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FENCHEL TYPE DUALITY THEOREMS IN FINITE DIMENSIONAL ORDERED VECTOR SPACES

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Summary.

In this paper we consider the minimization problem $\inf(f(x)-g(x))$, where f and $-g$ are generalized convex functions mapping a convex set of R^n into R^m, R^m an ordered vector space. We introduce conjugate R^m-valued functions f^c and g^c, defined on a set of $n \times m$-matrices Y, and associate with the minimization problem a dual maximization problem in R^m: $\sup(g^c(Y)-f^c(Y))$. For $\mathsf{R}^m = \mathsf{R}$ these two programs were considered by Fenchel. It is shown that under suitable assumptions the main results of Fenchel's duality theorem carry over to this more general case.

It should be noted, that our definition of conjugate functions differs from the one given by W. W. Breckner and I. Kolumbán [2], where the conjugates of f and g are real-valued functions.

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1. C-convex functions and their conjugates.

Let R^m be a (partially) ordered vector space and $C = \{x \in \mathsf{R}^m \mid x \geqq 0\}$ its positive convex cone. To denote this situation we write $\mathsf{R}^m{}_C$ for R^m. We assume that the *order cone* C is closed and its interior is nonvoid, that is,

$$(1.1) \qquad C = \bar{C} \quad \text{and} \quad C^\circ \neq \emptyset .$$

Now $C = \bar{C}$ implies $C = \bar{C} = C^{**}$, where

$$C^* = \{y \in \mathsf{R}^m \mid y^\top x \geqslant 0 \text{ for all } x \in C\}$$

is the dual cone of C (elements of R^m we always consider as column matrices). Since $C \cap -C = \{0\}$ for the order defining cone C, we get furthermore $(C^*)^\circ \neq \emptyset$ from $C = \bar{C}$, which will be used in the future. $C^\circ \neq \emptyset$ gives $C^* \cap -C^* = \{0\}$ and thus C^* defines a vector space order on R^m as well by $x \leqq y$ if $y - x \in C^*$. We write

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$$x \leqq y \quad if \quad y-x \in C\,, \qquad x \leqq^* y \quad if \quad y-x \in C^*\,,$$

and

$$x < y \quad if \quad y-x \in C^\circ, \qquad x <^* y \quad if \quad y-x \in (C^*)^\circ .$$

For example the *conical hull*

$$C = \{\textstyle\sum_{i=1}^m \lambda_i c_i \mid \lambda_i \in \mathsf{R}, \lambda_i \geqslant 0\}$$

of m linearly independent elements $c_1,\ldots,c_m$ of R^m is an order cone with $C=\bar{C}$ and $C^\circ \neq \emptyset$. It is known that $\mathsf{R}^m{}_C$ is an archimedean vector lattice iff C is the conical hull of m linear independent elements (cf. [1]). In this case $\mathsf{R}^m{}_C$ is even order complete.

Let $\emptyset \neq K \subseteqq \mathsf{R}^n$ be a convex set; we call a function f: $K \to \mathsf{R}^m{}_C$ *C-convex* if

$$f(\lambda x+(1-\lambda)y) \leqq \lambda f(x)+(1-\lambda)f(y) \tag{1.2}$$

for all $x,y \in K$ and $\lambda \in \mathsf{R}$, $0 \leqslant \lambda \leqslant 1$ (cf. [3]). The domain of definition K of f will also be denoted by $K(f)$. A function g is called *C-concave* if $-g$ is C-convex. For $m=1$, that is, $\mathsf{R}^m=\mathsf{R}$ and $C=\mathsf{R}^+ \cup \{0\}$, condition (1.2) is just the standard definition of convex functions. In this case we call f simply a convex function and g a concave function respectively. For the remainder of this paper we assume f to be C-convex and g to be C-concave. For f we define a family $\{f_v \mid v \geqq {}^*0\}$ of realvalued functions f_v with domain $K(f_v) := K(f)$ by

