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SETS OF DIVERGENCE OF FOURIER SERIES

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1. Introduction.

In 1965 Katznelson and Kahane [7] and [8], showed that to any set E of Lebesgue measure zero there is a continuous complex-valued function with Fourier series diverging in E.

A proof of this is also given in [9, ch. II. 3]. Later such real-valued functions were constructed [2]. From Carleson [3] follows that the Fourier series of a continuous function can diverge in a set of Lebesgue measure zero at most.

The purpose of this paper is to construct continuous 2π-periodic functions having *bounded* Fourier series diverging in some "large" set. Erdös, Herzog, and Piranian [4] have shown that to any set E of logarithmic measure zero, i.e. Hausdorff measure zero with regard to the function $h(t) = |\log t|^{-1}$, there exists a continuous function having bounded Fourier series diverging in E. Here we show that to any α less than one there is a set E of positive Hausdorff measure with regard to $h(t) = t^{\alpha}$ and a continuous function with bounded Fourier series diverging in E.

2. Notation and definitions.

T denotes the circle $\mathsf{R}/2\pi\mathsf{Z}$ (where R is the additive group of real numbers, Z the subgroup of integers). $C(\mathsf{T})$ is the Banach space of complex-valued continuous functions on T with the norm

$$\|f\| = \max_{x \in \mathsf{T}} |f(x)|$$

$S_n(f)$ is the n'th partial sum of the Fourier series of f and $S_n(f,x)$ is the value at the point $x \in \mathsf{T}$. The Fourier coefficients of f are denoted $\hat{f}(j)$. Define

$$S^*(f,x) = \sup_n |S_n(f,x)|$$

and

$$S^{**}(f,x) = \sup_{n \leq m} |\textstyle\sum_n^m \hat{f}(j) e^{ijx}|$$

The set $E \subset \mathsf{T}$ is a *set with divergence for* $C(\mathsf{T})$ if and only if there exists an $f \in C(\mathsf{T})$ such that the Fourier series of f diverges at every point of E.

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The Fourier series of f is *bounded* if $S^*(f,x)$ is a bounded function on T. We now introduce a new concept:

The Fourier series of f is *strongly bounded* if $S^{**}(f,x)$ is a bounded function on T.

The set E is a *set with (strongly) bounded divergence for* $C(\mathsf{T})$ if and only if there exists an $f \in C(\mathsf{T})$ such that the Fourier series of f is (strongly) bounded and divergent in E.

In Bari [1, ch. IV], Zygmund [14, ch. VIII], Tandori [13], Erdös, Herzog, and Piranian [4], and Šladkowska [11, 12] are given examples of sets with bounded divergence. Examination of these examples shows that they in fact are sets with strongly bounded divergence for $C(\mathsf{T})$.

Leth h be a continuous increasing function in $[0,\infty]$ such that $h(0)=0$. The set E has Hausdorff h-measure zero if it can be covered by a countable set of intervals I_j, of length $|I_j|$, such that $\sum h(|I_j|)$ is arbitrarily small. If this is not possible the set E is said to have positive Hausdorff measure with respect to the function h. The set E has Hausdorff dimension α if the Hausdorff measure of E is zero with respect to any function $h(t)=t^\beta$ with $\beta>\alpha$ and positive for any β less than α.

In this paper is studied the problem of strongly bounded divergence for $C(\mathsf{T})$. Theorem 2 shows that there exist sets of Hausdorff dimension arbitrarily near 1 with strongly bounded divergence for $C(\mathsf{T})$. Theorem 3 gives a necessary condition on a set E with strongly bounded divergence for $C(\mathsf{T})$. This condition is also sufficient by Theorem 1. Compare also proposition 2 in Katznelson [8]. As a corollary to Theorem 3 we find a necessary condition for bounded divergence without requiring strongly bounded divergence.

A survey of the question of sets admitting divergence for $C(\mathsf{T})$ and other spaces is also given in Katznelson [9, ch. II. 3].

3. Some lemmas.

LEMMA 1. *Let the set E be a union of a finite number of intervals on T of length 2ε and with midpoints (x_ν). If there exists a $K>0$ such that*

$$\sup_\nu \left(\sum_{\mu\neq\nu} |x_\nu - x_\mu|^{-1}\right) < C(K\varepsilon)^{-1}$$

for an absolute constant C then there exists a trigonometric polynomial P and positive constants A_1, B_1, A_2, and B_1 such that

(a) $\|P\| < 1$,
(b) $S^{**}(P,x) < A_1 \log K + B_1$ for all $x \in \mathsf{T}$,
(c) $S^{**}(P,x) > A_2 \log K - B_2$ for all $x \in E$.

