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ON ARBITRARILY TRACEABLE GRAPHS

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1. Introduction and summary.

We follow the terminology of D. König [4], in particular degree = Grad, path = Weg, circuit = Kreis, cut-vertex = Artikulation, member = Glied. Multiple edges and isolated vertices are allowed.

The concept of a graph which is arbitrarily traceable from a vertex was introduced by Ore [6], [7] and may be defined as follows:

DEFINITION 1. A graph is *arbitrarily traceable from V to V*, V being a vertex, if the graph has an edge and if the procedure of starting from V, choosing an edge incident with V and traversing it to its other end-vertex, choosing an edge incident with this vertex not already traversed and traversing it, and continuing this procedure arbitrarily as long as possible with the sole restriction that no edge may be traversed more than once, always causes one to finish at V having traversed every edge of the graph.

We will also require the following

DEFINITION 2. $d_\Gamma(V)$ denotes the degree of the vertex V in the graph Γ. A graph is called *Eulerian* if the set of its edges is finite and every vertex has even degree.

The principal results concerning the graphs of Definition 1 are due to Ore [6, p. 74–76], [7] and Harary [3], these may be stated as follows:

ORE'S THEOREM. *A graph is arbitrarily traceable from V to V if and only if it has an edge, is Eulerian, and each of its circuits contains V.*

HARARY'S THEOREM. *A graph which is arbitrarily traceable from V to V either has no cut-vertex or V is its only cut-vertex.*

Harary [3] investigated the corresponding properties of directed graphs

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and established the directed analogues of results of Ore [6], [7] and Bäbler [1].

The object of this paper is to give new very short proofs of the above and related theorems, to derive basic properties of graphs having a property similar to that of Definition 1, and of their directed counterparts, and to present a theory of randomly traceable infinite graphs undirected and directed.

2. Short proofs of Ore's and Harary's theorems.

These will be based on the following classical result of Veblen [8] (see also [4, p. 23, Satz 6]) (the sets mentioned may be empty).

VEBLEN'S THEOREM. *A graph is Eulerian if and only if it is the union of a finite set of circuits, no two of which have an edge in common, and a set of isolated vertices.*

PROOF OF ORE'S THEOREM. Suppose first that the graph Γ is arbitrarily traceable from V to V. Then by a well known easy reasoning Γ is Eulerian. Assume (reductio ad absurdum) that Γ contains a circuit Δ and $V \notin \Delta$. Let Γ' be the graph obtained from Γ by deleting all edges of Δ. Γ' is Eulerian. Let $\mathscr{C}'$ denote a set of circuits of Γ' as described in Veblen's Theorem. If we traverse successively from V to V all those circuits of $\mathscr{C}'$ which contain V, then we have traversed every edge of Γ incident with V because $V \notin \Delta$, but we have not traversed the edges of Δ, hence Γ is not arbitrarily traceable from V to V, which is contrary to hypothesis.

Suppose secondly that the graph Γ has an edge, is Eulerian, and every circuit of Γ contains the vertex V. By Veblen's Theorem there are edges incident with V. Let Γ be traced arbitrarily from V as described in Definition 1 until we must stop at a vertex V^*. But $V^* = V$, for if $V^* \neq V$ then we have traversed one more edge going towards V^* than going away from V^*, and can continue from V^* because Γ is Eulerian.

Because $V^* = V$ we have traversed every edge incident with V and we have traversed the edges of an Eulerian graph. Hence deleting from Γ all the edges which we have traversed gives an Eulerian graph, Γ'' say, in which V is isolated. Γ'' contains no circuit, because every circuit of Γ contains V. Hence by Veblen's Theorem Γ'' has no edge, that is, we have traversed every edge of Γ. Thus Γ is arbitrarily traceable from V to V. Ore's Theorem is now proved.

A consequence of Ore's and Veblen's theorems is

ORE'S LEMMA. *A graph which is arbitrarily traceable from V to V is the union of a finite non-empty set of circuits, no two of which have an edge in common, each of which contains V and no two of which have more than one vertex besides V in common, and a set (possibly empty) of isolated vertices* [6, p. 74–76].

PROOF OF HARARY'S THEOREM. By Ore's Lemma each vertex $\neq V$ of the connected component of the graph containing V is connected to V by two segments of one circuit. Hence if any vertex $\neq V$ is deleted the remaining graph is connected, because each vertex is still connected to V by a path. This proves Harary's Theorem.

A number of other properties of graphs arbitrarily traceable from a vertex V to V are known [1], [3], [7], they all follow at once from the above results of Ore and Harary.

3. Directed graphs arbitrarily traceable from V to V.

Cycle will denote König's term *Zyklus*, [4, p. 29].

DEFINITION 3. Following Harary [3] a directed graph is said to be *arbitrarily traceable from the vertex V to V* if it has an edge and if the procedure of Definition 1, continued as long as possible, with each edge being traversed only in the direction of its arrow, always causes one to finish at V having traversed every edge of the graph.

DEFINITION 4. X being a vertex of the directed graph Γ, the number of edges incident with and directed into and out of X will be denoted by $d_{i\Gamma}(X)$ and $d_{o\Gamma}(X)$, respectively. If $d_{i\Gamma}(X) = d_{o\Gamma}(X)$, then X is *balanced*. If every vertex is balanced then Γ is *balanced*, and if in addition the total number of edges is finite, then Γ is *Eulerian*. The degree of X in Γ is $d_\Gamma(X) = d_{i\Gamma}(X) + d_{o\Gamma}(X)$.

The directed analogue of Ore's Theorem is

HARARY'S THEOREM FOR DIRECTED GRAPHS. *A directed graph is arbitrarily traceable from V to V if and only if it has an edge, is Eulerian, and each of its cycles contains V* [3].

The directed analogue of Veblen's Theorem is the well-known result of König [4, p. 29, Satz 8] (the sets mentioned may be empty).

KÖNIG'S THEOREM. *A directed graph is Eulerian if and only if it is* *the union of a finite set of cycles no two of which have an edge in common, and a set of isolated vertices.*

PROOF OF HARARY'S THEOREM FOR DIRECTED GRAPHS. The above proof of Ore's Theorem almost word for word with "graph" changed into "directed graph", "circuit" into "cycle", "Veblen" into "König", and "Definition 1" into "Definition 3".

From König's and Harary's theorems for directed graphs we have the following analogue of Ore's Lemma:

COROLLARY 1. *A directed graph Γ which is arbitrarily traceable from V to V is the union of c ($c \geqq 1$ and finite) cycles, no two of which have an edge in common and each of which contains V, and a set (possibly empty) of isolated vertices.*

PROOF. By Harary's Theorem Γ is Eulerian. Therefore by König's Theorem it is the union of c ($c \geqq 1$ and finite) cycles, no two of which have an ege in common, and a set (possibly empty) of isolated vertices. By Harary's Theorem each of the c cycles contains V. This proves Corollary 1.

4. Undirected graphs arbitrarily traceable from one vertex to another.

The modification of Definition 1 suggests itself that the procedure of Definition 1 continued as long as possible should always cause one to traverse every edge of the graph, but without necessarily finishing at the vertex one started from. Now if there is one way of starting from the vertex V and arriving at the vertex $W \neq V$ having traversed every edge according to the procedure, then the total number of edges is finite and by a well known reasoning V and W have odd degree and all other vertices (if any) have even degree. Consequently traversing every edge starting from V, if this is possible, either causes one to finish at V in every case, or it causes one to finish at a fixed vertex $W \neq V$ in every case. Therefore we formulate

DEFINITION 5. A graph is *arbitrarily traceable from V to W* if it has an edge and if the procedure of Definition 1 continued arbitrarily as long as possible always causes one to finish at W having traversed every edge of the graph.

We will use

DEFINITION 6. V and W being two distinct vertices of a graph, the graph will be called *(V, W)-odd* if the number of edges is finite, V and W have odd degree, and all other vertices (if any) have even degree.

In this terminology the modification of Veblen's Theorem is (the sets mentioned may be empty):

THEOREM 1. *A graph is (V, W)-odd if and only if it is the union of one VW-path and a finite number of circuits, no two of this system having an edge in common, and a set of isolated vertices.*

PROOF. First, the union of a system as described and a set of isolated vertices is clearly a (V, W)-odd graph.

Secondly, suppose a graph is (V, W)-odd. Then by adding a new WV-edge not already there we obtain an Eulerian graph. By Veblen's Theorem this is the union of a finite non-empty set of circuits, no two having an edge in common, and perhaps a set of isolated vertices. When the added WV-edge is deleted, the circuit to which it belonged becomes a VW-path and the other circuits remain.

This proves Theorem 1.

The modification of Ore's Theorem is

THEOREM 2. *A graph is arbitrarily traceable from V to W, where $V \neq W$, if and only if it is (V, W)-odd and each of its circuits contains W.*

PROOF. Suppose first that the graph Γ is arbitrarily traceable from V to W, where $V \neq W$. Then Γ is (V, W)-odd. Assume (reductio ad absurdum) that Γ contains a circuit Δ and $W \notin \Delta$. Let Γ' be obtained from Γ by deleting all edges of Δ. Obviously Γ' is (V, W)-odd. Let $\mathscr{S}'$ be a set consisting of one VW-path and circuits of Γ' as in Theorem 1. From V traverse the VW-path to W and then traverse successively the circuits in $\mathscr{S}'$ containing W (if any). Then we have traversed every edge of Γ incident with W, since $W \notin \Delta$, but not the edges of Δ. Hence Γ is not arbitrarily traceable—contrary to hypothesis.

Suppose secondly that the graph Γ is (V, W)-odd and every circuit of Γ contains W. Let us trace Γ arbitrarily from V as in Definition 1 until we must stop at a vertex V^*.

$V^* = W$ since Γ is (V, W)-odd.

Because $V^* = W \neq V$ we have traversed the edges of a (V, W)-odd graph and we have traversed every edge incident with W. Hence deleting from Γ all the edges which we have traversed gives an Eulerian graph,

Γ'' say, in which W is isolated. Γ'' contains no circuit because every circuit of Γ contains W. Hence by Veblen's Theorem Γ'' has no edge, that is, we have traversed every edge of Γ. Thus Γ is arbitrarily traceable from V to W. Theorem 2 is now proved.

A consequence of Theorems 1 and 2 is the

COROLLARY 2. *A graph Γ which is arbitrarily traceable from V to W, where $V \neq W$, is the union of one VW-path and c circuits ($c \geqq 0$ and finite) each containing W, no two of this system having an edge in common, and no two having more than one vertex besides W in common, and a set (possibly empty) of isolated vertices. If $c \geqq 1$ then either W is the only vertex of maximal degree, or $d_\Gamma(V) = d_\Gamma(W) = 2c+1$ and the remaining vertices have degree 2 or 0, or else $d_\Gamma(V) = 1$, one vertex has degree $2c+2$, and the remaining vertices other than W have degree 2 or 0.*

PROOF. By Theorem 2, Γ is (V, W)-odd. Therefore by Theorem 1 Γ is the union of one VW-path and a finite number of circuits, no two of these having an edge in common, and possibly some isolated vertices. By Theorem 2 each of the circuits contains W. No two of the above system of path and c circuits have more than one vertex besides W in common, because otherwise the graph would include a circuit not containing W, contrary to Theorem 2.