$$f_v(x) := v^\top f(x) \quad \text{for } x \in K(f_v)\,. \tag{1.3}$$

For all $x,y \in K(f)$ and $0 \leqslant \lambda \leqslant 1$ the relation

$$\lambda f(x)+(1-\lambda)f(y)-f(\lambda x+(1-\lambda)y) \in C$$

implies

$$v^\top[\lambda f(x)+(1-\lambda)f(y)-f(\lambda x+(1-\lambda)y)] \geqslant 0$$

for all $v \geqq {}^*0$ and thus

$$f_v(\lambda x+(1-\lambda)y) \leqslant \lambda f_v(x)+(1-\lambda)f_v(y)\,.$$

Hence,

(1.4) *if f is C-convex, then the functions f_v: $K(f_v) \to \mathsf{R}$ are convex.*

An analogous result holds for the C-concave function g and the functions $g_v(x)=v^\top g(x)$, $v \geqq {}^*0$. In the following we will often state results for C-convex functions only as the corresponding statements for C-concave functions are obvious.

Let M be the set of $n \times m$-matrices Y, for which

$$\sup\{\, Y^\top x - f(x) \mid x \in K(f)\,\}$$

exists in $\mathsf{R}^m{}_C$. We define

$$(1.5) \qquad f^c(Y) = \sup\{Y^\top x - f(x) \mid x \in K(f)\} \quad \text{for } Y \in K(f^c) := M$$

and call f^c the *conjugate function* of the C-convex function f. Analogously one defines $K(g^c)$ and g^c, the conjugate function of the C-concave function g

$$(1.6) \qquad g^c(Y) = \inf\{Y^\top x - g(x) \mid x \in K(g)\} \quad \text{for } Y \in K(g^c)\,.$$

For $m=1$, that is, $\mathsf{R}^m = \mathsf{R}$, this is just the definition of conjugate functions given by Fenchel. In this special case Y is an n-vector and we write y for Y.

Let $K(f)^i$ denote the *relative interior* of $K(f)$, that is, the interior of $K(f)$ when $K(f)$ is regarded as a subset of its affine hull. Since $K(f)^i \neq \emptyset$ for $K(f) \neq \emptyset$, we know that $K(f^c) \neq \emptyset$ due to the following

(1.7) THEOREM. *If $x_0 \in K(f)^i$, then there is a Y_0 such that*

$$f^c(Y_0) = Y_0{}^\top x_0 - f(x_0)\,.$$

PROOF. See [6].

Thus definition (1.5) makes sense. In general $K(f^c)$ is not a convex subset of the space of $n \times m$-matrices and therefore $f^c\colon K(f^c) \to \mathsf{R}^m{}_C$ is not a C-convex function (unless for example $\mathsf{R}^m{}_C$ is an archimedean vector lattice).

2. A pair of dual programs and two duality theorems.

Given a C-convex function $f\colon K(f) \to \mathsf{R}^m{}_C, K(f) \subseteq \mathsf{R}^n$, a C-concave function $g\colon K(g) \to \mathsf{R}^m{}_C, K(g) \subseteq \mathsf{R}^n$, and their conjugates f^c and g^c, we consider the two programs:

P1 *Find* $\inf\{f(x) - g(x) \mid x \in K(f) \cap K(g)\}$,
P2 *Find* $\sup\{g^c(Y) - f^c(Y) \mid Y \in K(f^c) \cap K(g^c)\}$.

A point $x_0 \in K(f) \cap K(g)$ is called an *optimal solution* of P1, if

$$f(x_0) - g(x_0) = \inf\{f(x) - g(x) \mid x \in K(f) \cap K(g)\}\,.$$

Optimal solutions of P2 are defined analogously.

