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CHARACTERISTIC CLASSES OF n-MANIFOLDS IMMERSING IN R^{n+k}

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1. Introduction.

Let $f_r\colon X_r \to BO(r)$ be a sequence of fibrations with maps $g_r\colon X_r \to X_{r+1}$ such that the usual diagrams commute. For such a situation Lashof defines the concept of an X-structure on manifolds [7], and proves a Thom-isomorphism for the bordism groups of such manifolds. (The required Thom spectrum is $T(\gamma X) = \mathscr{X}$, γX, the pullback of $\gamma \to BO$.) Many of the usual classes of manifolds may be described in terms of X-structures, e.g. U, SO, Spin, etc., as well as some more esoteric classes of manifolds. For example, Hirsch's theorem reduces the study of manifolds which immerse with codimension k to an appropriate X-structure (Hirsch [6], Wells [11]).

In this paper we study X-characteristic classes mod p, i.e. the group $H^*(X) = \lim_{\leftarrow} H^*(X_r;\, \mathsf{Z}_p)$. In particular we are interested in those characteristic classes which go to zero by the normal map of all n-manifolds (by normal map I mean the lift of the Gauss map of $M \to BO(r)$). It is shown in theorems 2.7 and 3.4 that some assumption on the cohomology of $H^*(\mathscr{X}) = \lim_{\leftarrow} H^{*+r}(TX_r)$ through dimension s, implies that the set of relations looks like the set obtained in Brown and Peterson [2] through dimension s. This result implies that restricting the class of manifolds to those which immerse with codimension k, will not introduce any new relations among characteristic classes below dimension k.

Of course for $k = n$, we obtain the result that there are no new relations through all dimensions. This is consistent with the immersion theorem of Whitney [12]. We remark that using a theorem of R. L. Brown [4], one can prove that there are no new relations among manifolds which immerse with codimension $n - \alpha(n)$, where $\alpha(n)$ is the number of 1's in the dyacdic expansion of n. This is weaker than the conjecture concerning immersions of manifolds in this codimension.

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2. $\mathcal{A}_p/(Q_0)$-Free modules.

Let p be a prime, $\mathcal{A}_p$ the mod-p Steenrod algebra. In this section we consider spectra whose cohomology is a free $\mathcal{A}_p/(Q_0)$-module, where (Q_0) is the two-sided ideal generated by the Bockstein operation. As is well-known, $\mathcal{A}_p(Q_0)$ is generated by all $\mathcal{P}^I$, where I is an admissable sequence [5]. (For $p=2$, we use the convention $\mathcal{P}^i=\mathrm{Sq}^{2i}$). Let $I=(i_1,i_2,\ldots)$ be admissable, then $N(I)=2(p-1)\sum_k i_k$ and $\mathrm{exc}(I)=pi_1-\frac{1}{2}N(I)$ are defined.

LEMMA 2.1. *Let $MU(k)$ be the Thom space of the canonical bundle over $BU(k)$ and $U_k\in \tilde{H}^{2k}(MU(k))$, the Thom class, then*

$$\{\mathcal{P}^I(U_k)\mid \mathrm{exc}(I)\leqq k\}$$

form a linearly independent set of elements in $\tilde{H}^*(MU(k))$.

PROOF. In Milnor [8], a basis $\{\mathcal{P}^R\}$, for $\mathcal{A}_p/(Q_0)$ is defined. (Here $R=(r_1,r_2,\ldots)$ is a sequence of non-negative integers almost all of which are zero.) Let $l(R)=\sum r_i$. It follows from Brown and Peterson [2] that

$$\mathrm{span}\{\mathcal{P}^I\mid \mathrm{exc}(I)\leqq k\} = \mathrm{span}\{\mathcal{P}^R\mid l(R)\leqq k\}\,.$$

