition is now easy.

(φ). By the lemma it is sufficient to assume that M is flat, and hence we may use [11, Corollary 1.2] to get the desired result.

(ψ). Assume M injective and use [6, VI § 5].

Proposition 1.4. *If the* PG-*module* G *is in* Ψ_G *and* $\mathrm{Supp} M \subseteq \mathrm{Supp} G$, *then*

$$(\varphi) \quad M \in \Phi_G \quad \text{if and only if} \quad G\otimes M \in \Psi_G .$$
$$(\psi) \quad M \in \Psi_G \quad \text{if and only if} \quad [G, M] \in \Phi_G .$$

Proof. Since $\mathrm{Supp} M \subseteq \mathrm{Supp} G$ we may assume that A is local (with maximal ideal $\mathfrak{m}$), and hence H is free of rank $t = r_G(\mathfrak{m})^2$, that is $H = \oplus t A$ (the direct sum of t copies of A).

PROOF OF (φ). Consider the commutative diagram

$$\begin{array}{ccc}
G\otimes H\otimes M & \xrightarrow{1_G\otimes\varphi_M} & G\otimes[G, G\otimes M] \\
{\scriptstyle \psi_G\otimes 1_M}\downarrow \cong & & \downarrow{\scriptstyle \psi_{G\otimes M}} \\
[H, G]\otimes M & \underset{\cong}{\xrightarrow{\tau}} & [H, G\otimes M]
\end{array}$$

where τ is the canonical isomorphism (cf. Lemma 1.1.1 of [11]). First, assume $M\in\Phi_G$, that is φ_M is an isomorphism. Then $\psi_{G\otimes M}$ is an isomorphism too, by the diagram, and

$$\operatorname{Tor}_i(G, [G, G\otimes M]) = \operatorname{Tor}_i(G, H\otimes M) = \oplus t\operatorname{Tor}_i(G, M) = 0\,,$$

that is $G\otimes M \in \Psi_G$. If, on the other hand, $G\otimes M \in \Psi_G$, then $1_G\otimes\varphi_M$ is an isomorphism, and hence

$$G\otimes\operatorname{Coker}\varphi_M = \operatorname{Coker}(1_G\otimes\varphi_M) = 0\,,$$

therefore $\operatorname{Coker}\varphi_M=0$ (as in the proof of 1.3 (ψ)). Since $\operatorname{Tor}_1(G,[G,G\otimes M])=0$ we have

$$G\otimes\operatorname{Ker}\varphi_M = \operatorname{Ker}(1_G\otimes\varphi_M) = 0\,,$$

that is $\operatorname{Ker}\varphi_M=0$ and φ_M is an isomorphism. $\oplus t\operatorname{Tor}_i(G,M)=\operatorname{Tor}_i(G,[G,G\otimes M])=0$, therefore $M\in\Phi_G$.

The proof of (ψ) is similar using the commutative diagram

$$\begin{array}{ccc}
[G, G\otimes[G, M]] & \xrightarrow{[1_G,\psi_M]} & [G, [H, M]] \\
\uparrow & & \cong\uparrow{\scriptstyle \varrho} \\
{\scriptstyle \varphi_{[G,M]}} & & [G\otimes H, M] \\
 & & \cong\uparrow{\scriptstyle [\psi_G, 1_M]} \\
H\otimes[G, M] & \underset{\cong}{\xrightarrow{\sigma}} & [[H, G], M]
\end{array}$$

where σ and ϱ are the canonical isomorphism (cf. [6, VI, § 5]).

2. Modules of finite weak and of finite injective dimension.

If G is a Gorenstein module of rank 1 ($\mu^{\operatorname{ht}\mathfrak{p}}(\mathfrak{p},G)=1$ for all $\mathfrak{p}\in\operatorname{Spec}A$) and if A is indecomposable and of finite (Krull-)dimension (and hence A is Cohen-Macaulay and a homomorphic image of a Gorenstein ring, by Theorem 4.1 in section 4), then $H=[G,G]\cong A$ (see Sharp [17,(3.10.i)]). In this case we have simply φ_M: $M\to[G,G\otimes M]$ (respectively ψ_M:

$G\otimes[G,M] \to M$) defined by $\varphi_M(m)(g)=g\otimes m$ (respectively $\psi_M(g\otimes f)=f(g)$). In [17] Sharp has proved that if M is f.g. with $\operatorname{pd} M<\infty$ (respectively $\operatorname{id} M<\infty$), then $\operatorname{id} G\otimes M<\infty$ (respectively $\operatorname{pd}[G,M]<\infty$) and φ_M (respectively ψ_M) is an isomorphism. We want to prove the following extension of this:

THEOREM 2.1. *Let G be Gorenstein and assume* $\dim G<\infty$ (*e.g.* $\dim A<\infty$). *Then for any A-module M (not necessarily* f.g.)

(φ. i) *If* $\operatorname{wd} M<\infty$, *then* $\operatorname{id} G\otimes M<\infty$ *and* $M\in \Phi_G$.
(φ. ii) *If* $\operatorname{id} G\otimes M<\infty$ *and* $\operatorname{Supp} M\subseteqq \operatorname{Supp} G$, *then* $\operatorname{wd} M<\infty$.
(ψ. i) *If* $\operatorname{id} M<\infty$, *then* $\operatorname{wd}[G,M]<\infty$ *and* $M\in \Psi_G$.
(ψ. ii) *If* $\operatorname{wd}[G,M]<\infty$ *and* $\operatorname{Supp} M\subseteqq \operatorname{Supp} G$, *then* $\operatorname{id} M<\infty$.