For $m=1$ the above problems were considered by Fenchel [4], who proved the following duality theorem (see also [5, p. 179]):

(2.1) THEOREM. (a) *If* $K(f)^i \cap K(g)^i \neq \emptyset$ *and*

$$\mu = \inf\{f(x)-g(x) \mid x \in K(f)\cap K(g)\}$$

exists, then P2 *has an optimal solution* y_0, *and*

$$\mu = g^c(y_0)-f^c(y_0) = \max\{g^c(y)-f^c(y) \mid y \in K(f^c)\cap K(g^c)\}\,.$$

(b) *If* f *and* g *are closed,* $K(f^c)^i \cap K(g^c)^i \neq \emptyset$ *and*

$$\mu = \sup\{g^c(y)-f^c(y) \mid y \in K(f^c)\cap K(g^c)\}$$

exists, then P1 *has an optimal solution* x_0, *and*

$$\mu = f(x_0)-g(x_0) = \min\{f(x)-g(x) \mid x \in K(f)\cap K(g)\}\,.$$

NOTE: Here a convex function $f\colon K(f)\to \mathsf{R}$, $K(f)\subseteqq \mathsf{R}^n$, is called *closed* if

$$\tilde f(x) = \lim_{\varepsilon\downarrow 0} \inf\{\tilde f(y) \mid \|y-x\|<\varepsilon\} \quad \text{for all} \quad x \in \mathsf{R}^n,$$

where $\tilde f(x)=f(x)$ for $x \in K(f)$ and $\tilde f(x)=+\infty$ otherwise.

In the following we will try to generalize Fenchel's Theorem to the case, that the dimension m of the image space of f and g is greater than 1.

As a first relation between P1 and P2 we get

(2.2) LEMMA. *If* $x \in K(f)\cap K(g)$ *and* $Y \in K(f^c)\cap K(g^c)$, *then*

$$g^c(Y)-f^c(Y) \leqq f(x)-g(x)\,.$$

PROOF. For $Y \in K(f^c)\cap K(g^c)$ definitions (1.5) and (1.6) say that

$$f^c(Y) \geqq Y^\top x - f(x) \quad \text{for all } x \in K(f)$$

and

$$g^c(Y) \leqq Y^\top x - g(x) \quad \text{for all } x \in K(g)\,.$$

Thus we have $g^c(Y)-f^c(Y) \leqq f(x)-g(x)$ for $x \in K(f)\cap K(g)$.

For the following two propositions we assume that

$$\begin{aligned} & K(f)^i \cap K(g)^i \neq \emptyset\,, \\ (2.3) \qquad & \mu = \inf\{S\} \text{ exists and } \mu \in \bar S\,, \quad \text{where} \\ & S := \{f(x)-g(x) \mid x \in K(f)\cap K(g)\}\,. \end{aligned}$$

Programs P1 and P2 are closely connected with the families $\{\mathrm{P1}_v\}$, $\{\mathrm{P2}_v\}$, $v \geqq^* 0$, of one-dimensional Fenchel-problems:

$\mathrm{P1}_v$ *Find* $\inf\{f_v(x)-g_v(x) \mid x \in K(f_v)\cap K(g_v)\}$,

$\mathrm{P2}_v$ *Find* $\sup\{g_v{}^c(y)-f_v{}^c(y) \mid y \in K(f_v{}^c)\cap K(g_v{}^c)\}$.

Here f_v^c and g_v^c denote the conjugates of the realvalued functions f_v and g_v. Since $\mu \in \bar{S}$ we have

$$(2.4) \qquad v^\top \mu = \inf\{f_v(x) - g_v(x) \mid x \in K(f_v) \cap K(g_v)\}$$

for all $v \geqq^* 0$ and because of (1.4) Fenchel's Theorem yields

$$(2.5) \qquad v^\top \mu = \max\{g_v^c(y) - f_v^c(y) \mid y \in K(f_v^c) \cap K(g_v^c)\}.$$

Therefore, under assumption (2.3),

(2.6) *all sets* $M_v := \{y \mid g_v^c(y) - f_v^c(y) = v^\top \mu\}, v \geqq^* 0$, *are non-empty.*

Let $v_i \geqq^* 0$ for $i = 1, 2, \ldots, k$. If $y_i \in M_{v_i}$ and $v_0 = \sum_{i=1}^k \lambda_i v_i, \lambda_i \geqslant 0$, then we have for $y_0 = \sum_{i=1}^k \lambda_i y_i$

$$\begin{aligned} \inf\{y_0^\top x - g_{v_0}(x) \mid x \in K(g_{v_0})\} &\geqslant \textstyle\sum_{i=1}^k \lambda_i \inf\{y_i^\top x - g_{v_i}(x) \mid x \in K(g_{v_i})\} \\ &= \textstyle\sum_{i=1}^k \lambda_i g_{v_i}^c(y_i). \end{aligned}$$