PROOF. Let

$$\varphi(x) = \sum_{k=l}^{m} \frac{\sin kx}{k} = \sum_{-m}^{-l} \frac{e^{ikx}}{2ik} + \sum_{l}^{m} \frac{e^{ikx}}{2ik}.$$

In Mitrinović [10, pp. 248, 250] it is shown that

(1) $$\|\varphi\| < A < \pi+2\,,$$

(2) $$|\varphi(x)| < A/l|x| \quad \text{for } 0 < |x| \leqq \pi\,,$$

(3) $$S^{**}(\varphi, x) < 3A/l|x| \quad \text{for } 0 < |x| \leqq \pi\,.$$

A trivial estimate shows that

(4) $$S^{**}(\varphi, x) < C_1 \log ml^{-1} \quad \text{for all } x \in \mathsf{T}\,.$$

and

(5) $$S^{**}(\varphi, x) > \frac{\cos 1}{2} \sum_{l}^{m} k^{-1} > C_2 \log ml^{-1} \quad \text{for } |x| < m^{-1}.$$

Let $l = [(K\varepsilon)^{-1}]$ and $m = [\varepsilon^{-1}]$. Put

$$Q(x) = \sum_{\nu} \varphi(x - x_\nu)\,.$$

Let $x \in \mathsf{T}$ and let x_ν be the nearest midpoint of one of the intervals in E. From the conditions on E it follows that

(6) $$\sum_{\mu,\,\mu\neq\nu} |x - x_\mu|^{-1} < C \sup_\lambda \sum_{\mu,\,\mu\neq\lambda} |x_\lambda - x_\mu|^{-1} < C^2 (K\varepsilon)^{-1}$$

(1), (2), and (6) together give

$$|Q(x)| \leqq |\varphi(x - x_\nu)| + \sum_{\mu,\,\mu\neq\nu} |\varphi(x - x_\mu)| < A + AC^2/K\varepsilon l < C_4$$

In a similar way (3), (4), and (6) together give

$$S^{**}(Q, x) \leqq S^{**}(\varphi, x - x_\nu) + \sum_{\mu,\,\mu\neq\nu} S^{**}(\varphi, x - x_\mu) < C_1 \log K + C_6\,.$$

If $|x - x_\nu| < \varepsilon \leqq m^{-1}$ we have by (5) and (6)

$$S^{**}(Q, x) \geqq S^{**}(\varphi, x - x_\nu) - \sum_{\mu,\,\mu\neq\nu} S^{**}(\varphi, x - x_\mu) > C_2 \log K - C_7\,.$$

Letting $P(x) = C_4^{-1} Q(x)$ we note that the polynomial P satisfies conditions (a), (b), and (c) where the constants are nonnegative and independent of K and ε.

LEMMA 2. *Given* α, $0 < \alpha < 1$, *there exists a set* $E \subset \mathsf{T}$ *satisfying*

(a) *E has positive Hausdorff measure with respect to the function* $t^\alpha, 0 \leqq t$.

(b) *There exist two sequences* $(\varepsilon_n)_1^\infty$ *and* $(K_n)_1^\infty$ *such that* $\lim_{n\to\infty}\varepsilon_n = 0$ *and* $\limsup_{n\to\infty} K_n = \infty$ *and for every n the set E can be covered with a finite number of intervals of length* $2\varepsilon_n$ *with midpoints* x_k^n *such that*

$$\max_k \sum_{l,\, l\neq k} |x_k^n - x_l^n|^{-1} < C(K_n\varepsilon_n)^{-1}$$

where C is indepenent of n.

REMARK. The set E thus has "property L", which is a concept introduced in the theory of Kronecker and Dirichtlet sets (see Kahane [6, p. 90]). The lemma is proved by modifying a construction described in [6, p. 94].

PROOF. Let $\alpha < \beta < 1$. It is possible to construct a function h such that

(1) $0 < h(t) \leqq t^\alpha$,
(2) $h(t)t^\beta$ is decreasing in $]0, \pi]$.
(3) h is constant in some intervals $[\alpha_i, \beta_i]$, where $\alpha_i > 0, \beta_i > 0$,

$\lim_{i\to\infty}\alpha_i = 0$, $\lim_{i\to\infty}\alpha_i^{-1}\beta_i = \infty$.