PROOF. (φ. i). Let

$$0\to F_n\to\ldots\to F_0\to M\to 0$$

be a flat resolution for M. Since $\operatorname{Tor}_i(G,M)=0$ for $i>0$ (by Proposition 1.2 (φ)), we have an exact sequence

$$0\to G\otimes F_n\to\ldots\to G\otimes F_0\to G\otimes M\to 0\,.$$

We have id $G\otimes F_i<\infty$ for each i, by Corollary 1.2 of [11]. Therefore $\operatorname{id} G\otimes M<\infty$.

The proof of (ψ.i) is similar.

(φ.ii). By Proposition 1.2 (ψ) we have $G\otimes M\in\Psi$ and by Proposition 1.2 we have $G\in\Psi_G$, and hence we may use Proposition 1.4 (φ) and (ψ.i) above to get $n=\operatorname{wd} H\otimes M=\operatorname{wd}[G,G\otimes M]<\infty$, and hence $\operatorname{wd} M\leqq n<\infty$ (since H is projective).

Also the proof of (ψ. ii) is similar.

REMARK 1. Assume that G is Gorenstein. For all $\mathfrak{p}\in\operatorname{Spec} A$ let the i'th Betti number of the $A_{\mathfrak{p}}$-module $M_{\mathfrak{p}}$ be denoted by $\beta_i(\mathfrak{p},M)$ and put $s(\mathfrak{p})=\operatorname{depth} A_{\mathfrak{p}}$. If M is f.g. with $\operatorname{pd} M<\infty$ then for all i we have

$$\beta_i(\mathfrak{p},M)r_G(\mathfrak{p}) = \mu^{s(\mathfrak{p})-i}(\mathfrak{p},G\otimes M)\,.$$

This follows from the corresponding result (cf. [8, Proposition 3.1 (a)]): If M is f.g. with $\operatorname{id} M<\infty$ then for all i we have

$$\mu^i(\mathfrak{p},M)r_G(\mathfrak{p}) = \beta_{s(\mathfrak{p})-1}(\mathfrak{p},[G,M])\,.$$

COROLLARY 2.2 *Let G be Gorenstain and M non-zero* f.g. *with* $\operatorname{Supp} M\subseteqq\operatorname{Supp} G$. *Then:*

(a) *M is projective if and only if $G \otimes M$ is Gorenstein* (*and* $r_{G\otimes M} = r_G \cdot r_M$).
(b) *M is Gorenstein if and only if $[G,M]$ is projective* (*and* $r_{[G,M]} = r_G \cdot r_M$).

REMARK 2. If G is Gorenstein, $\operatorname{Supp} M \subseteq \operatorname{Supp} G$, and $\operatorname{wd} M < \infty$, then it is rather easy to prove that $\operatorname{Ass} M = \operatorname{Ass}(G\otimes M)$, $zM = z(G\otimes M)$, $\operatorname{Supp} M = \operatorname{Supp}(G\otimes M)$ and if moreover M is f.g. then the sequence $a_1,\ldots,a_p \in A$ is a M-regular sequence if and only if it is $G\otimes M$-regular. If $\operatorname{id} M < \infty$ we have a corresponding result.

REMARK 3. In Theorem 2.1 we may replace "wd" by "pd". This follows from results due to M. Raynaud and L. Gruson (Theorem 3.2.6 in "Critères de platitude et de projectivité" Invent.Math. 13 (1971), 1–89) and C.U. Jensen (Proposition 6 in "On the vanishing of $\lim^{(i)}$" J. Algebra 15 (1970), 151–166). However, R. Fossum informed me that it is easy to give a simple proof of this in the case in which we are interested, namely:

PROPOSITION 2.3. *Assume that G is a Gorenstein module with $d = \dim G < \infty$. Then for any A-module M with* $\operatorname{Supp} M \subseteq \operatorname{Supp} G$,

$$\operatorname{pd} M \leqq d \text{ if and only if } \operatorname{wd} M < \infty .$$

PROOF OF "IF". Put $A' = A/\operatorname{Ann} G$ (cf. Proposition 1.1) and let $F_0,\ldots,F_{d-1},F_d$ be free A'-modules in the two exact sequences

$$(1) \qquad 0 \to K \to F_{d-1} \to \ldots \to F_0 \to M \to 0$$
$$(2) \qquad 0 \to N \to F_d \to K \to 0$$

Here we have $\operatorname{Ext}^i(G\otimes F_j, G\otimes N) = 0$ for $i > 0$ and $j = 0,\ldots,d-1$, by Theorem 2.1 (φ.i), since F_j is A'-free and $\operatorname{wd} N < \infty$. From (1) we therefore obtain

$$\operatorname{Ext}^1(G\otimes K, G\otimes N) = \operatorname{Ext}^{d+1}(G\otimes M, G\otimes N) = 0 ,$$

since $\operatorname{id} G\otimes N \leqq d$ by Proposition 1.1, Theorem 2.1, and [2, Corollary 5.5]. This shows that we get from (2) a split-exact sequence

$$0 \to G\otimes N \to G\otimes F_d \to G\otimes K \to 0$$

since $\operatorname{Tor}_1(G,K) = 0$. Hence also (again by the Theorem)

$$0 \to H\otimes N \to H\otimes F_d \to H\otimes K \to 0$$

is splitting that is $\operatorname{Ext}^1(H\otimes K, H\otimes N) = 0$. Now, since $\operatorname{Supp} K$, $\operatorname{Supp} N \subseteq \operatorname{Supp} H$ and H is projective, we get $\operatorname{Ext}^1(K,N) = 0$ that is (2) splits. Therefore K is projective, that is $\operatorname{pd} M \leqq d$.