Suppose that $c \geqq 1$. Clearly $d_\Gamma(W) = 2c+1$. Therefore the degree of W is greater than the degree of every other vertex unless there is a vertex $Z \neq W$ which belongs to all c circuits and to the VW-path. If $Z = V$ then

$$d_\Gamma(V) = d_\Gamma(W) = 2c+1 ,$$

and the remaining vertices have degree 2 or 0 because any two of the path and circuits have only V and W in common. If $Z \neq V$ then

$$d_\Gamma(V) = 1, \quad d_\Gamma(Z) = 2c+2, \quad d_\Gamma(W) = 2c+1 ,$$

and the remaining vertices have degree 2 or 0 again. This proves Corollary 2.

Corresponding to Harary's Theorem we have

THEOREM 3. *Suppose that the graph Γ is arbitrarily traceable from V to W, where $V \neq W$, and has at least 3 vertices.*

If $d_\Gamma(V) > 1$ then no vertex except possibly W is a cut-vertex of Γ, and Γ has no bridge.

If $d_\Gamma(V) = 1$ then there is a unique VV'-path Π with $V \neq V'$ such that

$d_\Gamma(V') \neq 2$ and $d_\Gamma(X) = 2$ for each vertex X of $\Pi - V - V'$ (if any); every vertex of $\Pi - V - V'$ (if any) is a cut-vertex of Γ, V' is a cut-vertex of Γ unless Γ is a VW-path, and no other vertex except possibly W is a cut-vertex of Γ; the bridges of Γ are precisely the edges of Π.

PROOF. Theorem 3 is true with $V' = W$ if Γ is a VW-path. Assume that Γ is not a VW-path. Let Γ be the union of a system $\mathscr{S}$ as in Corollary 1 and possibly isolated vertices, Υ being the VW-path in $\mathscr{S}$. Obviously no vertex of any circuit of $\mathscr{S}$ is separated from W by any cut-vertex $\neq W$ of Γ. Therefore, if following Υ from W, the vertex V' is the last one encountered belonging to at least one circuit of $\mathscr{S}$, neither V' nor any vertex of Υ between V' and W (if any) is separated from W by a cut-vertex $\neq W$ of Γ (possibly $V' = V$ or $V' = W$). Consequently the only vertices of Γ which might be separated from W by a cut-vertex $\neq W$ are the vertices of Υ on the other side of V' from W (if any). Therefore, because each cut-vertex $\neq W$ separates a vertex from W, if $V' = V$ then no vertex except possibly W is a cut-vertex of Γ; by the Corollary $V' = V$ if and only if $d_\Gamma(V) \geqq 3$, hence if $d_\Gamma(V) > 1$ then no vertex except possibly W is a cut-vertex of Γ.

By the last sentence but one every cut-vertex $\neq W$ of Γ is on Υ and does not lie between V' and W.

Suppose that $d_\Gamma(V) = 1$, that is, $V' \neq V$. Let Π denote the part of Υ which connects V and V'. By the Corollary every vertex of $\Pi - V - V'$ (if any) has degree 2 in Γ, so all these vertices are cut-vertices of Γ and all the edges of Π are bridges of Γ and Π is unique. V' is a cut-vertex because $d_\Gamma(V') > 1$. V is not a cut-vertex of course. Hence the cut-vertices of Γ are as stated in Theorem 3. None of the edges of Υ between V' and W are bridges because V' belongs to a circuit of $\mathscr{S}$, and of course no edge of any of the circuits in $\mathscr{S}$ is a bridge. This proves Theorem 3.

Obviously not every graph which is arbitrarily traceable from V to W, where $V \neq W$, is arbitrarily traceable from W to V.

We prove

THEOREM 4. *A graph is arbitrarily traceable from V to W and from W to V, where $V \neq W$, if and only if it is the union of an odd number of VW-paths, any two of which have no edge and no vertex besides V and W in common, and a set (possibly empty) of isolated vertices.*

PROOF. Firstly, a graph with the above structure is obviously arbitrarily traceable from V to W and from W to V.

Secondly, suppose a graph is arbitrarily traceable from V to W and from W to V, where $V \neq W$. By Corollary 2 it is the union of a VW-path and a finite set C (possibly empty) of circuits containing W, no two of this system having an edge in common, and possibly some isolated vertices. By Theorem 2 with V and W exchanged, each circuit of C contains V. Therefore by Corollary 2 all vertices except V and W have degree 2. The theorem follows because each circuit of C is the union of two VW-paths which have nothing but V and W in common.

THEOREM 5. *If a graph is arbitrarily traceable from V to W, where $V \neq W$, then if the edges of any (V, W)-odd subgraph are deleted, the remaining graph is arbitrarily traceable from W to W or it has no edges.*

PROOF. The remaining graph is Eulerian and each of its cycles contains W.

THEOREM 6. *A graph is arbitrarily traceable from V to W, where $V \neq W$, if and only if, apart from isolated vertices, it is obtained by taking a finite circuitless graph Λ containing V but not W, and joining V to W by an odd or an even number of edges according as $d_\Lambda(V)$ is even or odd, and joining each vertex $X \neq V$ of Λ to W by a number of edges having the same parity as $d_\Lambda(X)$.*

PROOF. Any graph obtained in this way is (V, W)-odd and has a finite number of edges, and W belongs to each of its circuits. Therefore, by Theorem 2, the graph is arbitrarily traceable from V to W.

Conversely, if a graph is arbitrarily traceable from V to W, where $V \neq W$, then deleting W leaves a finite circuitless graph by Theorem 2 (apart from isolated vertices) and the graph is (V, W)-odd. From this it follows directly that the graph is obtained as described in Theorem 6.

5. Directed graphs arbitrarily traceable from one vertex to another.

In a directed graph Γ, if there is one way of starting from the vertex V and arriving at the vertex $W \neq V$ having traversed every edge just once and in the direction of its arrow, then the total number of edges is finite, and by a well-known reasoning

$$d_{o\Gamma}(V) - d_{i\Gamma}(V) = d_{i\Gamma}(W) - d_{o\Gamma}(W) = 1$$

and all the other vertices (if any) are balanced. Consequently starting from V and traversing every edge just once and in the direction of its arrow according to the procedure, if this is possible, either causes one to finish at V in every case, or it causes one to finish at a fixed vertex $W \neq V$ in every case. Therefore we formulate

DEFINITION 7. A directed graph is *arbitrarily traceable from V to W* if it has an edge and if the procedure of Definition 1 continued as long as possible with each edge traversed in the direction of its arrow always causes one to finish at W having traversed every edge of the graph.

DEFINITION 8. V and W being two distinct vertices of a directed graph, the graph will be called *$(V \to W)$-odd* if the number of edges is finite,

$$d_{o\Gamma}(V) - d_{i\Gamma}(V) = d_{i\Gamma}(W) - d_{o\Gamma}(W) = 1 ,$$

and all the other vertices (if any) are balanced.

In this terminology the analogue of Theorem 1 is (the sets mentioned may be empty)

THEOREM 7. *A directed graph is $(V \to W)$-odd if and only if it is the union of one directed $(V \to W)$-path and a finite number of cycles, no two of this system having an edge in common, and a set of isolated vertices.*

PROOF. The obvious analogue of the proof of Theorem 1. The analogue of Theorem 2 is

THEOREM 8. *A directed graph is arbitrarily traceable from V to W, where $V \neq W$, if and only if it is $(V \to W)$-odd and each of its cycles contains W.*

PROOF. The obvious analogue of the proof of Theorem 2. The result corresponding to Corollary 2 is

COROLLARY 3. *A directed graph Γ which is arbitrarily traceable from V to W, where $V \neq W$, is the union of one $(V \to W)$-path and c cycles ($c \geqq 0$ and finite) each containing W, no two of this system having an edge in common, and a set (possibly empty) of isolated vertices.*

PROOF. By Theorem 8 Γ is $(V \to W)$-odd. Therefore Γ is as described in Theorem 7. Each of the c cycles contains W by Theorem 8. This proves Corollary 3.

REMARK 1. There is no such restriction on the degrees of the vertices as in Corollary 2. This is because the cycles and the $(V \to W)$-path may have any number of vertices in common. When all vertices are common to all of them then V and W have degree $2c+1$ and all other vertices have degree $2c+2$. Correspondingly in an undirected graph which is arbitrarily traceable from V to V, and which is not a circuit, at most one vertex can have a degree as high as V, and in this case all the remaining vertices have degree 2 (Bäbler [1]), while in the directed case the degree of V is maximal, but every vertex may have the same degree as V.

REMARK 2. A directed graph which is arbitrarily traceable from V to W, where $V \neq W$, is of course not traceable from W to V.

The directed analogue of Theorem 5 is

THEOREM 9. *If a directed graph is arbitrarily traceable from V to W, where $V \neq W$, then if the edges if any $(V \to W)$-odd subgraph are deleted, the remaining graph is arbitrarily traceable from W to W, or it has no edges.*

PROOF. The remaining graph is Eulerian and each of its cycles contains W.

The directed analogue of Theorem 6 is

THEOREM 10. *A directed graph is arbitrarily traceable from V to W, where $V \neq W$, if and only if, apart from isolated vertices, it is obtained by taking a finite acyclic graph A containing V but not W, joining each vertex X of A other than V by edges to W in such a way that X becomes balanced, and joining V by edges to W in such a way that the outward degree of V exceeds the inward degree by 1, the number of new edges being finite.*

PROOF. Any graph obtained in this way is directed, has a finite number of edges, and is $(V \to W)$-odd, and W belongs to each of its cycles. Therefore by Theorem 8 the graph is arbitrarily traceable from V to W.

Conversely, if a graph is arbitrarily traceable from V to W, where $V \neq W$, then by Theorem 8 the graph is $(V \to W)$-odd and deleting W leaves a finite acyclic graph (apart from isolated vertices). From this it follows that the graph is obtained as described in Theorem 10.

6. Arbitrarily traceable infinite graphs.

For graphs with infinitely many edges the analogue of Definitions 1 and 5 is

DEFINITION 9. Γ being a graph with infinitely many edges and V_1 being a vertex of Γ, Γ is *arbitrarily traceable from* V_1 if the procedure of forming a sequence $V_1, E_1, V_2, E_2, \ldots, V_1, V_2, V_3, \ldots$ being vertices of Γ, and $E_1, E_2, E_3, \ldots$ being edges of Γ and being successively selected in any manner from among the edges of Γ, subject only to the restriction that for each $i \geqq 1$ E_i joins V_i and V_{i+1} and for each $i \geqq 2$ $E_i \neq E_{i-1}, \ldots, E_1$, always furnishes a sequence which includes all the edges of Γ.

It is also convenient to formulate

DEFINITION 10. Γ being a graph, any finite sequence $V_1, E_1, V_2, E_2 \ldots, E_{n-1}, V_n$ $(n \geqq 1)$ or 1-way infinite sequence $V_1, E_1, V_2, E_2, \ldots,$ where the V's are vertices and the E's are edges of Γ, for each i E_i joins V_i and V_{i+1}, and $E_i \neq E_j$ when $i \neq j$, is called a *chain* of Γ and is said to *start with* V_1. The chain $V_1, E_1, \ldots, V_n$ is said to be a $(V_1 V_n)$*-chain* and is called *open* if $V_1 \neq V_n$ and *closed* if $V_1 = V_n$. The number of edges in a chain is its *length*.