(The correspondence is defined by sending $I=(i_1,i_2,\ldots)$ to $R(I)=(r_1,r_2,\ldots)$, $r_k=i_k-p^{i_{k+1}}$.) Hence, we need only show that $\{\mathcal{P}^R\mid l(R)\leqq k\}$ are independent. By the splitting principle, one may consider $U_k=x_1\,x_2\,\ldots\,x_k$ (x_i are two-dimensional classes). Then

$$1)\quad \mathcal{P}^R(x_i)=0 \qquad l(R)>1$$
$$\qquad\qquad =(x_i)^{p^*} \qquad \text{if } r_j=\delta^*{}_j$$

and the product formula,

$$2)\quad \mathcal{P}^R(ab)=\textstyle\sum_{R_1+R_2=R}\mathcal{P}^{R_1}(a)\,\mathcal{P}^{R_2}(b)$$

imply the lemma.

We may now extend $\mathcal{P}^I(U_k)$ to an additive basis of $H^*(MU(k))$, which we denote by $\{\alpha_i\}$.

Let $\mathcal{X}$ be a spectrum such that $H^*(\mathcal{X})$ is a free module over $\mathcal{A}_p/(Q_0)$ through dimension s on generators $\{u_i\}$. Assume that $H^*(\mathcal{X})$ is also of finite type, then

LEMMA 2.2. *$\tilde{H}^*(\mathcal{X}\wedge MU(k))$ is a free module over $\mathcal{A}_p/(Q_0)$ (through dimension $s+2k$) on generators $\mu_i\otimes\alpha_j$. If $H^*(\mathcal{X};\,\mathsf{Z})$ has no p-torsions, then there exists maps*

$$f_{ij}\colon \mathcal{X}\wedge MU(k)\to S^{n(i,j)}BP$$

(where BP is the Brown-Peterson spectrum [3]*), such that* $F = \prod f_{ij}$ *induces an isomorphism on* mod-p *cohomology through dimension* $s+2k$.

PROOF. Because $MU(k)$ has only even cohomology, Q_0 acts trivially on $\tilde{H}^*(MU(k))$, *therefore* $\tilde{H}^*(\mathscr{X} \wedge MU(k))$ *is a module over* $\mathcal{a}_p/(Q_0)$.

Since $\{\alpha_i\}$ is an additive basis, it follows from the Cartan formula, and the connectivity of $MU(k)$, that $\mu_i \otimes \alpha_j$ freely generates $\tilde{H}^*(\mathscr{X} \wedge MU(k))$ for $* \leqq s+2k$.

$H^*(MU(k); \mathsf{Z})$ has no torsion, so it follows that $\tilde{H}^*(\mathscr{X} \wedge MU(k))$ has no p-torsion if $\tilde{H}^*(\mathscr{X})$ hasn't any. Therefore the maps f_{ij} exist by Brown and Peterson [3], and the lemma follows.

Let $h^*: H^*(\mathscr{X}) \to \mathrm{Hom}(\pi_*(\mathscr{X}); \mathsf{Z}_p)$ be the dual of the Hurewicz map, i.e.

$$h^*(\alpha)[f] = f^*(\alpha) \in \mathsf{Z}_p, \quad \text{for } [f] \in \pi_*(\mathscr{X}),\ \alpha \in H^*(\mathscr{X}) .$$

COROLLARY 2.3. *Suppose* $(\mathscr{X})$ *satisfies the assumptions of Lemma* 2.2. *Then* $\{h^*(\mu_i \otimes \alpha_j)\}$ *form a linearly independent set in*

$$\mathrm{Hom}\big(\pi_*(\mathscr{X} \wedge MU(k)), \mathsf{Z}_p\big) \quad \textit{for } * \leqq s+2k .$$

PROOF. Each $\mu_i \otimes \alpha_j$ is equal to $f_{ij}(\Sigma^{n(i,j)}1)$, where $1 \in H^0(BP)$. Furthermore, each f_{ij} maps to a different factor in $\prod S^{n(i,j)}BP$. Hence, since $h^*(\Sigma^{n(i,j)}1) \neq 0$, and F is an equivalence (through $s+2k$) mod the class of finite groups of order prime to p, the elements $h^*(\mu_i \otimes \alpha_j)$ are linearly independent, and the corollary is proven.