REMARK 4. It is easy to see that the following conditions are equivalent:
(a) *G is Gorenstein.*
(b) $\mathrm{id}\, G \otimes M < \infty$ *and* $\mathrm{Tor}_1(G,M) = 0$ *for all* (f.g.) *M with* $\mathrm{wd}\, M < \infty$.
(c) $\mathrm{wd}[G,M] < \infty$ *and* $\mathrm{Ext}^1(G,M) = 0$ *for all M with* $\mathrm{id}\, M < \infty$.

PROOF. (Assume (c). Then $\mathrm{id}\, G < \infty$ by [6, VI § 5], and it is easy to see that $\mathrm{Ext}^i(G,M) = 0$ for all $i > 0$ and all M with $\mathrm{id}\, M < \infty$. Thus it follows from [12, § 2] that $\mathrm{depth}_{A_\mathfrak{p}} G_\mathfrak{p} = \mathrm{depth} A_\mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Supp}\, G$.)

If in (c) we consider only finitely generated modules M we get the following result.

PROPOSITION 2.4. *When A is local, then G is Gorenstein if an only if A is Cohen-Macaulay and for all* f.g. *M with* $\mathrm{id}\, M < \infty$ *the following holds*: $\mathrm{pd}[G,M] < \infty$ *and* $\mathrm{Ext}^i(G,M) = 0$ *for* $i > 0$.

PROOF. By the theorem it is enough to prove the "if"-part and this will be done by induction on $s = \mathrm{depth}\, A = \dim A$.

$s = 0$. A is artinian, and hence $E = E(A/\mathfrak{m})$ is f.g. (where $\mathfrak{m}$ is the maximal ideal) and $\mathrm{pd}[G,E] < \infty$, therefore $\mathrm{id}\, G < \infty$ and G is Gorenstein.

Inductionstep $s-1 \to s$ for $s \geqq 1$. Since there exists a non-zero f.g. A-module of finite injective dimension (e.g. $N = [A/(a_1, \ldots, a_s), E]$, where $(a_1, \ldots, a_s)$ is a parameter system), we have $\mathrm{depth}\, G = s$ (by [12, § 2]). Choose $a \in M - (\mathfrak{m}^2 \cup zA \cup zG)$ (cf. [4, § 1, n° 2, Proposition 2]) and write $\bar{N} = N/aN$. We want to see that $\bar{G}$ as an $\bar{A}$-module satisfies the conditions in the proposition. Therefore let M be a f.g. $\bar{A}$-module with $\mathrm{id}_{\bar{A}} M < \infty$. Also $\mathrm{id}_A M < \infty$ (since

$$\mathrm{Ext}_A^{i+1}(k,M) = \mathrm{Ext}_A^{i}(\mathfrak{m},M) = \mathrm{Ext}_{\bar{A}}^{i}(\bar{\mathfrak{m}},M) = 0$$

for $i > \mathrm{id}_{\bar{A}} M$) and hence $[G,M]_A{}' = [\bar{G},M]_{\bar{A}}$ is of finite A-projective dimension, therefore $\mathrm{pd}_{\bar{A}}[\bar{G},M]_{\bar{A}} < \infty$ by Corollary (27.5) of [13]. Also $\mathrm{Ext}_{\bar{A}}^i(\bar{G},M) = \mathrm{Ext}_A^i(G,M) = 0$ for $i > 0$, and $\bar{G}$ is a Gorenstein $\bar{A}$-module by the inductive hypothesis, therefore G is a Gorenstein A-module.

We have already seen (Proposition 1.2) that $\mathrm{wd}\, M < \infty$ implies $M \in \Phi_G$ if G is a PG-module. If G is Gorenstein, then also another nice class of modules is contained in Φ_G, namely the class of f.g. modules of finite G-dimension (for definition see [1, Chapter 3] or [14, § 3.2]).

PROPOSITION 2.5. *If G is Gorenstein and M is* f.g. *with* G-dim $M < \infty$, *then* $M \in \Phi_G$.

PROOF. By induction on $d = G\text{-dim } M$.

$d=0$. $\mathrm{Tor}_i(M,N) = [\mathrm{Ext}^i(M,A),N] = 0$ for $i>0$ and N injective ([6, VI § 5]), and hence $\mathrm{Tor}_i(M,N)=0$ for $i>0$ if $\mathrm{id}\,N<\infty$, in particular $\mathrm{Tor}_i(G,M)=0$ for $i>0$. Now we want to show $\mathrm{Ext}^i(G,G\otimes M)=0$ for $i>0$. Assume A local with maximal ideal $\mathfrak{m}$ and use induction on $s=\mathrm{depth}\,A$:

$s=0$. G is injective and therefore (again by [6, VI § 5]) $\mathrm{Ext}^i(G,G\otimes M) = \mathrm{Ext}^i(G,[[M,A],G]) = [\mathrm{Tor}_i(G,[M,A]),G]=0$ for $i>0$, since $G\text{-dim}[M,A] = 0$.