A chain of a graph which includes every edge of the graph is called an *Euler-chain* of the graph. A graph with infinitely many edges is said to be *traceable from the vertex* V_1 if it has an Euler-chain starting with V_1 (which is necessarily 1-way infinite of course). If Δ is a subgraph or a chain of Γ then the graph obtained by deleting from Γ all the edges of Δ will be denoted by $\Gamma - \mathscr{E}(\Delta)$. Any chain starting with the vertex V and including the edge E will be called a *VE-chain.* If Γ has a VE-chain then E is *accessible from* V *in* Γ. The chain formed by $2m-1$ consecutive terms of a chain with at least $2m-1$ terms, m being a natural number, will be called a *segment*, the chain formed by the first $2m-1$ a *starting segment.*

THEOREM 11. *A graph with infinitely many edges is arbitrarily traceable from the vertex* V_1 *if and only if it is the union of a* 1*-way infinite path starting with* V_1 *and a set (possibly empty) of isolated vertices.*

PROOF. The union of a 1-way infinite path starting with the vertex V_1 and a set of isolated vertices is obviously arbitrarily traceable from V_1 according to Definition 9.

Suppose next that a graph with infinitely many edges is arbitrarily traceable from the vertex V_1. Then the graph has a 1-way infinite Euler chain starting from V_1, say $V_1, E_1, V_2, E_2, \ldots$. Assume (reductio ad absurdum) that the graph is not as described in Theorem 11. Then $V_i = V_j$ for at least one pair of suffixes i,j with $i<j$, because all vertices having degree d occur just $[(\frac{1}{2}(d+1))]$ times in every Euler chain. Then if $i=1$,

$$V_1, E_j, V_{j+1}, E_{j+1}, V_{j+2}, \ldots$$

and if $i>1$

$$V_1, E_1, \ldots, E_{i-1}, V_i, E_j, V_{j+1}, E_{j+1}, V_{j+2}, \ldots$$

is an infinite chain of the graph which does not include any of the edges $E_i, \ldots, E_{j-1}$. Hence the graph is not arbitrarily traceable, contrary to hypothesis. Theorem 11 is now proved.

Theorem 11 shows that the above definition of arbitrarily traceable graphs with infinitely many edges is very restrictive. In the other direction we will prove Lemmas 1–3 and Theorems 12 and 13. The presentation has been chosen with the directed analogues in mind.

LEMMA 1. *If V is a vertex of a closed chain Ξ of the graph Γ and if $d_\Gamma(V)$ is odd or infinite, then V is incident with an edge of Γ which does not belong to Ξ. If Ξ' is an open (XY)-chain of the graph Γ and if $d_\Gamma(Y)$ is even or infinite, then Y is incident with an edge of Γ which does not belong to Ξ'.*

PROOF. $d_{\Gamma-\mathscr{E}(\Xi)}(V)$ is odd or infinite because the number of edges of Ξ incident with V is even. $d_{\Gamma-\mathscr{E}(\Xi')}(Y)$ is odd or infinite because the number of edges of Ξ' indicent with Y is odd.

LEMMA 2. *If the vertex V_1 of the graph Γ has odd or infinite degree in Γ and if no vertex other than V_1 has odd degree in Γ then every finite chain of Γ starting with V_1 is a starting segment of a 1-way infinite chain of Γ.*

PROOF. By Lemma 1 every finite chain of Γ starting with V_1 can be continued indefinitely.

LEMMA 3. *Let Δ be any graph with a 1-way infinite Euler chain, let Δ' be obtained by deleting from Δ any finite set of edges, and let V be any vertex of Δ. Then either the set of edges accessible from V in Δ' is finite, or Δ' has a chain starting with V which includes all but at most a finite number of the edges of Δ. The second alternative holds for at least two vertices, and for all but at most a finite number of the vertices of Δ of non-zero degree.*

PROOF. Let $V_1, E_1, V_2, E_2, \ldots$ be a 1-way infinite Euler chain of Δ. V is a vertex of this chain unless $d_\Gamma(V)=0$. There exists an integer m such that Δ' is obtained by deleting from Δ a subset of $\{E_1, \ldots, E_{m-1}\}$. Then $V_m, E_m, V_{m+1}, E_{m+1}, \ldots$ is a chain of Δ'. If V is one of $V_m, V_{m+1}, V_{m+2}, \ldots$ there is nothing to prove. Suppose that $V \neq V_m, V_{m+1}, V_{m+2}, \ldots$ Either the edges accessible from V in Δ' are a subset of $\{E_1, \ldots, E_{m-1}\}$ or else for some $j \geqq m$, Δ' has a VE_j-chain. When we follow this chain from V let W be the first vertex encountered which is one of $V_m, V_{m+1}, V_{m+2}, \ldots$ and suppose that $W = V_k$, where $k \geqq m$. The portion of this chain up to and including W does not contain any of $E_m, E_{m+1}, E_{m+2}, \ldots$ Therefore this portion together with $V_k, E_k, V_{k+1}, E_{k+1}, \ldots$ is a chain of Δ' starting with V which includes all the edges of Δ other than $E_1, \ldots, E_{k-1}$. Lemma 3 is now proved.

THEOREM 12. *If the graph Γ has a 1-way infinite Euler chain starting with the vertex V_1 then if Ξ is any (V_1V_n)-chain of Γ of length $\geqq 1$, the vertices and edges of $\Gamma - \mathscr{E}(\Xi)$ which are not accessible from V_n in $\Gamma - \mathscr{E}(\Xi)$ and are not isolated vertices of Γ constitute an Eulerian subgraph Γ_Ξ of $\Gamma - \mathscr{E}(\Xi)$ and each vertex of Γ_Ξ is in Γ incident only with a subset of $\mathscr{E}(\Gamma_\Xi) \cup \mathscr{E}(\Xi)$. The graph Γ has a chain Ξ^* such that Ξ^* has Ξ as a starting segment, $\mathscr{E}(\Xi^*) = \mathscr{E}(\Gamma) - \mathscr{E}(\Gamma_\Xi)$, each connected component of Γ_Ξ (if any) contains a vertex of Ξ, and Γ_Ξ does not contain V_n or any vertex of Ξ^* which occurs beyond Ξ.*

The proof of Theorem 12 will use

LEMMA 4. *Any graph with a finite non-zero number of edges is the union of a finite set of circuits and/or paths of length $\geqq 1$ such that no two members of the set have an edge in common and no two paths of the set have an end-vertex in common, together with a set (possibly empty) of isolated vertices. The vertices of odd degree in the graph are precisely the end-vertices of these paths.*

PROOF OF LEMMA 4. Let Δ denote the graph. If Δ has no circuit then let $\Lambda = \Delta$. If Δ has a circuit then we can delete successively sets of edges of circuits from Δ until we are left with a graph Λ having no circuit. If Λ has no edge then Lemma 4 holds for Δ. If Λ has an edge then let Υ_1 be a path of maximal length in Λ. Clearly Υ_1 has length $\geqq 1$ and each end-vertex of Υ_1 has degree 1 in Λ. If $\Lambda - \mathscr{E}(\Upsilon_1)$ has no edge then Lemma 4 holds for Δ, otherwise let Υ_2 be a path of maximal length in $\Lambda - \mathscr{E}(\Upsilon_1)$. Clearly Υ_2 has length $\geqq 1$ and each end-vertex of Υ_2 has degree 1 in

$\Lambda - \mathscr{E}(\Upsilon_1)$ and each end-vertex of Υ_1 has degree 0 in $\Lambda - \mathscr{E}(\Upsilon_1)$. Since $\mathscr{E}(\Lambda)$ is finite, Λ is the union of a finite set of paths of length $\geqq 1$, no two of which have an edge or an end-vertex in common, together with a set (possibly empty) of isolated vertices. Furthermore the degrees of the end-vertices of these paths are odd in Λ and in Δ, while the degrees of all the other vertices are even in Λ and in Δ. This proves Lemma 4.

REMARK 3. From Lemma 4 there follows the well-known result that in any graph with a finite number of edges the number of vertices of odd degree is even.

PROOF OF THEOREM 12. Γ has a 1-way infinite Euler chain starting with V_1, therefore $\Gamma - \mathscr{E}(\Xi)$ satisfies the conditions of Lemma 2 with V_n in place of V_1, hence $\Gamma - \mathscr{E}(\Xi)$ has a 1-way infinite chain starting with V_n. Consequently by Lemma 3, $\Gamma - \mathscr{E}(\Xi)$ has a chain starting with V_n which includes all but at most a finite number of the edges of Γ. Therefore Γ has a chain Ξ_0 which has Ξ as a starting segment and includes all but at most a finite number of the edges of Γ. If Ξ_0 is an Euler chain of Γ then every edge of $\Gamma - \mathscr{E}(\Xi)$ is accessible from V_n in $\Gamma - \mathscr{E}(\Xi)$ and Theorem 12 holds with $\Xi^* = \Xi_0$. Suppose in what follows that Ξ_0 is not an Euler chain of Γ.

If $\Gamma - \mathscr{E}(\Xi_0)$ is Eulerian then let $\Xi' = \Xi_0$. Otherwise we obtain a chain Ξ' of Γ such that Ξ' starts with V_1, has Ξ as an initial segment, includes every edge of Ξ_0, and $\Gamma - \mathscr{E}(\Xi')$ is Eulerian, as follows: $\Gamma - \mathscr{E}(\Xi_0)$ has a finite non-zero number of edges, therefore it contains a finite set $\mathscr{S}$ of paths and/or circuits as given in Lemma 4, and $\mathscr{S}$ contains at least one path because $\Gamma - \mathscr{E}(\Xi_0)$ is not Eulerian. Let the paths of $\mathscr{S}$ be $\Upsilon_1, \ldots, \Upsilon_a$, where $a \geqq 1$, and let the end-vertices of Υ_i be X_i and Y_i for $i = 1, \ldots, a$; $X_1, Y_1, \ldots, X_a, Y_a$ are all distinct and constitute all the vertices having odd degree in $\Gamma - \mathscr{E}(\Xi_0)$. Clearly $d_\Gamma(X_1)$, $d_\Gamma(Y_1), \ldots, d_\Gamma(X_a)$, $d_\Gamma(Y_a)$ are all infinite. Therefore $X_1, Y_1, \ldots, X_a, Y_a$ each occur infinitely often on Ξ_0. It follows that Ξ_0 is the juxta-position of an infinite sequence of finite segments of itself $\Xi_{00}, \Xi_{01}, \Xi_{02}, \ldots$ where Ξ_{00} is a (V_1X_1)-chain having Ξ as a starting segment, and $\Xi_{01}, \Xi_{03}, \Xi_{05}, \ldots$ are (X_1Y_1)-chains, and $\Xi_{02}, \Xi_{04}, \Xi_{06}, \ldots$ are (Y_1X_1)-chains. Let Π_1 denote the chain whose terms are the vertices and edges of Υ_1 in turn starting with X_1. Let Ξ_1 denote the chain which is the juxtaposition of

$$\Xi_{00}, \Pi_1, \Xi_{02}, \Xi_{01}, \Xi_{04}, \Xi_{03}, \ldots$$

in this order. Clearly Ξ_1 is a chain of Γ having Ξ as an initial segment, and $\mathscr{E}(\Xi_1) = \mathscr{E}(\Xi_0) \cup \mathscr{E}(\Upsilon_1)$. Also, if $a = 1$ then $\Gamma - \mathscr{E}(\Xi_1)$ is Eulerian and if $a>1$ then the only vertices having odd degree in $\Gamma-\mathscr{E}(\Xi_1)$ are $X_2, Y_2,\ldots,X_a, Y_a$. If $a=1$ then put $\Xi'=\Xi_1$. If $a>1$ then we can obtain chains $\Xi_2,\ldots,\Xi_a$ which incorporate successively $\Upsilon_2,\ldots,\Upsilon_a$. Put $\Xi'=\Xi_a$. Now Ξ' is a chain of Γ having Ξ as an initial segment, and $\Gamma-\mathscr{E}(\Xi')$ is Eulerian.