For the explicit construction of h see [5].

We define for all positive integers n the number λ_n as the greatest real number λ such that

$$h(\pi 2^{-\lambda}) \geqq 2^{-n} .$$

Then for all n and k, $k \leqq n$ we have from (2) that

(4) $\lambda_{n-k} \leqq \lambda_n - k/\beta$

and by (3) (since for each i, $\pi 2^{-\lambda_n} \in [\alpha_i, \beta_i]$ for at most one n)

$$\limsup_{n\to\infty}(\lambda_{n+1} - \lambda_n) = \infty .$$

We define the set E as a Cantor-type (symmetric perfect) set:

$$E = \{x \mid x = \pi \textstyle\sum_{\nu=1}^\infty \varepsilon_\nu 2^{-\lambda_\nu}\} \quad (\varepsilon_\nu = 0 \text{ or } 1) .$$

Let μ be a measure on E such that if

$$x = \pi \textstyle\sum_1^\infty \varepsilon_\nu 2^{-\lambda_\nu}$$

then

$$\mu([0, x]) = \textstyle\sum_1^\infty \varepsilon_\nu 2^{-\nu} .$$

If the interval I of length $|I|$ satisfies

$$\pi 2^{-\lambda_{n+1}} < |I| < \pi 2^{-\lambda_n} ,$$

we have

(5) $$\mu(I) \leqq \textstyle\sum_{n+1}^\infty 2^{-\nu} = 2\cdot 2^{-n-1} = 2h(\pi\cdot 2^{-\lambda_{n+1}}) \leqq 2h(|I|) .$$

Let a union of intervals I_j, cover E. From (1) and (5) it follows that

$$\sum|I_j|^\alpha \geqq \sum h(|I_j|) \geqq \tfrac{1}{2}\sum\mu(I_j) \geqq \tfrac{1}{2}\mu(E) > 0 .$$

This implies that the Hausdorff measure of E with respect to the function t^α is positive, which proves (a).

Letting

$$\varepsilon_n = \pi 2^{-\lambda_{n+1}} \quad \text{and} \quad K_n = 2^{\lambda_{n+1}-\lambda_n}$$

we have

$$\lim_{n\to\infty} \varepsilon_n = 0 \quad \text{and} \quad \limsup_{n\to\infty} K_n = \infty .$$

Now, for some n, let $\{x_k\}$ be the set of midpoints of the covering intervals of length $2\varepsilon_n$.

For $k \neq l$ we have

$$|x_k - x_l| = \pi\left|\sum_{i=1}^{n} \varepsilon_i(k, l) 2^{-\lambda_i}\right|$$

where $\varepsilon_i(k,l) = -1, 0$, or 1, and is different from zero for at least one i. This together with (4) gives

$$\max_k \sum_{l,\, l\neq k} |x_k - x_l|^{-1} < C\sum_{i=0}^{n-1} 2^i 2^{\lambda_n - i} < C2^{\lambda_n}\sum_{i=1}^{n} 2^{(1-1/\beta)i} < C2^{\lambda_n} = C(K_n\varepsilon_n)^{-1}$$

where C is independent of n, which proves (b).

Lemma 3. *If the set E has properties as in lemma 2 there exists a sequence of trigomometric polynomials R_k and positive constants C_1 and C_2 such that*

(a) $\sum_{k=1}^{\infty} \|R_k\| < \infty$,
(b) $S^{**}(R_k, x) < C_1$ *for all x and k.*
(c) $\limsup_{k\to\infty} S^{**}(R_k, x) > C_2 > 0$ *for $x \in E$.*

Proof. To the set E in lemma 2 there is a sequence $(\varepsilon_n, K_n)_1^\infty$ such that $\varepsilon_n \to 0$ and $\limsup_{n\to\infty} K_n = \infty$. We can, if necessary by taking a subsequence, assume that $\sum(\log K_n)^{-1} < \infty$. For any n the set E can be covered by a finite union E_n of intervals of length $2\varepsilon_n$ and such that (b) in lemma 2 holds. We have

$$E = \bigcap_1^\infty E_n, \quad E_1 \supset E_2 \supset E \supset \dots .$$

For every n we construct the polynomial P_n as in lemma 1 with $\varepsilon = \varepsilon_n$ and $K = K_n$. Let

$$R_k = (\log K_k)^{-1} P_k .$$

This gives (a) and

$$S^{**}(R_k, x) < A_1 + B_1(\log K_k)^{-1} < C_1 \quad \text{for all } x \text{ and } k ,$$

which is (b).