Hence $\inf\{y_0^\top x - g_{v_0}(x) \mid x \in K(g_{v_0})\}$ exists, that is, $y_0 \in K(g_{v_0}^c)$, and

$$g_{v_0}^c(y_0) \geqslant \textstyle\sum_{i=1}^k \lambda_i g_{v_i}^c(y_i).$$

Analogously we get $y_0 \in K(f_{v_0}^c)$ and $-f_{v_0}^c(y_0) \geqslant -\sum_{i=1}^k \lambda_i f_{v_i}^c(y_i)$ and thus

$$\begin{aligned} g_{v_0}^c(y_0) - f_{v_0}^c(y_0) &\geqslant \textstyle\sum_{i=1}^k \lambda_i [g_{v_i}^c(y_i) - f_{v_i}^c(y_i)] \\ &= \textstyle\sum_{i=1}^k \lambda_i v_i^\top \mu = v_0^\top \mu. \end{aligned}$$

Because of (2.5) equality holds and we obtain:

PROPOSITION 1. *Let* $v_i \geqq^* 0$ *and* $\lambda_i \geqslant 0$ *for* $i = 1, \ldots, k$. *If* $y_i \in M_{v_i}$ *for* $i = 1, \ldots, k$ *then* $y_0 \in M_{v_0}$ *for* $y_0 := \sum_{i=1}^k \lambda_i y_i$, $v_0 := \sum_{i=1}^k \lambda_i v_i$.
Furthermore

$$f_{v_0}^c(y_0) = \textstyle\sum_{i=1}^k \lambda_i f_{v_i}^c(y_i) \quad \textit{and} \quad g_{v_0}^c(y_0) = \textstyle\sum_{i=1}^k \lambda_i g_{v_i}^c(y_i).$$

The first part of Proportion 1 basically says that

$$(2.7) \qquad M_{\Sigma \lambda_i v_i} \supseteqq \textstyle\sum \lambda_i M_{v_i} \text{ for all } v_i \geqq^* 0 \text{ and } \lambda_i \geqslant 0, i = 1, \ldots, k.$$

PROPOSITION 2. *If for every* $v \geqq^* 0$ *we can choose* $y = y(v) \in M_v$ *such that*

$$(2.8) \qquad y\big(\textstyle\sum_{i=1}^k \lambda_i v_i\big) = \textstyle\sum_{i=1}^k \lambda_i y(v_i) \quad \textit{for all} \quad v_i \geqq^* 0, \lambda_i \geqslant 0, k \geqslant 1,$$

then there exists a $n \times m$*-matrix* Y *and* $w \in \mathsf{R}^m$ *with*

$$y(v) = Yv \quad \textit{and} \quad v^\top w = f_v^c(y(v)) + v^\top \mu = g_v^c(y(v))$$

for all $v \geqq^* 0$.

PROOF. We choose m linearly independent vectors $v_1,\dots,v_m$ in C^*, which is possible because of $(C^*)^\circ \neq \emptyset$, and define with $y_i = y(v_i)$ an $n\times m$-matrix $Y := (y_1,\dots,y_m)(v_1,\dots,v_m)^{-1}$ and an m-vector w by

$$w^\top := \left(f_{v_1}{}^c(y_1),\dots,f_{v_m}{}^c(y_m)\right)(v_1,\dots,v_m)^{-1} + \mu^\top.$$

Then $Yv_i = y_i$ and $v_i{}^\top w = w^\top v_i = f_{v_i}{}^c(y_i) + v_i{}^\top \mu$ for $i = 1,\dots,m$. Moreover

$$Yv = \textstyle\sum_{i=1}^m \alpha_i y_i \quad \text{and} \quad v^\top w = \sum_{i=1}^m \alpha_i f_{v_i}{}^c(y_i) + \sum_{i=1}^m \alpha_i v_i{}^\top \mu$$

for all $v = \alpha_1 v_1 + \dots + \alpha_m v_m \in \tilde{C} := \{\sum_{i=1}^m \lambda_i v_i \mid \lambda_i \geqslant 0\} \subseteqq C^*$.