If $\mathscr{E}(\Xi')=\mathscr{E}(\Gamma)$ then put $\Xi^*=\Xi'$, in this case every edge of $\Gamma-\mathscr{E}(\Xi)$ is accessible from V_n in $\Gamma-\mathscr{E}(\Xi)$ and Ξ^* has the required properties. If $\mathscr{E}(\Xi')\neq\mathscr{E}(\Gamma)$ then let the connected components with edges of $\Gamma-\mathscr{E}(\Xi')$ be $\Phi_1,\ldots,\Phi_b$, where $b\geqq 1$. Of course $\Phi_1,\ldots,\Phi_b$ are Eulerian. Let $\Xi=V_1,E_1,\ldots,V_n$, where $n\geqq 2$, and let

$$\Xi' = V_1,E_1,\ldots,V_n,E_n,V_{n+1},E_{n+1},\ldots .$$

For $i=1,\ldots,b$ if Φ_i contains a vertex V_j of Ξ' with $j\geqq n$ then select one such vertex V_{j_i} and insert into Ξ' immediately after V_{j_i} in turn all edges and vertices of a closed Euler chain of Φ_i in cyclic order starting with an edge incident with V_{j_i}. Let Ξ^* denote the resulting chain, possibly $\Xi^*=\Xi'$. Let

$$\Xi^* = V_1,E_1,\ldots,E'_n,V'_{n+1},E'_{n+1},\ldots .$$

If $\mathscr{E}(\Xi^*)=\mathscr{E}(\Gamma)$ then every edge of $\Gamma-\mathscr{E}(\Xi)$ is accessible from V_n in $\Gamma-\mathscr{E}(\Xi)$ and Ξ^* has the required properties. If $\Gamma-\mathscr{E}(\Xi^*)$ has edges then it is the union of a subset of $\Phi_1,\ldots,\Phi_b$ and perhaps a set of isolated vertices, in this case we may assume that the connected components with edges of $\Gamma-\mathscr{E}(\Xi^*)$ are $\Phi_1,\ldots,\Phi_c$, where $1\leqq c\leqq b$. Clearly $\Gamma-\mathscr{E}(\Xi^*)$ and $\Phi_1\ldots\Phi_c$ are Eulerian.

Let W be any vertex of $\Phi_1\cup\ldots\cup\Phi_c$. Then $W\neq V_n, V'_{n+1}, V'_{n+2},\ldots$ because

$$\{V_n, V'_{n+1}, V'_{n+2},\ldots\} = \{V_n, V_{n+1}, V_{n+2},\ldots\}\cup\mathscr{V}(\Phi_{c+1}\cup\ldots\cup\Phi_b) .$$

Therefore in Γ, W is not incident with any of the edges $E'_n,E'_{n+1},E'_{n+2}, \ldots$. Consequently in Γ, W is incident with a subset of $\mathscr{E}(\Phi_1\cup\ldots\cup\Phi_c)\cup\mathscr{E}(\Xi)$. Each of $\Phi_1,\ldots,\Phi_c$ contains a vertex of Ξ^* because Γ is traceable from V_1, and none of them contains any of $V_n, V'_{n+1}, V'_{n+2},\ldots$. Therefore each of $\Phi_1,\ldots,\Phi_c$ contains a vertex of Ξ and none of V_n, $V'_{n+1}, V'_{n+2},\ldots$.

Because $V_n\notin\Phi_1\cup\ldots\cup\Phi_c$ and each vertex of $\Phi_1\cup\ldots\cup\Phi_c$ is incident only with edges of $\Phi_1\cup\ldots\cup\Phi_c$ and of Ξ, no vertex or edge of $\Phi_1\cup\ldots\cup\Phi_c$ is accessible from V_n in $\Gamma-\mathscr{E}(\Xi)$. On the other hand all edges of

$$\Gamma-\mathscr{E}(\Xi)-\mathscr{E}(\Phi_1\cup\ldots\cup\Phi_c)$$

are accessible from V_n in $\Gamma-\mathscr{E}(\Xi)$ of course, because they are just the edges of Ξ^* beyond V_n. Hence $\Phi_1 \cup \ldots \cup \Phi_c = \Gamma_{\Xi}$. Theorem 12 is now proved.

THEOREM 13. *A graph Γ with infinitely many edges is traceable from the vertex V_1 if and only if every edge of Γ is accessible from V_1, and Ξ being any finite chain of Γ starting with V_1, Γ has a chain starting with V_1 which includes all but at most a finite number of the edges of Γ and which has Ξ as a starting segment.*

PROOF. It was shown in the first part of the proof of Theorem 12 that if Γ has infinitely many edges and is traceable from V_1 then if Ξ is any finite chain of Γ starting with V_1, Γ has a chain which includes all but at most a finite number of the edges of Γ and which has Ξ as a starting segment. Theorem 12 in fact implies this. Obviously every edge is accessible from V_1 if Γ is traceable from V_1.

Next we prove the converse for graphs with infinitely many edges. There is no valid analogue of this converse for directed graphs!

(i) $d_\Gamma(V_1) \geqq 1$.

For by hypothesis Γ has edges and each edge is accessible from V_1.

(ii) Each edge of Γ is contained in a chain of Γ which starts with V_1 and includes all but at most a finite number of the edges of Γ.

For each edge is included in a finite chain of Γ starting from V_1, and each such chain is a starting segment of a chain as specified.

(iii) $d_\Gamma(V_1)$ is odd or infinite and no vertex other than V_1 has odd degree in Γ.

PROOF OF (iii). Let X be any vertex of Γ having finite degree $\neq 0$. Among all the chains of Γ which start with V_1 and include all but at most a finite number of the edges of Γ let Ξ_X be one which includes the greatest number of edges incident with X. By (ii) this number is >0 because $d_\Gamma(X) > 0$ is assumed.

We will show that Ξ_X includes all the edges of Γ incident with X. Assume (reductio ad absurdum) that the edge E' is incident with X and $E' \notin \Xi_X$. Let E'' denote the last edge in Ξ_X incident with X, and let Ψ_X denote the starting segment of Ξ_X immediately preceding E''; the last term of Ψ_X is X. Γ has a chain Φ_X which starts with V_1, includes all but at most a finite number of the edges of Γ, has Ψ_X as a starting segment, and has the edge E' immediately following Ψ_X. Clearly Ξ_X

and Φ_X have Ψ_X as a starting segment and have infinitely many edges in common; $E' \notin \Xi_X$ by hypothesis and $E'' \notin \Phi_X$ from the maximal property of Ξ_X. Let E^* be the first edge of Ξ_X occurring after the edges of Ψ_X which is also in Φ_X and let V^* be the vertex represented by the vertex-term of Ξ_X immediately preceding E^*. We have $E^* \neq E''$ since $E'' \notin \Phi_X$, therefore E^* is not incident with X, consequently $V^* \neq X$. The segment of Φ_X starting with X and E' and continuing until V^* is reached for the first time, followed by the segment of Ξ_X immediately preceding E^* and going back as far as E'' and X, is a chain of Γ because of the definition of E^*, and it contains E' and E'' and has no edge in common with Ψ_X. Therefore Ψ_X starting with V_1, followed by this chain, is a chain of Γ which includes more edges incident with X than does Ξ_X, and this chain is the starting segment of a chain of Γ which includes all but at most a finite number of the edges of Γ. This is contrary to the definition of Ξ_X. Therefore Ξ_X includes all the edges of Γ which are incident with X.

Because Ξ_X is a 1-way infinite chain starting with V_1 it follows that if $Y = V_1$, then $d_\Gamma(Y)$ is odd or infinite, and if $Y \neq V_1$, then $d_\Gamma(Y)$ is even or infinite. The isolated vertices of Γ (if any) all have zero degree. Now (iii) is proved.

(iv) The number of edges of Γ is enumerably infinite, and if any finite set of edges is deleted from Γ then only one connected component of the remaining graph is infinite.

Γ has an enumerable infinity of edges because it has infinitely many edges and a 1-way infinite chain which includes all but at most a finite number of the edges of Γ. By Lemma **3** therefore, if any finite set of edges is deleted from Γ then the remaining graph has a vertex from which all but at most a finite number of the edges are accessible. This proves (iv).

Erdös, Grünwald and Vázsonyi [2] proved that Γ has a 1-way infinite Euler chain starting with V_1 if and only if every edge of Γ is accessible from V_1 and Γ satisfies (iii) and (iv). The converse of (1) follows from this.

Theorem **13** is now proved.

Theorem **11** suggests that it is too much to require that every infinite chain starting with V_1 shall be an Euler chain. Theorems **11** and **13** together suggest the following

Definition 11. A graph will be called *semi-arbitrarily traceable from the vertex* V_1 if it has infinitely many edges and if every finite chain of the graph starting with V_1 is a starting segment of at least one Euler chain of the graph starting with V_1. This is equivalent to the following property in the terminology of Definition 1: the procedure of starting with the vertex V_1 and traversing edges at random as described in Definition 1 can be continued without end, and at each stage, whatever finite set of edges have been traversed, the procedure can be continued in such a way that every edge of the graph is traversed exactly once.

Obviously if a graph is semi-arbitrarily traceable from the vertex V_1 then it is traceable from V_1. That Definition 11 is far less restrictive than Definition 9 is shown by Theorems 14 and 15.

Theorem 14. *A graph Γ is semi-arbitrarily traceable from the vertex V_1 if and only if $\mathscr{E}(\Gamma)$ is enumerably infinite, and Ξ being any (V_1V_n)-chain of the graph, in $\Gamma-\mathscr{E}(\Xi)$ every edge is accessible from V_n.*

Proof. It is easy to see that if Γ is semi-arbitrarily traceable from V_1 then it has the properties stated.

Suppose next that the graph Γ has the stated properties. Let Ξ_0 be any finite chain of Γ starting with V_1. We have to prove that Γ has an Euler chain starting with V_1 of which Ξ_0 is a starting segment. Let the edges of Γ be enumerated $E_1, E_2, E_3, \ldots$ For $i=1,2,3,\ldots$ Γ has a finite chain Ξ_i starting with V_1, of which Ξ_{i-1} is a starting segment and which contains E_i. Therefore the chain $\Xi_0 \cup \Xi_1 \cup \Xi_2 \cup \ldots$ is an Euler chain of Γ starting with V_1 and having Ξ_0 as a starting segment. Theorem 14 is now proved.