COROLLARY 2.4. *Let*

$$\mu_i \otimes \alpha_j \in \tilde{H}^*(\mathscr{X} \wedge MU(k)/\emptyset) ,$$

where $MU(k)/\emptyset = MU(k) \cup \{+\}$, *then* $h^*(\mu_i \otimes \alpha_j) \neq 0$ *for* $* \leqq s+2k$.

PROOF. Consider $\bar{g} \circ \bar{f}$: $MU(k) \to MU(k)/\emptyset \to MU(k)$, where $\bar{f}$ is the inclusion, and $\bar{g} \mid MU(k)$ is the identity, $\bar{g}(+) = * =$ the base point of $MU(k)$. Smashing with $\mathscr{X}$ gives maps $f = 1 \wedge \bar{f}$, $g = 1 \wedge \bar{g}$. Clearly $g \circ f = id$. We have the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}\big(\pi_*(\mathscr{X} \wedge MU(k)/\emptyset), \mathsf{Z}_p\big) & \xrightarrow{g^*} & \mathrm{Hom}\big(\pi_*(\mathscr{X} \wedge MU(k)), \mathsf{Z}_p\big) \\ \uparrow{\scriptstyle h^*} & & \uparrow{\scriptstyle h_1^*} \\ H^*(\mathscr{X} \wedge MU(k)/\emptyset) & \xrightarrow{g^*} & H^*(\mathscr{X} \wedge MU(k)) \end{array}$$

with $h^*_1(\mu_i \otimes \alpha_i) \neq 0$, $g^*(\mu_i \otimes \alpha_i) = \mu_i \otimes \alpha_i$. But $f^* \circ g^* = 1$, so g^* is injective and the result follows.

REMARK. All the results remain true if $MU(k)$ is replaced by $SMU(k)$, and U_k is replaced by ΣU_k. All results are then true through dimension $s + 2k + 1$.

We shall apply 2.4. to the Thom spectrum TX, so we assume for the remainder of this section, that TX satisfies the assumption of Lemma 2.2. (For example MU (all p) and MSO (odd primes) with $s = \infty$). We assume TX is oriented for $p \neq 2$.

Let Y be a space, and $\Phi^u \colon H^*(X \times Y) \to H^*(TX \wedge Y/\emptyset)$ the Thom isomorphism of the bundle $\gamma X \times Y \to X \times Y$ (i.e. Φ^u is cupping with $U \otimes 1 \in H^*(TX \wedge Y/\emptyset)$. We then have the composite $\lambda = h^* \circ \Phi^u$,

$$\lambda \colon H^*(X \times Y) \to \tilde{H}^*(TX \wedge Y/\emptyset) \to \mathrm{Hom}(\pi_*(TX \wedge Y/\emptyset), \mathsf{Z}_p)\,.$$

LEMMA 2.5. *Suppose $\theta \in H^q(X)$ vanishes on all n-dimensional X-manifolds. If $\alpha \in H^{n-q}(Y)$, then*

$$\lambda(\theta \otimes \alpha) = 0\,.$$

PROOF. One may proceed by remarking that λ is the evaluation of characteristic numbers Stong [9]. For completeness, we indicate another proof.