Inductionstep $s-1\to s$ for $s\geqq 1$. We have $\mathrm{depth}\,M = s \geqq 1$, by Theorem (4.13.6) of [1]. Choose $a\in\mathfrak{m}-(zM\cup zA)$ and write $\bar{N}=N/aN$. Then

$$0\to G\otimes_A M \xrightarrow{a\cdot} G\otimes_A M \to \bar{G}\otimes_{\bar{A}}\bar{M}\to 0$$

is exact (since $\mathrm{Tor}^A{}_1(\bar{G},\bar{M}) = \mathrm{Tor}^{\bar{A}}{}_1(\bar{G},\bar{M})=0$ and $G\text{-dim}_{\bar{A}}\bar{M}=0$) and hence also

$$\mathrm{Ext}_A{}^i(G,G\otimes_A M) \xrightarrow{a\cdot} \mathrm{Ext}_A{}^i(G,G\otimes_A M)\to \mathrm{Ext}_A{}^i(G,\bar{G}\otimes_{\bar{A}}\bar{M})$$

is exact. Now $\mathrm{Ext}_A{}^i(G,\bar{M}\otimes_{\bar{A}}\bar{M}) = \mathrm{Ext}_{\bar{A}}{}^i(\bar{G},\bar{G}\otimes_{\bar{A}}\bar{M})=0$ for $i>0$, and hence $\mathrm{Ext}_A{}^i(G,G\otimes_A M)=0$ for $i>0$, by Nakayama's lemma.

It still remains to show that φ_M is an isomorphism. Let $F_1\to F_0\to M\to 0$ be exact with F_0,F_1 f.g. free. Since both $\mathrm{Ker}(F_1\to F_0)$ and $\mathrm{Ker}(F_0\to M)$ are of G-dimension zero, it is easy to see that the rows in the commutative diagram

$$\begin{array}{ccccccc}
[G,\, G\otimes F_1] & \to & [G,\, G\otimes F_0] & \to & [G,\, G\otimes M] & \to & 0\\
\uparrow{\scriptstyle \varphi_{F_1}}\,\cong & & \uparrow{\scriptstyle \varphi_{F_0}}\,\cong & & \uparrow{\scriptstyle \varphi_M} & & \\
H\otimes F_1 & \longrightarrow & H\otimes F_0 & \longrightarrow & H\otimes M & \to & 0
\end{array}$$

are exact and therefore φ_M is an isomorphism, by the 5-lemma. Now the proposition is established in the case $d=0$.

The inductive step is easy using Lemma 1.3.

3. Some remarks on duality with respect to a PG-module.

In this section G is a PG-module, and we will use the following notation:

$$M^* = [M,G], \quad E^i = \mathrm{Ext}^i(-,G), \quad L_i = E^iE^i\,.$$

Let

$$\ldots\to P_i(M)\to\ldots\to P_0(M)\to M\to 0$$

$(P_{-1}(M)=M, P_{-2}(M)=0,\ldots)$ be a fixed projective resolution for M (with each $P_i(M)$ f.g. if M is f.g.) (cf. p. 51 of [1]) and define

$$D(M) = \operatorname{Coker}(P_0(M)^* \to P_1(M)^*),$$
$$\Omega^i(M) = \operatorname{Ker}(P_{i-1}(M) \to P_{i-2}(M)).$$

PROPOSITION 3.1. *There is an exact sequence*

$$0 \to E^1D(M) \to M\otimes H \xrightarrow{\sigma_M{}^0} M^{**} \to E^2D(M) \to 0,$$

where $\sigma_M{}^0(m\otimes h)(f) = hf(m)$ *for* $f: M \to G$ *and* $h: G \to G$.

PROOF. This is well-known in the case $G=A$, cf. Proposition 1 of [14, § 3.1] (or (2.1) of [1]), and the proof is similar using that $E^i(P^*)=0$ for $i>0$, if P is f.g. projective.

It is now possible to define concepts like "k-torsion free with respect to G" and "G-dimension with respect to G" and to prove many analogues to the results in [14, Chapitre 3] and [1, Chapters 2–4], if $G \in \Psi_G$. (Here we use the isomorphisms

$$M^*\otimes H \to [M, G\otimes H] \to [M, [H, G]] \to (M\otimes H)^*.)$$

In particular, if A is local then G is Gorenstein if and only if each f.g. M has finite G-dimension with respect to G (the analogue to Theorème 3 of [14] and Theorem (4.20) of [1]). We have furthermore that $\Omega^k(M)$ is k-torsion free with respect to G, if G is Gorenstein (the analogue to a part of Theorem (4.21) of [1]). A result in the opposite direction is the following:

PROPOSITION 3.2. *If G is Gorenstein and $k \geqq 0$ is an integer, then the following are equivalent*:

(i) $A_\mathfrak{p}$ *is a Gorenstein ring for all* $\mathfrak{p} \in \operatorname{Supp} G$ *with* $\operatorname{depth} A_\mathfrak{p} < k$ (*that is* $A' = A/Ann\ G$ *is a* G_k*-ring, cf. Proposition* 1.1 (ii) *and* [10, 3.16 *Definition.*])

(ii) *G is a k'th syzygy* (*that is* $G=\Omega^k(M)$ *for a suitable* f.g. M)

(iii) *G is k-torsion free with respect to A*

(iv) *For all* f.g. M *with* $\operatorname{Supp} M \subseteqq \operatorname{Supp} G$ *the following holds*: *If M is k-torsion free with respect to G then M is a k'th syzyg.*

COROLLARY 3.3. *If G is a Gorenstein module of finite G-dimension with respect to A, then* $A' = A/\operatorname{Ann} G$ *is Gorenstein ring* (*and G is projective*).

PROOF OF THE PROPOSITION. Since a k-torsion free module always is a k'th syzygy, and since the converse of this also holds if A is a G_{k-1}-ring (cf.

[1, Theorem 4.21]), it is enough to show the equivalence of (i), (iv), (ii), and (i).