Theorem 15. *A graph is semi-arbitrarily traceable from the vertex V_1 if and only if it has infinitely many edges and is traceable from V_1, and every circuit of the graph contains at least one vertex having infinite degree in the graph.*

Because of the criterion of Erdös, Grünwald and Vázsonyi quoted at the end of the proof of Theorem 13, Theorem 15 is equivalent to

Theorem 15′. *The graph Γ is semi-arbitrarily traceable from the vertex V_1 if and only if*

(a) *only one connected component of Γ has edges,*
(b) *the number of edges is enumerably infinite,*

(c) $d_\Gamma(V_1)$ *is odd or infinite and no vertex other than* V_1 *has odd degree in* Γ,

(d) *if the edges of any finite chain of* Γ *starting with* V_1 *are deleted from* Γ, *the remaining graph has only one infinite connected component,*

(e) *every circuit of* Γ *contains at least one vertex of infinite degree.*

Proof of Theorem 15. First we will prove that if the graph Γ has a 1-way infinite Euler chain starting with the vertex V_1 and if every circuit of Γ contains at least one vertex having infinite degree then Γ is semi-arbitrarily traceable from V_1. Let Ξ be any (V_1V_n)-chain of Γ. In the terminology of Theorem 12 $\mathscr{E}(\Gamma_\Xi)=\emptyset$, because if $\mathscr{E}(\Gamma_\Xi)\neq\emptyset$ then by Theorem 12 and Veblen's Theorem Γ_Ξ is the union of one or more circuits, and in Γ each vertex of each of these circuits has finite degree contrary to hypothesis. Therefore by Theorem 12 Γ has an Euler chain Ξ^* which has Ξ as a starting segment. Hence Γ is semi-arbitrarily traceable from V_1.

Next we will prove that if the graph Γ is semi-arbitrarily traceable from the vertex V_1 then it has infinitely many edges and is traceable from V_1, and every circuit of Γ contains at least one vertex having infinite degree in Γ. Definition 11 implies that the set of edges of Γ is enumerably infinite and Γ is traceable from V_1. Assume (reductio ad absurdum) that the circuit Θ of Γ contains no vertex having infinite degree in Γ.

Γ has an Euler chain starting with V_1, and this contains the vertices of Θ. Among the starting segments of this Euler chain which contain a vertex of Θ let Ξ_1 be the one with the least number of edges, and let

$$\Xi_1 = V_1,E_1,\ldots,V_m ,$$

where $m\geqq 1$ and $V_i\notin\Theta$ if $i<m$. Let the vertices and edges of Θ in cyclic order round Θ be $V_m,E_m,V_{m+1},\ldots,E_{n-1},V_n$, where $V_n=V_m$ and $n\geqq m+2$, and let Ξ_2 denote the chain

$$V_1,E_1,\ldots,V_m,E_m,V_{m+1},\ldots,E_{n-1},V_n .$$

Because Γ is semi-arbitrarily traceable from V_1, Γ has an Euler chain which starts with V_1 and has Ξ_2 as a starting segment, Ξ^* say. Let

$$\Xi^* = V_1,E_1,\ldots,V_m,E_m,V_{m+1},\ldots,E_{n-1},V_n,E_n,V_{n+1},E_{n+1},V_{n+2},\ldots .$$

The total number of edges incident with at least one vertex of Θ is finite because each vertex of Θ has finite degree in Γ. Therefore there is a term V_p in Ξ^* such that $p\geqq n+1$ and none of the edges $E_p,E_{p+1},E_{p+2},\ldots$ is incident with any vertex of Θ. Now let Ξ_3 denote the chain

$$V_1, E_1, \ldots, V_m, E_n, V_{n+1}, \ldots, V_p\,,$$

and consider $\Gamma - \mathscr{E}(\Xi_3)$. The edges of $\Gamma - \mathscr{E}(\Xi_3)$ are precisely $E_m, E_{m+1}, \ldots, E_{n-1}$ and $E_p, E_{p+1}, E_{p+2}, \ldots$. In $\Gamma - \mathscr{E}(\Xi_3)$ none of the edges $E_m, \ldots, E_{n-1}$ are accessible from V_p, because $V_p \notin \Theta$ since E_p is not incident with any vertex of Θ, by definition $E_m, \ldots, E_{n-1} \in \Theta$, and none of $E_p, E_{p+1}, E_{p+2}, \ldots$ are incident with any vertex of Θ. However, by Theorem 14 in $\Gamma - \mathscr{E}(\Xi_3)$ every edge is accessible from V_p. This contradiction completes the proof of Theorem 15.

From Theorem 15′ we easily deduce the following

Corollary 4. *The only locally finite graphs which are semi-arbitrarily traceable from a vertex are those which consist of a* 1-*way infinite path and a set* (*possibly empty*) *of isolated vertices.*

Three further kinds of arbitrary traceability may be distinguished. For this purpose we will formulate some new definitions.

Definition 12. Let $\Xi = W_1, E_1, W_2, E_2, \ldots, E_{n-1}, W_n$ be a closed chain ($W_1 = W_n$). Then the closed chain $W_a, E_a, \ldots, E_{n-1}, W_1 E_1, \ldots, E_{a-1}, W_a$, where $2 \leqq a \leqq n-1$, will be called a *rotation* of Ξ.

Let $\Xi = W_1, E_1, \ldots, E_{n-1}, W_n$ be any chain. Then the chain $W_n, E_{n-1}, \ldots, E_1, W_1$ will be called the *reverse* of Ξ and denoted by Ξ^{rev}.

If $\Xi = W_1, E_1, \ldots, W_n$ and $\Xi' = W'_1, E'_1, \ldots, W'_{n'}$ are two chains which have no edge in common and if $W_n = W'_1$ then $W_1, E_1, \ldots, W_n, E'_1, \ldots, W'_{n'}$ is a chain, it will be denoted by Ξ, Ξ'.

Definition 13. Let Γ be a graph with infinitely many edges and V_1 a vertex of Γ. We will say that Γ is

a) *type* 1 *traceable from* V_1 if, for Ξ any finite chain of Γ starting with V_1, Γ has an Euler chain starting with V_1 which has Ξ as a segment,

b) *type* 2 *traceable from* V_1 if, for Ξ any open chain of Γ, Γ has an Euler chain starting with V_1 which has Ξ or Ξ^{rev} as a segment, and Ξ being any closed chain of Γ other than an isolated vertex of Γ, Γ has an Euler chain starting with V_1 which has Ξ as a segment,

c) *type* 3 *traceable from* V_1 if Ξ being any finite chain of Γ other than an isolated vertex, Γ has an Euler chain starting with V_1 which has Ξ as a segment.

Obviously if a graph is type 3 traceable from the vertex V_1 then it is also type 2 and type 1 traceable from V_1, but not conversely. We also have

LEMMA 5. *If a graph is type 2 traceable from the vertex V_1 then it is type 1 traceable from V_1, but not conversely.*

PROOF. Let Γ denote the graph which is type 2 traceable from V_1. Let Ξ be a finite chain of Γ starting with V_1. By definition Γ has an Euler chain starting with V_1 which has either Ξ or Ξ^{rev} as a segment. In the first case there is nothing to prove. In the second case let Ξ' be an Euler chain of Γ starting with V_1 which has Ξ^{rev} as a segment. Let Ξ_1 be the starting segment of Ξ' which ends with V_1 having just described Ξ^{rev}, and let Ξ'' be the remainder of Ξ'. Then clearly $\Xi_1{}^{\text{rev}}$, Ξ'' is an Euler chain of Γ starting with V_1 which has Ξ as a starting segment. Therefore Γ is type 1 traceable from V_1.

Let Γ consist of the vertices $V_1, V_2, V_3, \ldots$ and edges $E_1, E_2, E_3, \ldots$ joining the pairs $(V_1, V_2), (V_2, V_3), (V_3, V_1), (V_1, V_4), (V_4, V_5), (V_5, V_6), \ldots$ respectively. Γ is type 1 traceable from V_1 but not type 2 traceable from V_1. (Γ has no Euler chain which has the chain V_2, E_1, V_1, E_3, V_3 or the reverse of this chain as a segment). Lemma 5 is now proved.

THEOREM 16. *A graph is type 1 traceable from the vertex V_1 if and only if it has infinitely many edges and is traceable from V_1, and every circuit of the graph which does not include V_1 contains a vertex having infinite degree in the graph.*

PROOF. First we will prove that if the graph Γ satisfies the conditions of the theorem then Γ is type 1 traceable from V_1. Let Ξ be any $(V_1 V_n)$-chain of Γ of length > 0. In the terminology of Theorem 12, Γ_Ξ is Eulerian. If Γ_Ξ has no edge then by Theorem 12, Γ has an Euler chain having Ξ as a starting segment. If Γ_Ξ has an edge then by Veblen's Theorem it is the union of a finite number of circuits no two of which have an edge in common, and perhaps a set of isolated vertices. None of these circuits includes a vertex having infinite degree in Γ by Theorem 12, therefore each of them includes V_1. By Theorem 12, Γ has a chain Ξ^* which has Ξ as a starting segment and includes every edge of Γ not in Γ_Ξ. If we first describe every circuit of Γ_Ξ from V_1 to V_1 in turn and then Ξ^*, we have an Euler chain of Γ starting with V_1 which has Ξ as a segment. Hence Γ is type 1 traceable from V_1.

Next we will prove that if the graph Γ is type 1 traceable from the vertex V_1 then it has the required properties. From Definition 13, Γ has infinitely many edges and is traceable from V_1. Assume (reductio ad absurdum) that the circuit Θ of Γ does not include V_1 and contains no vertex having infinite degree in Γ. Let Ξ_2 be defined as in the last paragraph of the proof of Theorem 15. Γ has an Euler chain Ξ' which starts with V_1 and has Ξ_2 as a segment. Because every vertex of Θ has finite degree in Γ, there is a vertex-term in Ξ' such that none of the edges of Ξ' beyond it are incident with a vertex of Θ. Let Ξ'_3 denote the starting segment of Ξ' up to and including this vertex-term. Of course Ξ_2 is a segment of Ξ'_3. Let Ξ_3 denote the chain obtained from Ξ'_3 by leaving out $E_m, V_{m+1}, \ldots, E_{n-1}, V_n$.

Γ contains no chain which has Ξ_3 as a segment and includes an edge of Θ. For neither the first nor the last vertex-term of Ξ_3 belongs to Θ, and Ξ_3 contains every edge of Γ incident with every vertex of Θ except only the edges of Θ. But this contradicts the definition of Γ. Therefore every circuit of Γ which does not include V_1 contains a vertex having infinite degree in Γ. Theorem 16 is now proved.

Corollary 5. *If a graph is type* 1 *traceable from a vertex whose degree is* 1 *or infinite, then the graph is semi-arbitrarily traceable from that vertex.*

Proof. Let V_1 denote the vertex. It follows from Theorem 16 and the degree of V_1 that the graph has infinitely many edges and is traceable from V_1 and every circuit of the graph contains a vertex of infinite degree in the graph. Therefore by Theorem 15 the graph is semi-arbitrarily traceable from V_1. This proves Corollary 5.

Corollary 6. *There are only two kinds of locally finite graphs which are type* 1 *traceable from the vertex* V_1, *apart from isolated vertices*:

1. *a* 1-*way infinite path starting with* V_1,
2. *the union of a* 1-*way infinite path starting with* V_1 *and a finite graph arbitrarily traceable from* V_1 *to* V_1, *such that the two graphs have no edge in common and every circuit of their union contains* V_1.

Proof. Let Γ denote a locally finite graph without isolated vertices which is type 1 traceable from the vertex V_1. It follows from Theorem 16 that Γ has a 1-way infinite Euler chain starting with V_1 and every circuit of Γ includes V_1. Let $\Xi^* = V_1, E_1, V_2, \ldots$ be an Euler chain of Γ starting with V_1. Only a finite number of the vertex-terms of Ξ^* coincide with V_1 because $d_\Gamma(V_1)$ is finite. Let V_p be the vertex-term of Ξ^* with largest suffix such that $V_p = V_1$ $(p \geq 1)$ and let $\Xi^{**} = V_p, E_p, V_{p+1} \ldots$.