Clearly, (2.8) and Proposition 1 give

$$Yv = y(v) \quad \text{and} \quad v^\top w = f_v{}^c(y(v)) + v^\top \mu \quad \text{for all } v \in \tilde{C}.$$

We show next that this is also true for all $v \in C^*$. Indeed, let $v^1 \in C^*$ but $v^1 \notin \tilde{C}$. We choose $v^2 \in (\tilde{C})^\circ$ and $\lambda > 0$ small enough such that $v_\lambda := \lambda v^1 + (1-\lambda)v^2 \in \tilde{C}$. By (2.8) and by what has been shown so far we obtain

$$\begin{aligned}\lambda Y v^1 + (1-\lambda) Y v^2 &= Y v_\lambda = y(v_\lambda) = \lambda y(v^1) + (1-\lambda) y(v^2)\\ &= \lambda y(v^1) + (1-\lambda) Y v^2\end{aligned}$$

and thus $Yv^1 = y(v^1)$ for all $v^1 \in C^*$. Similarly we get

$$\begin{aligned}\lambda w^\top v^1 + (1-\lambda) w^\top v^2 &= w^\top v_\lambda = f_{v_\lambda}{}^c(y(v_\lambda)) + v_\lambda{}^\top \mu\\ &= \lambda f_{v^1}{}^c(y(v^1)) + (1-\lambda) f_{v^2}{}^c(y(v^2)) + v_\lambda{}^\top \mu\\ &= \lambda f_{v^1}{}^c(y(v^1)) + (1-\lambda) w^\top v^2 + \lambda (v^1)^\top \mu\end{aligned}$$

and thus $(v^1)^\top w = f_{v^1}{}^c(y(v^1)) + (v^1)^\top \mu$ for all $v^1 \geqq^* 0$. Since $y(v) \in M_v$, we have furthermore $v^\top w = g_v{}^c(y(v))$ for $v \geqq^* 0$.

For the following duality theorem it will be important to know that a function $v \to y(v) \in M_v, v \geqq^* 0$, exists which satisfies (2.8). The existence, though, does not follow from Proposition 1. There are examples where assumption (2.3) holds and thus Proposition 1, but where no function $v \to y(v) \in M_v$ exists which satisfies (2.8) (see [6]). However,

(2.9) *if in addition to* (2.3) *either one of the following conditions holds, then the assumption of Proposition* 2 *holds*:

(a) M_{v_0} *consists of one element only for some* $v_0 >^* 0$.

(b) $\mu = f(x_0) - g(x_0)$ *for some* $x_0 \in K(f) \cap K(g)$, *and* f *or* g *is differentiable at* x_0.

(c) $\mathsf{R}^m{}_C$ *is an archimedean vector lattice.*

PROOF. (a): (2.6) shows that $M_v \neq \emptyset$ for all $v \geqq^* 0$. Suppose M_{v_1} contains more than one element. Since $v_0 >^* 0$ we can choose $v_2 \geqq^* 0$ and $0 < \lambda < 1$ such that $v_0 = \lambda v_1 + (1-\lambda)v_2$. Proposition 1 implies $\lambda M_{v_1} + (1-\lambda)M_{v_2} \subseteqq M_{v_0}$, a contradiction to assumption (a). Thus every $M_v, v \geqq^* 0$, contains but one element and equality holds in (2.7). It is easy to see, that (2.8) holds for $v \to y(v) := \{M_v\}$.

(b): See [6].

(c): If $\mathsf{R}^m{}_C$ is an archimedean vector lattice, then C is the conical hull of m linear independent elements. The same is true for C^*, that is,

$$C^* = \{\textstyle\sum_{i=1}^m \lambda_i v_i \mid \lambda_i \geqslant 0\}$$

where $v_1, \ldots, v_m$ are linear independent elements of R^m. We choose $y_i \in M_{v_i}$ for $i = 1, \ldots, m$ and define $y(v) := \sum_{i=1}^m \lambda_i y_i$ for $v = \sum_{i=1}^m \lambda_i v_i \in C^*$. Proposition 1 shows that $y(v) \in M_v$. Furthermore, (2.8) holds.