(i) implies (iv). Assume M is k-torsion free with respect to G (that is $E^iD(M)=0$ for $1 \leqq i \leqq k$, and hence $E^i(M^*)=0$ for $1 \leqq i \leqq k-2$). We have an exact sequence

$$0 \to M \otimes H \to P_0(M^*)^* \to \ldots \to P_{k-1}(M^*)^* ,$$

where each $P_i(M^*)^*$ is a Gorenstein module (or zero). Therefore M is a b_k-module (as defined in [10, Definition 4.3]) and hence a k'th syzygy (by Satz 4.2 und 4.6 of [10]).

That (iv) implies (ii) is obvious, since G is of G-dimension zero with respect to G (because $\sigma_G{}^0=\psi_G$ is an isomorphism, by Proposition 1.2, and $E^i(G)=E^i(G^*)=0$ for $i>0$) and hence G is k-torsion free with respect to G for all k.

(ii) implies (i). Assume A is local with $\operatorname{depth} A<k$ and use induction on $d=\operatorname{depth} A$ to prove that A is Gorenstein:

$d=0$. G is injective and a submodule of a (f.g.) free A-module F (since G is a kth syzygy, $k>0$). Therefore $F=G \oplus M$ for a suitable M, that is G is projective and hence A is self-injective, that is A is Gorenstein.

Inductionstep $d-1 \to d$ for $d \geqq 1$. Assume $G=\Omega^k(M)$, choose $a \in \mathfrak{m}-zA$ and write $\bar{N}=N/aN=N \otimes A/(a)$. Since

$$\operatorname{Tor}^A{}_1(\Omega^i(M),\bar{A}) = \operatorname{Tor}^A{}_{i+1}(M,\bar{A}) = 0$$

for $i>0$, we have an exact sequence

$$0 \to \bar{G} \to \overline{P_{k-1}(M)} \to \ldots \to \overline{P_1(M)} \to \overline{\Omega^1(M)} \to 0 .$$

Therefore $\bar{G}$ is a $(k-1)$th syzygy as an $\bar{A}$-module and hence $\bar{A}$ (and thereby A) is Gorenstein by the inductive hypothesis.

PROPOSITION 3.4. *Let $g \geqq 0$ be an integer and assume* $\operatorname{grade}_A M \geqq g$ *and* $\operatorname{Supp} M \subseteq \operatorname{Supp} G$ *(or just* $\operatorname{grade}_G M \geqq g$*). Then there exists a natural homomorphism $\sigma_M{}^g$ which can be included in an exact sequence*

$$0 \to E^{g-1}E^g(M) \to E^{g+1}D\Omega^g(M) \to M \otimes H \xrightarrow{\sigma_M{}^g} L_g(M) \to E^{g+2}D\Omega^g(M) \to 0 .$$

REMARK. Now it is not hard to state and prove the analogue to Proposition 7 of [7]. In particular $E^i(M \otimes H)=E^iL_i(M)$ if M is f.g. with $\operatorname{grade} M \geqq i$ and $\operatorname{Supp} M \subseteq \operatorname{Supp} G$.

PROOF OF THE PROPOSITION. Let $0 \to G \to Q^0 \to \ldots$ be an injective resolution for G and consider the double complex:

$$K^{pq} = \begin{cases} P_{-p}(M)\otimes[G,Q^q] \\ =[[P_{-p}(M),G],Q^q] & \text{for } -(g+1)\leqq p\leqq 0 \\ 0 & \text{otherwise.}\end{cases}$$

The two corresponding spectral sequences have initial terms $I_2{}^{**}$ and $II_2{}^{**}$:

$$I_2{}^{pq} = \begin{cases} \mathrm{Tor}_{-p}(M,E^q(G)) & \text{for } -g\leqq p\leqq 0 \text{ and } q\geqq 0 \\ \text{some kernel} & \text{for } p=-g-1' \text{ and } q\geqq 0 \\ 0 & \text{otherwise.}\end{cases}$$

In particular

$$I_2{}^{pq} = \begin{cases} M\otimes H & \text{for } p=0 \text{ and } q=0 \\ \mathrm{Ker}(P_{g+1}(M)\to P_g(M))\otimes H & \text{for } p=-g-1 \text{ and } q=0 \\ 0 & \text{otherwise.}\end{cases}$$

$$II_2{}^{pq} = \begin{cases} E_pE^g(M) & \text{for } p\geqq 0 \text{ and } q=-g \\ E_pD\Omega^g(M) & \text{for } p\geqq 0 \text{ and } q=-g-1 \\ 0 & \text{otherwise.}\end{cases}$$

Now the desired exact sequence follows by use of the usual technique.

Next, we will consider Cohen-Macaulay modules (i.e. non-zero f.g. modules M such that $\mathrm{depth}_{A_\mathfrak{p}} M_\mathfrak{p} = \dim_{A_\mathfrak{p}} M_\mathfrak{p}$ for all $\mathfrak{p}\in \mathrm{Supp}\, M$, cf. Chapitre IV B of [15]). First two easy lemmas.

LEMMA 3.5. *Assume that M is a Cohen-Macaulay module and that $A_\mathfrak{p}$ is Cohen-Macaulay for all $\mathfrak{p}\in \mathrm{Supp}\, M$. Then:*

(i) *The map g_M: $\mathrm{Supp}\, M\to \mathsf{Z}$ defined by $g_M(\mathfrak{p})=\mathrm{grade}_{A_\mathfrak{p}} M_\mathfrak{p}$ is continuous. In particular, g_M is constant (on $\mathrm{Supp}\, M$) if A is local.*

(ii) *There exist Cohen-Macaulay modules $M_1,\ldots,M_p$ such that $M=M_1\oplus\ldots\oplus M_p$ and such that g_{M_i} is constant on $\mathrm{Supp}\, M_i$ for each i, $1\leqq i\leqq p$.*