Let $\theta_i \otimes \alpha = \theta \otimes \alpha|_{X_i}$. Then we have the fibration

$$E_{\theta_i \otimes \alpha} \xrightarrow{F_i} X_i \times Y \xrightarrow{\theta_i} K(n, \mathsf{Z}_p)\,.$$

There are maps $\zeta_i \colon E_{\theta_i} \to E_{\theta_{i+1} \otimes \alpha}$. Hence $\{E_{\theta_i \otimes \alpha}\} = E_{\theta \otimes \alpha}$ form a structure on manifolds. Clearly $M^n \xrightarrow{g} X \times Y$ lifts to $E_{\theta \otimes \alpha}$ if and only if $g^*(\theta \otimes \alpha) = 0$ (which is the case if θ vanishes on all n-manifolds). We then have the diagram:

$$\begin{array}{ccccc}
\pi_*(TE_{\theta \otimes \alpha}) & \xrightarrow{F} & \pi_*(TX \wedge Y/\emptyset) & \xrightarrow{\quad\psi\quad} & \\
\downarrow & & \downarrow & & \\
\tilde{H}_*(TE_{\theta \otimes \alpha}) & & \tilde{H}_*(TX \wedge Y/\emptyset) & & \\
\downarrow & & \downarrow & & \downarrow \\
H_*(E_{\theta \otimes \alpha}) & \longrightarrow & H_*(X \times Y) & \xrightarrow{(\theta \otimes \alpha)_*} & H_*(K(n; \mathsf{Z}_p))
\end{array}$$

By assumption on θ, F (which is the forgetful functor) is onto for $* \leqq n$. Also, the bottom composite is zero (because

$$E_{\theta\otimes\alpha} \to X \times Y \to K(n, Z_p)$$

is a fibration). Therefore $0 = \psi^{\text{dual}}(\iota) = \lambda(\theta\otimes\alpha)$.

We now define

$$\begin{aligned} Y_k &= MU(\tfrac{1}{2}k) && \text{if } k \text{ is even} \\ &= SMU\big(\tfrac{1}{2}(k-1)\big) && \text{if } k \text{ is odd}. \end{aligned}$$

Let $U_k \in \tilde{H}^k(Y_k)$ be the Thom class if k is even or Σ(Thom class) if k is odd. Then Lemma 2.1. implies

$$\{\mathscr{P}^I(U_k) \mid \text{exc}(I) \leqq \tfrac{1}{2}k\}$$

is a linearly independent set in $\tilde{H}^*(Y_k)$.

In Brown and Peterson [1], a right operation of $\mathfrak{a}_p$ on $H^*(X)$ is defined. Namely, if $\alpha \in \mathfrak{a}_p$ and $U \in H^*(TX)$ then $x\alpha$ is the unique element such that $(x\alpha) \cup U = c(\alpha)(x \cup U)$ where c is the canonical antiisomorphism of the Steenrod algebra. Since $c(Q_0) = -Q_0$ it follows that $H^*(X)$ is a free right-module over $\mathfrak{a}_p/(Q_0)$ if $H^*(TX)$ is a free left-module. (If u_i are right $\mathfrak{a}_p/(Q_0)$ generators for $H^*(X)$ then $u_i \cup U$ are left $\mathfrak{a}_p/(Q_0)$ generators of $H^*(TX)$.)

The following is essentially proven in [1] in a different form.

LEMMA 2.6.

$$\lambda(\theta\alpha\otimes\gamma) = \lambda(\theta\otimes\alpha\gamma) \quad \textit{where} \quad \theta \in H^*(X),\ \gamma \in H^*(Y_k),\ \alpha \in \mathfrak{a}_p/(Q_0).$$

PROOF. For $f \in \pi_*(TX \wedge Y_k/\emptyset)$ we have

$$\begin{aligned} \lambda(\theta\alpha\otimes\gamma)[f] &= f^*[(\theta\alpha\otimes\gamma) \cup (U_k\otimes 1)] \\ &= \pm f^*[(\theta\alpha \cup U_k)\otimes\gamma] = \pm f^*[c(\alpha)(\theta \cup U_k)\otimes\gamma] \\ &= \pm f^*[(\theta \cup U_k)\otimes\alpha\gamma] = \pm f^*[(\theta\otimes\alpha\gamma) \cup (U_k\otimes 1)] \\ &= \pm\lambda(\theta\otimes\alpha\gamma), \end{aligned}$$

where we have used Lemma 6.2 in [1].