If $x \in E_k$ we have by (c) in lemma 1 that

$$S^{**}(R_k, x) > A_2 - B_2(\log K_k)^{-1}$$

and

$$\limsup_{k\to\infty} S^{**}(R_k, x) > C_2 > 0$$

which is (c). (It also follows that $\liminf_{k\to\infty} S^{**}(R_k, x) \geqq A_2 > 0$).

4. Main results.

THEOREM 1. *If for a set $E \subset \mathsf{T}$ there exists a sequence of trigonometric polynomials P_j such that*

(a) $\sum_{j=1}^{\infty} \|P_j\| < \infty$,
(b) $\sup_j S^{**}(P_j, x) < C$ *for all* $x \in \mathsf{T}$,
(c) $\limsup_{j\to\infty} S^{**}(P_j, x) > 0$ *for* $x \in E$,

then E is a set with strongly bounded divergence for $C(\mathsf{T})$.

PROOF. (Compare Katznelson [8], [9, p. 56]). Let

$$P_j(x) = \sum_{-M_j}^{M_j} C_n{}^j e^{inx}, \quad j = 1, 2, 3, \ldots$$

Let $(N_i)_1^{\infty}$ be a sequence of integers such that for all j greater than one

$$N_j - M_j > N_{j-1} + M_{j-1} .$$

Let

$$f(x) = \sum_{j=1}^{\infty} e^{iN_j x} P_j(x) .$$

By (a) the series is uniformly convergent and f continuous.

If $N_j - M_j \leqq p \leqq q \leqq N_j + M_j$ then

$$\sup_{p,q} |S_q(f,x) - S_p(f, x)| = S^{**}(P_j, x) .$$

It follows that

$$\limsup_{m,n\to\infty} |S_n(f, x) - S_m(f, x)| \geqq \limsup_{j\to\infty} S^{**}(P_j, x) > 0$$

for $x \in E$, which menans that E is a set with divergence for $C(\mathsf{T})$. By (a) and (b) we have for any n and $x \in \mathsf{T}$

$$S^{**}(f, x) < \sum_{j=1}^{\infty} \|P_j\| + 2 \sup_j S^{**}(P_j, x) < C_1 < \infty .$$

Thus the Fourier series of f is strongly bounded.

THEOREM 2. *Given α, $0 < \alpha < 1$, there exist a set $E \supset \mathsf{T}$ having positive Hausdorff measure with respect to t^{α} and with strongly bounded divergence for $C(\mathsf{T})$.*

PROOF. Follows from lemmas 2 and 3 together with theorem 1.

REMARK 1. We even have a function $f \in C(\mathsf{T})$ such that $S^{**}(f,x) < C_1$ for all x and $\limsup_{m,n\to\infty} |S_n(f,x) - S_m(f,x)| \geqq C_2 > 0$ for $x \in E$.

REMARK 2. By modifying lemmas 1 and 2 it is possible to construct a set E with divergence for $C(\mathsf{T})$ having positive Hausdorff h-measure for any function h such that

$$\lim_{t\to 0+} h(t)/t \log t = -\infty .$$

5. Necessary conditions for bounded divergence.

THEOREM 3. *If the set $E \subset \mathsf{T}$ is a set with strongly bounded divergence for $C(\mathsf{T})$ there exists a sequence of trigonometric polynomials P_j such that*

(a) $\sum_j \|F_j\| < \infty$,
(b) $\limsup_{j\to\infty} S^*(P_j,x) > 0$ when $x \in E$,
(c) $\sup_j S^{**}(P_j,x) < C$ *for all* x.

REMARK. If the set E only admits bounded divergence (but not strongly bounded) we have instead of (c)

(c′) $\sup_j S^*(P_j,x) < C$ for all x.

PROOF. Let $f \in C(\mathsf{T})$ such that the Fourier series of f diverges in E but $\sup_n |S_n(f,x)| < C$ in T. We define the function φ such that

$$\limsup_{m,n\to\infty} |S_n(f,x) - S_m(f,x)| = \varphi(x) . \tag{1}$$

Then $\varphi(x) > 0$ for $x \in E$. (From Carleson's result mentioned in the introduction it follows that $\varphi(x)$ is zero a.e. Also $0 \leqq \varphi(x) \leqq 2C$ everywhere.)