PROOF. For prime ideals $\mathfrak{p}$ and $\mathfrak{q}$, $\mathfrak{p}\subseteqq \mathfrak{q}$, we have

$$\dim_{A_\mathfrak{p}} M_\mathfrak{p} + \dim_{A_\mathfrak{q}} (A/\mathfrak{p})_\mathfrak{q} = \dim_{A_\mathfrak{q}} M_\mathfrak{q}$$

(see [15] or [16, (2.3. iii)]). This gives

$$\mathrm{grade}_{A_\mathfrak{p}} M_\mathfrak{p} = \dim_{A_\mathfrak{p}} A_\mathfrak{p} - \dim_{A_\mathfrak{p}} M_\mathfrak{p} = \dim_{A_\mathfrak{q}} A_\mathfrak{q} - \dim_{A_\mathfrak{q}} M_\mathfrak{q} = \mathrm{grade}_{A_\mathfrak{q}} M_\mathfrak{q},$$

by §1.4 Lemme 4 of [14]. Now the first assertion follows by the "connected spectrum principle", see [4, § 4, exercise 14], or [17, (3.7)].

Let $X_1, \ldots, X_p$ be the connected components of $\operatorname{Supp} M \simeq \operatorname{Spec}(A/\operatorname{Ann} M)$, and let $A/\operatorname{Ann} M = A_1 \oplus \ldots \oplus A_p$ be the corresponding direct product decomposition (see [4, § 4] or [17, § 1]). We have now $M = M_1 \oplus \ldots \oplus M_p$ where M_i is an A_i-(and A-)module. For all i, j, $1 \leqq i, j \leqq p$, and all $\mathfrak{p} \in X_i$ we have $(M_i)_{\mathfrak{p}} = M_{\mathfrak{p}}$ and $(M_j)_{\mathfrak{p}} = 0$ if $i \neq j$, therefore M_i is a Cohen-Macaulay A-module of constant grade g_{M_i} on $\operatorname{Supp} M_i = X_i$.

The assumption on the ring in the above lemma is in particular satisfied if there exists a Gorenstein module G with $\operatorname{Supp} G \supseteqq \operatorname{Supp} M$, and the conclusion reduces in this case the study of Cohen-Macaulay modules to the study of Cohen-Macaulay modules of constant grade g (that is $g_M(\mathfrak{p}) = g$ for all $\mathfrak{p} \in \operatorname{Supp} M$).

We have the following well-known lemma.

Lemma 3.6. *Let G be a Gorenstein module, let M be a non-zero* f.g. *module with* $\operatorname{Supp} M \subseteqq \operatorname{Supp} G$, *and let g be a non-negative integer. Then M is Cohen-Macaulay of constant grade g if and only if $E^i(M) = 0$ for $i \neq g$.*

Proof. See Lemma 3.3 of [8].

The following result is expected from Proposition 3.4 of [8]:

Proposition 3.7. *If G is Gorenstein and M is Cohen-Macaulay of constant grade g and* $\operatorname{Supp} M \subseteqq \operatorname{Supp} G$, *then $\sigma_M{}^g$: $M \otimes H \to L_g(M)$ is an isomorphism.*

Proof. Assume A local of depth d and divide into two cases.

$g = 0$. See the proof of Proposition 4 of [14, Chap. 3]. In fact, M is of G-dimension zero with respect to G, in particular $E^i D(M) = 0$ for all $i > 0$.

$g > 0$. We have $\operatorname{depth} \Omega^g(M) = d$ since $\operatorname{depth} M = d - g$. Hence $E^i D \Omega^g(M) = 0$ for $i > 0$ by the first case.

The following example shows that there exists in general a f.g. module M such that $\sigma_M{}^i$ is an isomorphism and such that M is not Cohen-Macaulay.

Example 3.8. If $G \in \Psi_G$ is a PG-module with $\operatorname{Supp} G = \operatorname{Spec} A$ then the following two conditions are equivalent:

(a) Both A and G are Cohen-Macaulay of (the same) dimension less than 3.

(b) M^* is either Cohen-Macaulay or zero for all f.g. M.

PROOF. We may assume A is local.

(a) implies (b). The sequence

$$0 \to M^* \to P_0(M)^* \to P_1(M)^*$$

is exact, therefore $\operatorname{depth} M \geqq \min\{2, n\}$, $n = \dim A = \operatorname{depth} P_i(M)^*$, and hence M^* is Cohen-Macaulay (of dimension n).

(b) implies (a). $H^* = G^{**}$ is Cohen-Macaulay, therefore A and G are Cohen-Macaulay. Assume $n = \dim A \geqq 3$ and choose an A-regular sequence $a_1, a_2, a_3 \in \mathfrak{m} =$ the maximal ideal of A. Write $M = A/(a_1, a_2, a_3)$ and $K = \Omega^2(M)$. Then $\operatorname{grade} M = 3$ and hence $E^i(M) = 0$ for $i \leqq 2$ (by Proposition 1.1 (x)) and

$$0 \to P_0(M)^* \to P_1(M)^* \to K^* \to 0$$

is exact. This shows that the G-dimension of K^* with respect to G is less than 2. By Proposition 3.4 we have that

$$E^3D(K) \to H \otimes M \to L_2(M) = 0$$

is exact, and hence $E^1(K^*) = E^3D(K) \neq 0$. Therefore the G-dimension of K^* with respect to G is 1 by the analogue to Theorem (4.13. a.iv) of [1]. Hence depth $K^* = n - 1$ by the analogue to Theorem (4.13.b) of [1]. Since K is a submodule of a free module, it is 1-torsion free with respect to G. Therefore $0 \to K \otimes H \to K^{**}$ is exact and hence $K^{**} \neq 0$, that is $\operatorname{grade} K^* = 0$. This shows that K^* is not Cohen-Macaulay because $\operatorname{depth} K^* = n - 1 < n = \dim K^*$, contradicting our assumption.