We are now able to state our first duality theorem:

THEOREM 1. *Assume*

(1) $K(f)^i \cap K(g)^i \neq \emptyset$,

(2) $\mu = \inf\{S\}$ *exists and* $\mu \in \bar{S}$ *with* $S := \{f(x) - g(x) \mid x \in K(f) \cap K(g)\}$,

(3) *for some* $v_0 >^* 0$ *there is only one* y_0 *such that* $g_{v_0}{}^c(y_0) - f_{v_0}{}^c(y_0) = v_0{}^\top \mu$.

Then P2 *has an optimal solution* Y_0 *and*

$$\mu = g^c(Y_0) - f^c(Y_0) = \max\{g^c(Y) - f^c(Y) \mid Y \in K(f^c) \cap K(g^c)\}.$$

PROOF. Assumptions (1) and (2) are exactly condition (2.3). Because of (3), M_{v_0} contains but one element and (2.9) (a) implies that there is a function $v \to y(v) \in M_v, v \geqq^* 0$, which satisfies (2.8). Let Y and w be chosen as in Proposition 2. We want to show

$$Y^\top x - f(x) + \mu \leqq w \leqq Y^\top x' - g(x')$$

for all $x \in K(f)$ and $x' \in K(g)$. If we assume $Y^\top \tilde{x} - f(\tilde{x}) + \mu \nleqq w$ for some $\tilde{x} \in K(f)$, that is, $z := w - (Y^\top \tilde{x} - f(\tilde{x}) + \mu) \notin C$, then the compact convex set $\{z\}$ and the closed convex set C can be strictly separated; hence there exist $0 \neq v \in \mathsf{R}^m$ and $\alpha \in \mathsf{R}$ such that

$$v^\top z < \alpha \leqslant v^\top c \quad \text{for all } c \in C\,.$$

C being a cone we get $v^\top c \geqslant 0$ for all $c \in C$ and consequently $v \geqq^* 0$. Now $0 \in C$ and thus $v^\top z < \alpha \leqslant v^\top 0 = 0$. Hence

$$v^\top w - v^\top \mu < v^\top Y^\top \tilde{x} - v^\top f(\tilde{x})$$

or

$$f_v^c(y(v)) < y(v)^\mathsf{T}\tilde{x} - f_v(\tilde{x})$$

by construction of Y and w. This contradicts

$$f_v^c(y(v)) = \sup\{y(v)^\mathsf{T}x - f_v(x) \mid x \in K(f_v)\}.$$

Hence $w-\mu$ is an upper bound for $\{Y^\mathsf{T}x - f(x) \mid x \in K(f)\}$. Let $w'-\mu$ be any upper bound and $v \geqq^* 0$. Proposition 2 then shows that

$$v^\mathsf{T}(w'-\mu) \geqslant \sup\{v^\mathsf{T}Y^\mathsf{T}x - f_v(x) \mid x \in K(f_v)\} = f_v^c(y(v)) = v^\mathsf{T}(w-\mu)$$

and consequently $v^\mathsf{T}(w'-w) \geqslant 0$ for all $v \in C^*$, that is,

$$w'-w \in C^{**} = \bar{C} = C.$$

Hence $w-\mu$ is the least upper bound and this yields $Y \in K(f^c)$ and $f^c(Y) = w-\mu$.

Analogously one shows $Y \in K(g^c)$ and $g^c(Y) = w$. For $Y_0 = Y$ we get $g^c(Y_0) - f^c(Y_0) = \mu$. Together with Lemma (2.2) this proves the theorem.

REMARK. Assumptions (1) and (2) of Theorem 1 guarantee $M_v \neq \emptyset$ for all $v \geqq^* 0$. Condition (3) is needed to show the existence of a function $v \to y(v) \in M_v$ which satisfies (2.8). Therefore (3) can be exchanged for instance by (2.9) (b) or (c).