Let K_n be the the Fejér kernel and V_n the de la Valleé-Poussin kernel

$$V_n = 2K_{2n+1} - K_n$$

Since V_n and K_n are summability kernels we have

$$\|f - V_n * f\| = \max_x |f(x) - V_n * f(x)| \to 0 \quad \text{as } n \to \infty$$

(where $*$ denotes convolution). Choose a sequence of natural numbers $(\lambda_n)_1^\infty$ such that

$$\lambda_{n+1} > 2\lambda_n + 1 \quad \text{and} \quad \|f - V_{\lambda_n} * f\| < 2^{-n-1} .$$

Let $P_n = (V_{\lambda_{n+1}} - V_{\lambda_n})*f$. Then

$$\|P_n\| \leqq \|V_{\lambda_{n+1}}*f - f\| + \|f - V_{\lambda_n}*f\| < 2^{-n}.$$

P_n is a trigonometric polynomial of order $2\lambda_{n+1}+1$ and

$$\hat{P}_n(j) \neq 0 \rightarrow \lambda_n + 2 \leqq |j| \leqq 2\lambda_{n+1}+1.$$

Also $\sum_1^\infty \|P_n\| < \infty$, which is (a).

$S_j(P_n)$ is a linear combination of four Fejér kernels convoluted with $S_j(f)$. Since these are positive and $S_j(f)$ is uniformly bounded we have for all j and n

(2) $$|S_j(P_n, x)| \leqq 6C.$$

It is no restriction to assume $\hat{f}(0) = 0$. Since we have strongly bounded divergence of the Fourier series of f we have for all n

$$\|\textstyle\sum_0^{2n} \hat{f}(j)e^{ijx}\| < C \quad \text{and} \quad \|\sum_0^{-2n} \hat{f}(j)e^{ijx}\| < C.$$

Since $f \in C(\mathsf{T})$ the conjugate function $\tilde{f}$ exists and has Fourier series

$$\textstyle\sum_j - i \operatorname{sign}(j) \hat{f}(j) e^{ijx}.$$

The functions f^b and f^{-b} defined by

$$f^b = \tfrac{1}{2}(f + i\tilde{f}) \quad \text{and} \quad f^{-b} = \tfrac{1}{2}(f - i\tilde{f})$$

have Fourier series

$$\textstyle\sum_{j>0} \hat{f}(j)e^{ijx} \quad \text{and} \quad \sum_{j<0} \hat{f}(j)e^{ijx}.$$

Their partial sums are bounded by the constant C and $f = f^b + f^{-b}$.

$$P_n(x) = \textstyle\sum_{j<0} \hat{P}_n(j)e^{ijx} + \sum_{j>0} P_n(j)e^{ijx} = P_n^{-b} + P_n^{b}.$$

By the same arguments that led to (2) we have for all j and n

$$\|S_j(P_n^b)\| < 6C \quad \text{and} \quad \|S_j(P_n^{-b})\| < 6C.$$

If $k \leqq l$ we have for any n

$$\textstyle\sum_k^l \hat{P}_n(j)e^{ijx} = S_l(P_n^b, x) - S_k(P_n^b, x) + S_l(P_n^{-b}, x) - S_k(P_n^{-b}, x)$$

which gives

$$S^{**}(P_n, x) \leqq 24C.$$

Thus (c) holds.

Given a positive ε there is a K such that

$$S_M(f, x) - S_N(f, x)$$

can be written as a linear combination of at most four partial sums of polynomials of type P_n plus a polynomial with norm less than ε for all M and N greater than K. (1) then implies

$$\limsup_{n\to\infty} S^*(P_n, x) \geqq \tfrac{1}{4}\varphi(x)$$

For details of this, see [5].
Since $\varphi(x) > 0$ for $x \in E$ we have (b), which completes the proof.

REMARK. We can also prove that

$$\limsup_{n\to\infty} S^*(P_n, x) \leqq 6 \limsup_{m,k\to\infty} |S_m(f, x) - S_k(f, x)| = 6\varphi(x) .$$

COROLLARY. *Let E_j be sets with strongly bounded divergence for $C(\mathsf{T})$. Then*

$$E = \bigcup_{j=1}^{\infty} E_j$$

is a set with strongly bounded divergence for $C(\mathsf{T})$.

PROOF. Modification of theorem II. 3. 3. in Katznelson [8].

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