We end this section with a remark concerning Gorenstein modules and local cohomology. Assume A local with maximal ideal $\mathfrak{m}$, $n = \operatorname{depth} A$. Let $H_{\mathfrak{m}}{}^d$ be the dth derived of the local cohomology functor (that is $H_{\mathfrak{m}}{}^d = \varinjlim_j \operatorname{Ext}^d(A/\mathfrak{m}^j, -)$, cf. [9, Th. 2.8]). It is easy to prove that G is Gorenstein if and only if G is Cohen-Macaulay of dimension d and $H_{\mathfrak{m}}{}^d(G)$ is injective (cf. Proposition 4.14 of [9] for the case $G = A$). Let now G be Gorenstein. Then $H_{\mathfrak{m}}{}^n(G) = \oplus rE$, when $r = r_G(\mathfrak{m}) = \mu^n(\mathfrak{m}, G)$ and $E = E(A/\mathfrak{m})$. Let $\varrho_i : H_{\mathfrak{m}}{}^n(G) \to E$ be the ith projection, that is $x = (\varrho_1(x), \ldots, \varrho_r(x))$ for all $x \in H_{\mathfrak{m}}{}^n(G)$ (after the identification $H_{\mathfrak{m}}{}^n(G) = \oplus rE$). The natural homomorphism

$$\xi_M{}^n : \oplus r H_{\mathfrak{m}}{}^n(M) \to [[M, G], E]$$

is defined by

$$\xi_M{}^n(x_1,\ldots,x_r)(\gamma) = \sum \varrho_i H_{\mathfrak{m}}{}^n(\gamma)(x_i), \quad \gamma \in [M,G].$$

The homomorphism $\xi_A{}^n$ is an isomorphism (by induction on n) and we have natural isomorphisms for all i and all f.g. M:

$$\oplus r H_{\mathfrak{m}}{}^i(M) \cong [E^{n-i}(M), E] \quad \text{and} \quad \oplus r T^i(M) = E^{n-i}(M)^\wedge$$

where $T^i = [H_{\mathfrak{m}}{}^i, E]$ and ^ denotes completion with respect to the $\mathfrak{m}$-adic topology. (Cf. Theorem 6.3 of [9] for the case $G = A$.)

If A (or G) is complete, then Proposition 6.6.8 of [9] gives a natural homomorphism α_M: $\oplus r^2 M \to L_{n-i}(M)$, where $i = \dim M$. Therefore $\operatorname{grade} M = n - i$ (cf. Proposition 3.4). The homomorphism α_M is an isomorphism if M is Cohen-Macaulay, by the exercise p. 94 of [9] (cf. Proposition 3.7).

4. On the existence of a rank 1 Gorenstein module.

It is well-known that if A is Cohen-Macaulay and a homomorphic image of a Gorenstein ring B, then there exists a Gorenstein A-module G of rank 1 (see Sharp [17, Th. 3.9]). It turns out that if the conditions on A are satisfied and $A = A_1 \oplus \ldots \oplus A_p$ is a decomposition of the Cohen-Macaulay B-module A in constant grade Cohen-Macaulay B-modules A_i (cf. Lemma 3.5), then $G = \oplus \operatorname{Ext}_B{}^{g_i}(A_i, B)$ has the desired properties if $g_i = \operatorname{grade} A_i$. In fact, if $0 \to B \to Q^0 \to \ldots$ is a minimal B-injective resolution for B, then

$$0 \to \operatorname{Ext}_B{}^{g_i}(A_i, B) \to [A_i, Q^{g_i}]_B \to [A_i, Q^{g_i+1}]_B \to \ldots$$

is a minimal A_i-injective resolution for $\operatorname{Ext}_B{}^{g_i}(A_i, B)$ (cf. Lemma 3.6 and [2, Lemma 2.1]).

Let R be a commutative ring with identity and let M be an R-module. We will now catalogue some simple properties of "the principle of idealization" (introduced by Nagata on p. 2 of [13]):

(0) The cartesian product $R \times M$ has a structure of a commutative ring with identity (addition is componentwise and multiplication is $(r,x)(r',x') = (rr', rx' + r'x)$).

(1) $r \in zR \cup zM$ if and only if $(r,0) \in z(R \times M)$; and if so:

$$(R \times M)/(r,0)(R \times M) = (R/(r)) \times (M/rM).$$

(2) $\operatorname{Spec}(R \times M) = \{\mathfrak{p} \times M \,;\, \mathfrak{p} \in \operatorname{Spec} R\}$

(3) $R \times M$ is noetherian if and only if R is noetherian and M is f.g.