From the definition of p, and since every circuit of Γ includes V_1, it follows that Ξ^{**} is the chain of a 1-way infinite path starting with V_1, say Π. If $p=1$ then $\Pi = \Gamma$. If $p>1$ then let Γ_1 denote the subgraph of Γ whose vertices and edges are $V_1, E_1, \ldots, V_p$. Then Γ_1 has edges, is Eulerian, and every circuit of Γ_1 includes V_1. Therefore by Ore's Theorem Γ_1 is arbitrarily traceable from V_1 to V_1. Also $\Gamma = \Pi \cup \Gamma_1$. Furthermore Π and Γ_1 have no edge in common, and by Theorem 16 every circuit of Γ contains V_1. This proves Corollary 6.

In what follows we will use

LEMMA 6. *If a graph is traceable from a vertex of infinite degree then it is traceable from every vertex of infinite degree.*

PROOF. Suppose that the graph Γ has an Euler chain starting with the vertex V_1, where $d_\Gamma(V_1)$ is infinite, say Ξ^*. Let W be any vertex of Γ such that $V_1 \neq W$ and $d_\Gamma(W)$ is infinite. The chain Ξ^* is the juxtaposition of an infinite sequence of segments of itself $\Xi_1, \Xi_2, \Xi_3, \ldots$ where $\Xi_1, \Xi_3, \Xi_5, \ldots$ are (VW)-chains and $\Xi_2, \Xi_4, \Xi_6, \ldots$ are (WV)-chains. Therefore

$$\Xi^{**} = \Xi_2, \Xi_1, \Xi_4, \Xi_3, \Xi_6, \Xi_5, \ldots$$

is an Euler chain of Γ starting with W. This proves Lemma 6.

DEFINITION 14. A graph will be called *locally infinite traceable* if it is traceable from a vertex and each of its vertices has infinite degree.

REMARK 4. A graph without isolated vertices is locally infinite traceable if and only if it is enumerably infinite, and remains connected when any finite set of edges is deleted from it. For if $V_1, E_1, V_2, E_2, \ldots$ is an Euler chain of a locally infinite graph Γ and $\mathscr{E}$ is any finite subset of the edges of Γ, then $V_m, E_m, V_{m+1}, E_{m+1}, \ldots$ with m chosen so that $\mathscr{E} \subseteqq \{E_1, \ldots, E_{m-1}\}$, is a chain of $\Gamma - \mathscr{E}$, and it contains every vertex of Γ because Γ is locally infinite, so $\Gamma - \mathscr{E}$ is connected. Conversely, if Γ is an enumerably infinite graph which remains connected when any finite set of edges is deleted from it, then let V_1 be any vertex and Ξ any $(W_1 W_n)$-chain of Γ. $\Gamma - \mathscr{E}(\Xi)$ contains a $(V_1 W_1)$-chain, Ξ_1 say. Enumerate $\mathscr{E}(\Gamma) - \mathscr{E}(\Xi) - \mathscr{E}(\Xi_1)$ as $E'_1, E'_2, E'_3, \ldots$. From the hypothesis $\Gamma - \mathscr{E}(\Xi) - \mathscr{E}(\Xi_1)$ contains a path Υ_1 starting with W_n and ending with E'_1 and the vertex X_1 say; if E'_i is the first of $E'_2, E'_3, \ldots$ in $\Gamma - \mathscr{E}(\Xi) - \mathscr{E}(\Xi_1) - \mathscr{E}(\Upsilon_1)$ then this graph contains a path Υ_2 starting with X_1 and ending with E'_i and the vertex X_2 say, etc. Thus Γ is type 3 and semi-arbitrarily traceable from each vertex. Obviously Γ is also locally infinite.

THEOREM 17. *A graph is type* 3 *traceable from the vertex* V_1 *if and only if it is locally infinite and traceable from* V_1, *apart from isolated vertices. It is then type* 3 *and semi-arbitrarily traceable from all its vertices except the isolated ones.*

PROOF. If the graph Γ is locally infinite and traceable from V_1, then by Remark 4, Γ is type 3 traceable from each vertex. Otherwise thus: Γ has a 1-way infinite Euler chain starting with V_1. By Theorem 16, Γ is type 1 traceable from V_1. Let Ξ be a $(W_1 W_n)$-chain of length >0 of Γ. Since $d_\Gamma(V_1)$ is infinite, by Lemma 3, $\Gamma - \mathscr{E}(\Xi)$ has a chain Ξ^* starting with V_1 which includes all but at most a finite number of the edges of $\Gamma - \mathscr{E}(\Xi)$. The chain Ξ^* includes W_1 because $d_\Gamma(W_1)$ is infinite. Therefore Ξ^* has a starting segment which is a $(V_1 W_1)$-chain, say Ξ_1. Then Ξ_1, Ξ is a finite chain of Γ starting with V_1. Now Γ has an Euler chain starting with V_1 which has Ξ_1, Ξ as a segment because Γ is type 1 traceable from V_1. Hence Γ is type 3 traceable from V_1.

Next we will prove that if the graph Γ is type 3 traceable from the vertex V_1 then it satisfies the conditions of the theorem. From Definition 13, Γ has an infinite Euler chain Ξ^* starting with V_1.

Now $d_\Gamma(V_1)$ is infinite. For otherwise Ξ^* has as a starting segment an open chain Ξ which contains all the edges of Γ incident with V_1. But then Γ has no chain starting with V_1 which has Ξ^{rev} as a segment, contrary to the hypothesis that Γ is type 3 traceable from V_1.

Suppose (reductio ad absurdum) that W is a vertex of Γ with non-zero finite degree. Of course $W \neq V_1$. Then Ξ^* has as a starting segment a closed chain Ξ' which contains all edges of Γ incident with W. But then Γ has no chain starting with V_1 which has as a segment the rotaion of Ξ' which starts and ends with W, contrary to the hypothesis that Γ is type 3 traceable from V_1. So every vertex of Γ except the isolated vertices has infinite degree.

Let X be any vertex of infinite degree in Γ. Then Γ is traceable from X by Lemma 6. Therefore by the part of Theorem 17 which is already proved Γ is type 3 traceable from X. By Corollary 5, Γ is therefore semi-arbitrarily traceable from X. Theorem 17 is now proved.

THEOREM 18. *A graph is type* 2 *traceable from the vertex* V_1 *if and only if it has infinitely many edges and is traceable from* V_1, *every circuit of the graph which does not include* V_1 *contains a vertex having infinite degree in the graph, and if the edges of any connected Eulerian subgraph are deleted then in the remaining graph every vertex which is not an isolated vertex of the original graph is accessible from* V_1.

The proof of Theorem 18 will use

LEMMA 7. *Let Γ be a graph with a 1-way infinite Euler chain starting with the vertex V_1 and let Ξ be an open (W_1W_n)-chain of Γ. If in $\Gamma-\mathscr{E}(\Xi)$ neither W_1 nor W_n is accessible from V_1 then W_1 and W_n belong to the same connected component of $\Gamma-\mathscr{E}(\Xi)$ and this connected component is finite and (W_1, W_n)-odd.*

PROOF. By Lemma 1 it follows that any finite chain of $\Gamma-\mathscr{E}(\Xi)$ starting with V_1 can be continued. Therefore $\Gamma-\mathscr{E}(\Xi)$ contains an infinite chain starting with V_1. Therefore by Lemma 3 the connected component of $\Gamma-\mathscr{E}(\Xi)$ to which V_1 belongs contains all but a finite number of the edges of Γ. But $d_\Gamma(W_1)$, $d_\Gamma(W_n)$ are not odd, hence $d_{\Gamma-\mathscr{E}(\Xi)}(W_1)$, $d_{\Gamma-\mathscr{E}(\Xi)}(W_n)$ are not even. Therefore W_1 and W_n belong to the same connected component of $\Gamma-\mathscr{E}(\Xi)$. This proves Lemma 7.

PROOF OF THEOREM 18. First we will prove that if the graph Γ satisfies the conditions of the theorem then it is type 2 traceable from the vertex V_1. By Theorem 16, Γ is type 1 traceable from V_1.

Let Ξ be any closed (W_1W_1)-chain of Γ of length >0. In $\Gamma-\mathscr{E}(\Xi)$ the vertex W_1 is accessible from V_1 by hypothesis, so let Ξ_1 denote a (V_1W_1)-chain of $\Gamma-\mathscr{E}(\Xi)$. (If $V_1=W_1$ then $\Xi_1=V_1$ of course.) Ξ_1,Ξ is a finite chain of Γ starting with V_1. Since Γ is type 1 traceable from V_1 by Theorem 16, therefore Γ has an Euler chain starting with V_1 which has Ξ_1,Ξ as a segment.

Let Ξ be any open (W_1W_n)-chain of Γ. In $\Gamma-\mathscr{E}(\Xi)$ either W_1 or W_n is accessible from V_1. For if neither W_1 nor W_n is accessible from V_1 in $\Gamma-\mathscr{E}(\Xi)$ then by Lemma 7 $\Gamma-\mathscr{E}(\Xi)$ contains a (W_1W_n)-path, say Π, therefore $\mathscr{E}(\Xi)\cup\mathscr{E}(\Pi)$ constitutes the set of edges of an Eulerian subgraph of Γ and in $\Gamma-\mathscr{E}(\Xi)-\mathscr{E}(\Pi)$ neither W_1 nor W_n is accessible from V_1, which is contrary to hypothesis. So in $\Gamma-\mathscr{E}(\Xi)$ either W_1 or W_n is accessible from V_1. It follows easily, since Γ is type 1 traceable from V_1, that Γ has an Euler chain starting with V_1 which has Ξ or Ξ^{rev} as a segment. This proves that if Γ satisfies the conditions of the theorem then Γ is type 2 traceable from V_1.

Next we will prove that if the graph Γ is type 2 traceable from the vertex V_1 then it satisfies the conditions of the theorem. From Definition 13, Γ has infinitely many edges and is traceable from V_1. Now Γ is type 1 traceable from V_1 by Lemma 5, therefore every circuit of Γ which does not include V_1 contains a vertex having infinite degree in Γ.

Suppose (reductio ad absurdum) that if the edges of the connected Eulerian subgraph Φ of Γ are deleted from Γ then the connected component of Γ to which V_1 belongs becomes disconnected. Of course $\mathscr{E}(\Phi) \neq \emptyset$. Every connected component of the abovementioned graph has a vertex in common with Φ. Therefore there is a vertex X_1 of Φ such that in $\Gamma - \mathscr{E}(\Phi)$, X_1 is not accessible from V_1. Then if Ξ denotes a closed Euler chain of Φ starting and ending with X_1, there is no (V_1X_1)-chain in $\Gamma - \mathscr{E}(\Xi)$, hence Γ has no chain starting with V_1 which has Ξ as a segment. This contradicts the hypothesis that Γ is type 2 traceable from V_1. Theorem 18 is now proved.

An alternative form of Theorem 18 is

THEOREM 18′. *A graph without isolated vertices is type 2 traceable from the vertex V_1 if and only if it has infinitely many edges and is traceable from V_1, every circuit of the graph which does not include V_1 contains a vertex having infinite degree in the graph, and if the edges of any connected Eulerian subgraph are deleted then the remaining graph has no isolated vertex.*

This follows directly from the following

LEMMA 8. *If a (finite or infinite) graph has an Euler chain starting with the vertex V_1, and if it has an Eulerian subgraph such that when its edges are deleted then the remaining graph is disconnected, then the graph has a connected Eulerian subgraph such that when its edges are deleted then the remaining graph has an isolated vertex.*

PROOF. If the graph has a closed Euler chain then deleting its edges leaves only isolated vertices. Suppose that Γ is a graph without an isolated vertex which has an open or an infinite Euler chain starting with V_1, and that Φ is an Eulerian subgraph of Γ such that $\Gamma - \mathscr{E}(\Phi)$ is disconnected.