THEOREM 2.7. *Let $H^*(TX)$ be as in Lemma 2.2, then through dimension s, all relations among X-characteristic classes are given by*

$$\textstyle\sum_{2pj>n-i} H^i(X)\mathscr{P}^j.$$

PROOF. If $\theta^q \in H^q(X)$ is a universal relation for n-dimensional X-manifolds, then with $k=n-q$, $\lambda(\theta \cup U_k)=0$ by Lemma 2.5. Let $\theta=\sum \nu_i \alpha_i$ $\alpha_i \in \mathcal{A}_p/(Q_0)$, $\{\nu_i\}$ a right $\mathcal{A}_p/(Q_0)$ basis for $H^*(X)$, then by Lemma 2.6

$$0 = \lambda(\sum \nu_i \alpha_i \otimes U_k) = \lambda(\sum \nu_i \otimes \alpha_i U_k) .$$

But, as long as $\dim(\nu_i \otimes \alpha_i U_k)=n \leqq s+(n-q)$, this can only happen if $\alpha_i=\sum_j \mathscr{P}^{I_j}$, and $\operatorname{exc}(I_j) > \frac{1}{2}(n-q)$ by Corollary 2.4.

We have now reduced the theorem to [2, Lemma 4.2], and the result follows for $q \leqq s$.

Now consider the cobordism of all oriented manifolds which immerse in codimension k Euclidean space (Wells [11]). The Thom space of this structure is given by $TX_r^{(k)}=S^{r-k}MSO(k)$. For $p \neq 2$, $H^*(TX^{(k)})$ is a free module over $\mathcal{A}_p/(Q_0)$ through dimension $\leqq k$, and $H^*(TX^{(k)}; \mathrm{Z})$ has no p-torsion. In a similar way, one can form $TX_r^{(k^c)}=S^{2r-2k}MU(k)$. This is the cobordism of weakly, almost complex manifolds which immerse in codimension $2k$, with a complex normal bundle. Then $H^*(TX^{(k^c)})$ is a free $\mathcal{A}_p/(Q_0)$ module through dimension $2k$. Hence we have

THEOREM 2.8. *Among oriented (complex) manifolds which immerse in codimension k Euclidean space (with a complex structure on the normal bundle) there are no new relations among the normal characteristic classes in dimension k, for* mod p *cohomology. (p odd for SO-manifolds; all p for U-manifolds).*

PROOF. The relations given in 2.7. agree with the universal relations given in [2].

3. Relations among Stiefel-Whitney classes.

In order to obtain results similar to Theorem 2.8 for Stiefel–Whitney classes of SO- and O-manifolds, we must study spectra $\mathscr{X}$ such that for $* \leqq s$, $H^*(\mathscr{X})$ is of finite type and

CONDITION 3.1. $H^*(\mathscr{X})$ is a module over $\mathcal{A}_p$ on generators $\{\mu_i\}$ and $\{\beta_i\}$ subject only to the relation $Q_0\beta_i=0$ where β_i are integral classes.

MSO and *MO* are examples of such spectra for $p=2$ [10].

Since we are interested in the prime 2, all cohomology in this section shall be assumed to have Z_2-coefficients, and $\mathcal{A}$ will denote the algebra of Steenrod squares. We remark that all results have mod p analogous for odd p.

DEFINITION 3.2. Let R_q be the Z_2-vector space contained in $H^*(K(Z_2,q))$ spanned by the image of Sq^1 or some higher order Bockstein, δ_m. The generators of this vector space were characterized by Brown and Peterson [1].

Let $\{\alpha_i\} \in H^*(K(Z_2,q))$ be the additive basis consisting of monomials in the $Sq^I(\iota)$'s such that $\mathrm{exc}(I) < q$.