(4) If S is a multiplicative system in R then $S \times M$ is a multiplicative system in $R \times M$, and

$$(r, x)/(s, y) \mapsto (r/s, (sx-ry)/s^2)$$

defines an isomorphism

$$(S \times M)^{-1}(R \times M) \to (S^{-1}R) \times (S^{-1}M) .$$

(5) If R is noetherian local with maximal ideal $\mathfrak{m}$ and M is f.g., then

$$\mathrm{Ann}_{R\times M}(\mathfrak{m} \times M) = (\mathrm{Ann}\,\mathfrak{m} \cap \mathrm{Ann}\,M) \times (0:\mathfrak{m})_M ,$$

and hence

$$\mu_{R\times M}{}^0(\mathfrak{m} \times M, R \times M) = \mu_R{}^0(\mathfrak{m}, M) + \mathrm{length}(\mathrm{Ann}\,\mathfrak{m} \cap \mathrm{Ann}\,M).$$

Now assume G is a rank 1 Gorenstein module. Then A is Cohen-Macaulay. We want to prove that $A \times G$ is Gorenstein. By (4) above we may assume A local, and by (1) $\dim A = 0$, therefore $A \times G$ is artinian by (2), and $\mu_{A\times G}{}^0(A \times G) = 1$ by (5). Hence $A \times G$ is Gorenstein by (2.8) of [3]. We have now obtained:

THEOREM 4.1. *A necessary and sufficient condition for the existence of a rank* 1 *Gorenstein module is that A is Cohen-Macaulay and a homomorphic image of a Gorenstein ring.*

That this condition is necessary has recently been proved independently by Idun Reiten ("The converse to a theorem of Sharp on Gorenstein modules", Proc. Amer. Math. Soc. 32 (1972) 417–420).

PROPOSITION 4.2. *If A is artinian and local with maximal ideal* $\mathfrak{m}$, *then the following are equivalent*:

(a) *G is a Gorenstein module of rank* 1 (*that is* $G = E(A/\mathfrak{m})$),

(b) $A \times G$ *is a Gorenstein ring* (*that is a* QF *ring*),

(c) $\mu^0(\mathfrak{m}, G) = 1$ *and* $\mathrm{Ann}_A\, G = 0$.

Proof. (a) ⇒ (b) is proved above. (b) ⇒ (c) follows from (5) above, since $\mathrm{Ann}\ G \neq 0$ implies $\mathrm{Ann}\ G \cap \mathrm{Ann}\,\mathfrak{m} \neq 0$. For (c) → (a) note that $\mu^i(\mathfrak{m}, G) = \beta_i(\mathfrak{m}, [G, E])$, when $E = E(A/\mathfrak{m})$ (cf. [6, VI, § 5]). Therefore $\beta_0(\mathfrak{m}, [G, E]) = 1$ and hence $[G, E] = A/\mathrm{Ann}[G, E] = A/\mathrm{Ann}\,G = A$ is free. Therefore $\mu^i(\mathfrak{m}, G) = 0$ for $i > 0$.

From this we get the following analogue to (2) and (3) in Theorem (4.1) of Bass [3] (cf. also (3.12) Remark of Sharp [16]).

COROLLARY 4.3. *If G is a* PG*-module and A is local, then the following are equivalent:*

(i) *G is Gorenstein of rank* 1.

(ii) *$(a_1,\ldots,a_s)G$ is an irreducible submodule of G for all parameter systems $(a_1,\ldots,a_s)$ of G.*

(iii) *G is Cohen-Macaulay of dimension d and $\mu^d(\mathfrak{m},G)=1$.*

REFERENCES

1. M. Auslander and M. Bridger, *Stable module theory,* Memoirs Amer. Math. Soc. 94 (1969).
2. H. Bass, *Injective dimension in Noetherian rings,* Trans. Amer. Math. Soc. 102 (1962), 18–29.
3. H. Bass, *On the ubiquity of Gorenstein rings,* Math. Z. 82 (1963), 8–28.
4. N. Bourbaki, Éléments de Mathématique, fasc. XXVII (Act. Sci. Ind. 1290) *Algèbre commutative,* Chapitre 2, Hermann, Paris, 1961.
5. N. Bourbaki, Éléments de Mathématique, fasc. XXVIII (Act. Sci. Ind. 1293) *Algèbre commuative,* Chapitre 4, Hermann, Paris, 1967.
6. H. Cartan and S. Eilenberg, Homological Algebra (Princeton Math. Ser. 19), Princeton University Press, Princeton, N.J., 1956.
7. R. Fossum, *Duality over Gorenstein ring,* Math. Scand. 26 (1970), 165–176.
8. H.-B. Foxby, *On the μ^i in a minimal injective resolution,* Math. Scand. 29 (1971), 175–186.
9. R. Hartshorne, *Local Cohomology,* A seminar given by A. Grothendieck, Harvard University, Fall 1961 (Lecture Notes in Math. 41), Springer-Verlag, Berlin · Heidelberg · New York, 1967.
10. F. Ischebeck, *Eine Dualität zwischen den Funktoren Ext und Tor,* J. Algebra 11 (1969), 510–531.
11. T. Ishikawa, *On injective modules and flat modules,* J. Math. Soc. Japan 17 (1965), 291–296.
12. G. Levin and W. Vasconcelas, *Homological dimensions and Macaulay rings,* Pacific J. Math. 25 (1968), 315–323.
13. M. Nagata, *Local Rings,* (Interscience Tracts in Pure and Applied Mathematics 13) Interscience Publishers, New York · London, 1962.
14. P. Samuel, Séminaire d'algèbre commutative, 1966/67, *Anneaux de Gorenstein et torsion en algèbre commutative,* Secrétariat mathématique, 11 rue Pierre Curie, Paris, 1967.
15. J.-P. Serre, *Algèbre Locale · Multiplicités* (Lecture Notes in Math. 11), Springer-Verlag, Berlin · Heidelberg · New York, 1965.
16. R. Y. Sharp, *Gorenstein modules,* Math. Z. 115 (1970), 117–139.
17. R. Y. Sharp, *Finitely-generated modules of finite injective dimension over certain Cohen-Macaulay rings,* Proc. London Math. Soc. 25 (1972), 303–328.
18. W. V. Vasconcelos, *Reflexive modules over Gorenstein rings,* Proc. Amer. Math. Soc. 19 (1968), 1349–1355.

UNIVERSITY OF COPENHAGEN, DENMARK