If there is an isolated vertex W in $\Gamma - \mathscr{E}(\Phi)$ then, in case Φ is disconnected, W is an isolated vertex of the graph obtained by deleting from Γ the edges of the connected component of Φ to which W belongs. Lemma 8 is therefore true if there is an isolated vertex in $\Gamma - \mathscr{E}(\Phi)$. In what follows we assume that this is not the case.

The connected components of $\Gamma - \mathscr{E}(\Phi)$ which do not contain V_1, are finite in number and Eulerian. For if $\mathscr{E}(\Gamma)$ is finite then V_1 and just one other vertex have odd degree in $\Gamma - \mathscr{E}(\Phi)$, and they both belong to the same connected component of $\Gamma - \mathscr{E}(\Phi)$, and if $\mathscr{E}(\Gamma)$ is infinite then by Lemmas 2 and 3 applied to $\Gamma - \mathscr{E}(\Phi)$, all but a finite number of edges belong to the connected component containing V_1, also no vertex other than V_1 has odd degree in $\Gamma - \mathscr{E}(\Phi)$. This proves the assertion.

Let Λ be a connected component of $\Gamma - \mathscr{E}(\Phi)$ not containing V_1. The vertices of Λ are in Γ incident only with a subset of $\mathscr{E}(\Lambda \cup \Phi)$, and by the above Λ is Eulerian and contains edges. The union of Λ and all those connected components of Φ which have a vertex in common with Λ is a connected Eulerian subgraph of Γ, and when its edges are deleted then in the remaining graph the vertices of Λ are isolated. This proves Lemma 8.

The type 2 traceable graphs can be characterised in another way, using the second part of the following

THEOREM 19. *There is only one category of finite graphs without isolated vertices which have an Euler chain starting with a vertex X_1 and are such that if the edges of any connected Eulerian subgraph are deleted then the remaining graph does not have an isolated vertex, namely all those graphs consisting of the distinct vertices $X_1, \ldots, X_n$, where $n \geqq 2$, and an odd number of (X_iX_{i+1})-edges for $1 \leqq i \leqq n-1$. If the edges of any Eulerian subgraph are deleted, then the remaining graph is in the same category.*

There are only three categories of infinite graphs with the above properties, namely

I. *All locally infinite traceable graphs.*

II. *All those which consist of the infinite set of distinct vertices $X_1, X_2, X_3, \ldots$ together with an odd number of (X_iX_{i+1})-edges for each $i \geqq 1$, and no other edges.*

III. *All those which consist of the distinct vertices $X_1, \ldots, X_n$, where $n \geqq 1$, and an odd number of (X_iX_{i+1})-edges for $1 \leqq i \leqq n-1$ if $n > 1$, together with a locally infinite traceable graph not containing any of $X_1, \ldots, X_n$, and an odd number of edges joining X_n to vertices of the locally infinite traceable graph, and no other edges.*

If we delete from any of these graphs any finite set of edges, such that each vertex having finite degree in the graph is incident with an even number of edges of the set, then the remaining graph is in the same category as the original one.

The proof of Theorem 19 will use the following four lemmas.

LEMMA 9. *If a graph has no isolated vertex and is traceable from the vertex V_1, then if any finite set of edges whose endvertices all have infinite degree in the graph are deleted or added, the resulting graph always has an Euler chain starting with V_1 which includes every vertex.*

PROOF. Let Γ denote the graph and let E be a (YZ)-edge of Γ such that $d_\Gamma(Y)$ and $d_\Gamma(Z)$ are infinite. Let Ξ^* be an Euler chain of Γ starting with V_1. Then Ξ^* includes every vertex of Γ and Ξ^* is the juxtaposition of an infinite sequence of finite segments of itself $\Xi_0, \Xi_1, \Xi_2, \Xi_3, \ldots$ where Ξ_0 is a (V_1Y)-chain, $\Xi_1, \Xi_3, \Xi_5, \ldots$ are (YZ)-chains, $\Xi_2, \Xi_4, \Xi_6, \ldots$ are (ZY)-chains, and for some value of a $(a \geqq 1)$, $\mathscr{E}(\Xi_a) = E$. Then

$$\Xi_0, \ldots, \Xi_{a-1}, \Xi_{a+2}, \Xi_{a+1}, \Xi_{a+4}, \Xi_{a+3}, \ldots$$

is an Euler chain of $\Gamma - E$ starting with V_1, which includes every vertex of Γ.

Y' and Z' being any two distinct vertices having infinite degree in Γ, let a new $(Y'Z')$-edge E' be added to Γ. The chain Ξ^* is the juxtaposition of an infinite sequence of finite segments of itself $\Xi'_0, \Xi'_1, \Xi'_2, \Xi'_3, \ldots$ where Ξ'_0 is an (V_1Y')-chain, $\Xi'_1, \Xi'_3, \Xi'_5, \ldots$ are $(Y'Z')$-chains, and $\Xi'_2, \Xi'_4, \Xi'_6, \ldots$ are $(Z'Y')$-chains. Let Ξ' denote the chain Y', E', Z'. Then $\Xi'_0, \Xi', \Xi'_2, \Xi'_1, \Xi'_4, \Xi_3', \Xi'_6, \Xi'_5, \ldots$ is an Euler chain of $\Gamma \cup \{E'\}$ starting with V_1 which includes every vertex of Γ.

The truth of Lemma 9 follows by iteration.

REMARK 5. It can be shown that if an enumerably infinite number of edges whose end-vertices all have infinite degree in the original graph are added, then the resulting graph always has an Euler chain starting with V_1 which includes every vertex. This result will not be required in the sequel, however.

LEMMA 10. *If Ω is a locally infinite traceable graph and Ω' is any graph obtained from Ω by adding a vertex A not belonging to Ω and an odd number of edges joining A to vertices of Ω, then Ω' is semi-arbitrarily and type* 2 *traceable from A and is not traceable from any other vertex.*

PROOF. Ω' is not traceable from any vertex other than A because $d_{\Omega'}(A)$ is odd. We will show that Ω' is traceable from A by induction over $d_{\Omega'}(A)$. If $d_{\Omega'}(A) = 1$ then Ω' is traceable from A because by Lemma 6, Ω is traceable from each of its vertices. Let $d_{\Omega'}(A) = 2\varkappa + 1$, where $\varkappa \geqq 1$, and suppose that if the degree of A is $2\varkappa - 1$ then the graph is traceable from A. If in Ω' two edges incident with A are incident with the same vertex of Ω then Ω' is clearly traceable from A on the basis of the induction hypothesis. The alternative is that two edges F and G are incident with the two distinct vertices C and D of Ω, respectively. In this case let Ξ^* be an Euler chain of $\Omega' - F - G$ starting with A. Since C and D have infinite degree, Ξ^* is the juxtaposition of an infinite sequence of finite segments $\Xi_0, \Xi_1, \Xi_2, \Xi_3, \ldots$ of itself, where Ξ_0 is an (AC)-chain, $\Xi_1, \Xi_3, \Xi_5, \ldots$ are (CD)-chains and $\Xi_2, \Xi_4, \Xi_6, \ldots$ are (DC)-chains. Then

$$\Xi_0, F, A, G, \Xi_2, \Xi_1, \Xi_4, \Xi_3, \Xi_6, \Xi_5, \ldots$$

is an Euler chain of Ω' starting with A. It follows from Theorems 15 and 18′ that Ω' is semi-arbitrarily and type 2 traceable from A. This proves Lemma 10.

REMARK 6. In fact, if Ω is any graph with a 1-way infinite Euler chain and Ω' is any graph obtained from Ω by adding a vertex A not belonging to Ω and an odd or enumerably infinite number of edges joining A to vertices of Ω so that A is joined to the vertex having odd valency in Ω (if there is one) by an odd number of edges, and to each vertex having even valency in Ω (if there are any) by an even number of edges, then Ω' is traceable from A. Also, if an even number of edges join A to the vertices of Ω, and to the vertex having odd valency in Ω (if there is one), then Ω' is traceable from the vertex having odd valency in Ω (if there is one), or from every vertex having infinite valency, as the case may be, but Ω' is not traceable from A. These results will not be required in the sequel, however.

LEMMA 11. *If a graph Γ (finite or infinite) is traceable from a vertex, and if deleting from Γ the edges of any connected Eulerian subgraph of Γ never results in a graph with an isolated vertex, then every circuit of length >2 in Γ has a vertex having infinite degree in Γ.*

PROOF. Suppose on the contrary (reductio ad absurdum) that Θ is a circuit of length >2 contained in Γ and every vertex of Θ has finite degree in Γ. Let $\Xi^* = V_1, E_1, V_2, \ldots$ be an Euler chain of Γ. Let V_i and V_j, respectively, be the vertex terms of Ξ^* with smallest and largest suffix belonging to Θ, and let $\Xi = V_i, E_i, \ldots, V_j$. Let Φ denote the subgraph of Γ formed by the vertices and edges of Ξ. Every vertex of Θ belongs to Φ, and also every edge of Γ incident with every vertex of Θ, except possibly E_{i-1} and E_j.

Now $V_i \neq V_j$. For if $V_i = V_j$, then Φ is Eulerian and connected and in $\Gamma - \mathscr{E}(\Phi)$ every vertex of Θ other than V_i is isolated, which is contrary to hypothesis.

From $V_i \neq V_j$ it follows that Φ is (V_iV_j)-odd. Let Λ denote a shortest (V_iV_j)-path contained in Θ, then Θ has a vertex W not in Λ. Clearly $\Phi - \mathscr{E}(\Lambda)$ is Eulerian. If we delete from Γ the edges of the connected component of $\Phi - \mathscr{E}(\Lambda)$ to which W belongs, then in the remaining graph

W is isolated. This is contrary to hypothesis, therefore Lemma 11 is proved.

REMARK 7. The converse of Lemma 11 is not true of course.

LEMMA 12. *If a (finite or infinite) graph Γ with an edge is traceable from a vertex V_1 of odd degree, if deleting from Γ the edges of any connected Eulerian subgraph never results in a graph with an isolated vertex, and if Γ contains no circuit of length >2, then V_1 is joined to only one vertex of Γ.*

PROOF. Clearly Γ does not have a closed Euler chain. Lemma 12 is obviously true if $d_\Gamma(V_1)=1$. Suppose that $d_\Gamma(V_1)>1$ and let $\Xi^*= V_1,E_1,V_2,\ldots$ be an Euler chain of Γ, and let V_δ be the vertex-term of Ξ^* with largest suffix which coincides with V_1; $\delta \geqq 3$ because $d_\Gamma(V_1)>1$. Let

$$\Xi = V_1,E_1,\ldots,V_\delta \quad \text{and} \quad \Xi^{**} = V_\delta,E_\delta,V_{\delta+1},\ldots .$$

Ξ^{**} has an edge because Γ does not have a closed Euler chain. Then $V_i \neq V_1$ for $i \geqq \delta+1$, and Ξ contains every vertex joined to V_1 except possibly $V_{\delta+1}$. Now

$$\{V_2,\ldots,V_{\delta-1}\} \subseteqq \{V_\delta,V_{\delta+1},V_{\delta+2},\ldots\},$$

for otherwise one of $V_2,\ldots,V_{\delta-1}$ would be isolated in $\Gamma-\mathscr{E}(\Xi)$. Therefore the vertices joined to V_1 in Γ all belong to Ξ^{**}. Suppose that Lemma 12 does not hold for Γ (reductio ad absurdum) and that V_1 is joined to a vertex Z of Ξ^{**} by an edge E_Z, where $Z\neq V_{\delta+1}$. Clearly $E_Z \notin \Xi^{**}$, because in the subgraph of Γ consisting of the vertices and edges of Ξ^{**} the only edge incident with V_1 is E_δ. Thus this graph contains a (V_1Z)-path of length >1. Such a path and E_Z together constitute a circuit of length >2, whereas Γ contains no circuit of length >2. This contradiction shows that V_1 is joined to no vertex different from $V_{\delta+1}$ in Γ. Lemma 12 is now proved.