Since $\mathbf{R}$ with its natural order is trivially an archimedean vector lattice our theorem generalizes part (a) of Fenchel's duality theorem.

But notice, that Theorem 1 does in general not hold without condition (3) or a similar condition. A counter-example is given in [6].

THEOREM 2. *Assume*

(1) $\mu = \sup\{S\}$ *exists and* $\mu \in \bar{S}, S := \{g^c(Y) - f^c(Y) \mid Y \in K(f^c) \cap K(g^c)\}$.
Furthermore assume that for some $v_0 >^* 0$

(2) f_{v_0} *and* g_{v_0} *be closed,*

(3) $K(f_{v_0}{}^c)^i \cap K(g_{v_0}{}^c)^i \neq \emptyset$,

(4) $g_{v_0}{}^c(y) - f_{v_0}{}^c(y) \leqslant v_0{}^\mathsf{T}\mu$ *for all* $y \in K(f_{v_0}{}^c)) \cap K(g_{v_0}{}^c)$.

Then P1 *has an optimal solution* x_0 *and*

$$\mu = f(x_0) - g(x_0) = \min\{f(x) - g(x) \mid x \in K(f) \cap K(g)\}.$$

PROOF. For $Y \in K(f^c) \cap K(g^c)$ we have

$$\begin{aligned} v_0{}^\mathsf{T}f^c(Y) &= v_0{}^\mathsf{T}\sup\{Y^\mathsf{T}x - f(x) \mid x \in K(f)\} \\ &\geqslant \sup\{v_0{}^\mathsf{T}Y^\mathsf{T}x - f_{v_0}(x) \mid x \in K(f_{v_0})\} \end{aligned}$$

and similarly

$$v_0^\top g^c(Y) \leqslant \inf\{v_0^\top Y^\top x - g_{v_0}(x) \mid x \in K(g_{v_0})\},$$

that is,

$$Yv_0 \in K(f_{v_0}{}^c) \cap K(g_{v_0}{}^c),$$

$$v_0^\top f^c(Y) \geqslant f_{v_0}{}^c(Yv_0) \quad \text{and} \quad v_0^\top g^c(Y) \leqslant g_{v_0}{}^c(Yv_0).$$

Together with $\mu \in \bar{S}$ this gives

$$\begin{aligned} v_0^\top \mu &= \sup\{v_0^\top g^c(Y) - v_0^\top f^c(Y) \mid Y \in K(f^c) \cap K(g^c)\} \\ &\leqslant \sup\{g_{v_0}{}^c(Yv_0) - f_{v_0}{}^c(Yv_0) \mid Y \in K(f^c) \cap K(g^c)\} \end{aligned}$$

and (4) shows

$$v_0^\top \mu = \sup\{g_{v_0}{}^c(y) - f_{v_0}{}^c(y) \mid y \in K(f_{v_0}{}^c) \cap K(g_{v_0}{}^c)\}.$$

Because of (2) and (3) part (b) of Fenchel's duality theorem yields the existence of an $x_0 \in K(f_{v_0}) \cap K(g_{v_0}) = K(f) \cap K(g)$ such that $v_0^\top \mu = f_{v_0}(x_0) - g_{v_0}(x_0)$ or

$$v_0^\top (f(x_0) - g(x_0) - \mu) = 0.$$

Lemma (2.2) shows that $f(x_0) - g(x_0) - \mu \in C$ and since $C = \bar{C} = C^{**}$, we have for all $v \in C^*$

$$v^\top (f(x_0) - g(x_0) - \mu) \geqslant 0.$$

If we interpret $f(x_0) - g(x_0) - \mu$ as a linear form $l(u)$ on R^m by $l(u) := u^\top (f(x_0) - g(x_0) - \mu)$, then l is positive on C^* and vanishes for a $v_0 \in (C^*)^\circ$. By continuity we get $l = 0$ or $f(x_0) - g(x_0) = \mu$. Together with (2.2) this proves Theorem 2.

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