LEMMA 3.3. *$\tilde{H}^*(\mathscr{X} \wedge K(Z_2,q))$ is a module over $\mathcal{a}$ generated by $\{\mu_i \otimes \alpha_j\} \cup \{\beta_i \otimes \alpha_j\}$. There are maps $f_{ij}: \mathscr{X} \wedge K(Z_2,q) \to S^{n_{ij}}K(Z_2)$ realizing the classes $\mu_i \otimes \alpha_j$ and maps*

$$g_{ij}: \mathscr{X} \wedge K(Z_2,q) \to S^{m_{ij}}K(Z_{2^{r_{ij}}})$$

realizing the classes $\beta_i \otimes \alpha_j$ for $\alpha_j \notin R_q$.

$T = \prod f_{ij} \times \prod g_{ij}$ induces an isomorphism on cohomology through dimension $s+q$.

PROOF. That $\{\mu_i \otimes \alpha_j\} \cup \{\beta_i \otimes \alpha_j\}$ generate, is similar to 2.2. Let f_{ij} be the maps realizing the $\mu_i \otimes \alpha_j$'s. Then 3.1. implies that $F = \prod f_{ij}$ induces an injection in cohomology onto the free $\mathcal{a}$-module generated by $\mu_i \otimes \alpha_j$.

Let h_{ij} be the maps of

$$\mathscr{X} \wedge K(Z_2,q) \to S^{m(ij)}K(Z_2)$$

realizing $\beta_i \otimes \alpha_j$, $\alpha_j \notin R_q$. Then

$$Sq^1(\beta_i \otimes \alpha_j) = \beta_i \otimes Sq^1 \alpha_j, \quad Sq^1 \alpha_j \in R_q .$$

If $Sq^1 \alpha_j = 0$, then one may lift to a map

$$g_{ij}: \mathscr{X} \wedge K(Z_2,q) \to S^{m(ij)}K(Z_{2^r}) ,$$

where r is determined by

$$\delta_l \alpha_j = 0, \quad l < r, \quad \delta_r \alpha_j \neq 0 .$$

(If $\delta_l \alpha_j = 0$ for all l, then we have a lift to $S^{m(ij)}(K(Z))$. But $\tilde{H}^*(K(Z_2,q): Z)$ has only torsion elements, and we infer that $\alpha_j \in R_q$). Since $\delta_r \alpha_j \in R_q$, we have: The map $G = \prod g_{ij}$ induces an injection in cohomology onto the $\mathcal{a}$-module spanned by $\{\beta_i \otimes \alpha_j\}$ for all i,j. Therefore $T = F \times G$ induces an isomorphism in cohomology through dimension $s+q$, which proves the lemma.

One may then proceed as in section 2.

LEMMA 3.5.

$$\lambda: H^n(X \times K(Z_2, n-q)) \to \mathrm{Hom}\big(\pi_n(TX \wedge K(Z_2, n-q)/\emptyset), Z_2\big)$$

satisfies $\lambda(\theta \otimes \iota^{n-q}) = 0$ if θ^q vanishes on all n-manifolds.

THEOREM 3.6. *Through dimension s, all relations among X-manifolds are given by*:

$$\sum_{2j>n-i} H^i(X)\,\mathrm{Sq}^j + \beta^j a^{q-j}\,,$$

where $a^{n-q}(\iota^{n-q}) \in R_{n-q}$.

Since these relations coincide with the relations for all SO- or O-manifolds we obtain:

THEOREM 3.7. *Among oriented (unoriented) manifolds which immerse in codimension k Euclidean space, there are no new relations among Stiefel–Whitney classes in dimension* $\leqq k$.

It should be remarked that our results are interesting only in the metastable range (that is $k > \frac{1}{2}n$). Because one can show that there are never any relations among mod q-characteristic classes of n-dimensional X-manifolds below the middle dimension for any X.

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