PROOF OF THEOREM 19. To prove the part of the theorem concerning finite graphs we first verify that each graph in the category described has the properties stated in the theorem. It clearly has an Euler chain starting with X_1 and has no isolated vertex. Any Eulerian subgraph contains an even number of (X_iX_{i+1})-edges for $1\leqq i\leqq n-1$, therefore if the edges of such a subgraph are deleted then the remaining graph is in the same category. Next we show that if Γ is any graph having the properties stated in the theorem then Γ belongs to the category described. Γ does not have a closed Euler chain obviously. Therefore Γ has an open

Euler chain starting with X_1. Hence, by Lemma 4, Γ is the union of a path Π of length $\geqq 1$ having X_1 as an end-vertex, and possibly some circuits, no two of this system having an edge in common. Every such path Π contains every vertex of Γ, because if W is a vertex of Γ not in Π, then in the graph obtained by deleting from Γ all the edges of all the above circuits containing W, this vertex is isolated, which is contrary to hypothesis. Let the vertices of Π in order be $X_1, \ldots, X_n$, where $n \geqq 2$. By Lemma 11, X_i and X_j are not joined by an edge if $i-j>1$. Also Γ is (X_1X_n)-odd. It follows directly that Γ contains an odd number of (X_iX_{i+1})-edges for $1 \leqq i \leqq n-1$. The first part of Theorem 19 is now proved.

To prove the part of the theorem concerning infinite graphs we first verify that every graph in each of the categories I, II and III has the properties stated in the theorem. If a graph is locally infinite traceable then it has no isolated vertices, and by Lemma 9 if any finite number of edges are deleted from it then the remaining graph is locally infinite traceable. A graph in category II obviously has no isolated vertices and has an infinite Euler chain starting with X_1. If any finite set of edges as described at the end of Theorem 19 are deleted, then the remaining graph contains an odd number of (X_iX_{i+1})-edges for $i=1,2,3,\ldots$ and is therefore in category II. A graph in category III obviously has no isolated vertex, and by Lemma 10 it has an infinite Euler chain starting with X_1. Let Γ denote the graph, and Ω the locally infinite traceable graph, and Ω' the graph consisting of Ω and X_n together with the edges joining X_n to vertices of Ω. Assume first that $n=1$, that is $\Omega'=\Gamma$. If any finite set of edges as described at the end of Theorem 19 are deleted from Γ, then by Lemma 9 the remaining graph consists of a locally infinite traceable graph Ω^* together with X_n and an odd number of edges joining X_n to the vertices of Ω^*, and therefore it is in category III. Assume secondly that $n>1$. Any finite set of edges as described at the end of Theorem 19 contains an even number of (X_iX_{i+1})-edges for $i=1,\ldots,n-1$, and an even number of edges joining X_n to vertices of Ω. From this and Lemma 9 it follows that when any such set of edges is deleted, the remaining graph is in category III. It has now been proved that every graph in each of the categories I, II, III has the properties mentioned in Theorem 19.

To complete the proof of the theorem it remains to show that if Γ is any infinite graph having the properties stated in the first paragraph of the theorem, then Γ belongs to one of the three categories described. We note that since Γ is infinite and without isolated vertices, and has an Euler chain, therefore $\mathscr{E}(\Gamma)$ is enumerably infinite.

One of the following three possibilities holds: (i) $d_\Gamma(X_1)$ is infinite, (ii) the degree of every vertex of Γ is finite, (iii) $d_\Gamma(X_1)$ is finite and a vertex of Γ has infinite degree.

Suppose that (i) holds. In this case every vertex of Γ has infinite degree. For otherwise let Y denote a vertex of finite degree. Because $d_\Gamma(X_1)$ is infinite, any Euler chain of Γ starting with X_1 has as a starting segment a closed chain Ξ_Y which includes every edge of Γ incident with Y. Then in $\Gamma - \mathscr{E}(\Xi_Y)$ the vertex Y is isolated, which is contrary to the hypothesis. This proves that if (i) holds then Γ is in category I.

Suppose that (ii) holds. Then by Lemma 11 the graph Γ contains no circuit of length >2. Therefore by Lemma 12 the vertex X_1 is joined to only one vertex of Γ, say X_2. Obviously there are an odd number of (X_1X_2)-edges in Γ. It follows that $\Gamma - X_1$ is traceable from X_2. If the edges of any Eulerian subgraph Φ of $\Gamma - X_1$ are deleted from $\Gamma - X_1$ then the remaining graph has no isolated vertex, for any isolated vertex would be different from X_2 (because $d_{\Gamma - X_1}(X_2)$ is odd), and therefore, because X_1 is joined only to X_2 in Γ, it would be an isolated vertex of $\Gamma - \mathscr{E}(\Phi)$, contrary to hypothesis. $\Gamma - X_1$ contains no circuit of length >2 because Γ does not. Therefore by Lemma 12 applied to $\Gamma - X_1$ with X_2 in place of V_1, X_2 is joined to only one vertex of $\Gamma - X_1$, say X_3. It is easy to see that there are an odd number of (X_2X_3)-edges in Γ. It follows that $\Gamma - X_1 - X_2$ is traceable from X_3. By a repetition of the argument just given we have that X_3 is joined to only one vertex of $\Gamma - X_1 - X_2$, say X_4, and there are an odd number of (X_3X_4)-edges in Γ. This reasoning can be repeated without end, proving that Γ belongs to category II if (ii) holds.

Suppose that (iii) holds. Let $\Xi^* = V_1, E_1, V_2, \ldots$ be an Euler chain of Γ with $V_1 = X_1$, and suppose that the vertex-term of Ξ^* with smallest suffix having infinite degree in Γ is V_{p+1}, here $p \geqq 1$ because $d_\Gamma(V_1)$ is finite. Let $\Xi = V_1, E_1, \ldots, V_p$ (if $p=1$ then $\Xi = V_1$) and let $\Xi^{**} = V_{p+1}, E_{p+1}, V_{p+2}, E_{p+2}, \ldots$.

$V_1, \ldots, V_p$ are the only vertices of Γ having finite degree. For if not, then there is a vertex-term V_q such that $q > p+1$, which has even degree and is different from each of $V_1, \ldots, V_{p+1}$. Because $d_\Gamma(V_{p+1})$ is infinite, Ξ^{**} has as a starting segment a closed chain Ξ_q which contains every edge of Γ incident with V_q, so that V_q is isolated in $\Gamma - \mathscr{E}(\Xi_q)$, contrary to hypothesis. This contradiction shows that $V_1, \ldots, V_p$ are the only vertices having finite degree in Γ.

If $p=1$ then X_1 is the only vertex of finite degree in Γ, of course $d_\Gamma(X_1)$ is odd. Let V_r be the vertex-term with largest suffix in Ξ^* such that $V_r = V_1 = X_1$, here $r \geqq 1$. The subgraph Γ_r of Γ whose vertices and edges are $V_{r+1}, E_{r+1}, V_{r+2}, E_{r+2}, \ldots$ is a locally infinite traceable graph. Γ_r includes every vertex of Γ except X_1 because Γ_r includes every edge of Γ except $E_1, \ldots, E_r$, and $V_1 = X_1$ is the only vertex having finite degree in Γ. Let Ω be the graph obtained by adding to Γ_r those of $E_1, \ldots, E_r$ both of whose end-vertices have infinite degree in Γ, possibly $\Omega = \Gamma_r$. By Lemma 9, Ω is locally infinite traceable. Γ is obtained from Ω by adding the vertex X_1 and an odd number of edges joining X_1 to vertices of Ω. Therefore Γ is in category III when $p=1$.

In what follows $p>1$ will be assumed.

As $d_\Gamma(V_{p+1})$ is infinite, Ξ^{**} has as a starting segment a closed chain Ξ' which includes every edge of Γ incident with each of $V_1, \ldots, V_p$ other than $E_1, \ldots, E_p$, if any; if the only edges of Γ incident with $V_1, \ldots, V_p$ are $E_1, \ldots, E_p$ then for convenience we write $\Xi' = \emptyset$. Of course $\mathscr{E}(\Xi) \cap \mathscr{E}(\Xi') = \emptyset$.

Now $V_1 \neq V_p$. For suppose not (reductio ad absurdum). Then $\mathscr{E}(\Xi) \cup \mathscr{E}(\Xi')$ is the set of edges of an Eulerian subgraph of Γ. In $\Gamma - \mathscr{E}(\Xi) - \mathscr{E}(\Xi')$ the only edge incident with $V_1, \ldots, V_p$ is E_p. Now E_p is a $(V_p V_{p+1})$-edge, and $V_{p+1} \neq V_1, \ldots, V_p$ because $d_\Gamma(V_{p+1})$ is infinite while $d_\Gamma(V_1), \ldots, d_\Gamma(V_p)$ are finite. Therefore in $\Gamma - \mathscr{E}(\Xi) - \mathscr{E}(\Xi')$ the vertices $V_2, \ldots, V_{p-1}$ are isolated. It follows by Lemma 8 that Γ has a connected Eulerian subgraph such that when its edges are deleted from Γ the remaining graph has an isolated vertex, which is contrary to hypothesis. This contradiction proves that $V_1 \neq V_p$.

Let Γ_p denote the subgraph of Γ consisting of $V_1, E_1, \ldots, V_p$. Then Γ_p is $(V_1 V_p)$-odd because $V_1 \neq V_p$, and therefore by Lemma 4 Γ_p is the union of a $(V_1 V_p)$-path, say Υ, and a set $\mathscr{S}$ (possibly empty) of circuits, no two of this system of path and circuits having an edge in common. The path Υ contains every vertex of Γ_p. For suppose not (reduction ad absurdum), and let W be a vertex of Γ_p not belonging to Υ. Of course $W \neq V_1$ and $W \neq V_p$. Let Φ denote the union of the circuits of $\mathscr{S}$, then Φ is Eulerian, and therefore $\mathscr{E}(\Phi) \cup \mathscr{E}(\Xi')$ is the set of edges of an Eulerian subgraph of Γ, say Δ, since $\mathscr{E}(\Phi) \subset \mathscr{E}(\Xi)$ and $\mathscr{E}(\Xi) \cap \mathscr{E}(\Xi') = \emptyset$. In $\Gamma - \mathscr{E}(\Delta)$ the set of edges incident with $V_1, \ldots, V_p$ is $\{E_p\} \cup \mathscr{E}(\Upsilon)$, therefore W is an isolated vertex. By Lemma 8 this is again contrary to hypothesis. This contradiction proves that Υ contains all of $V_1, \ldots, V_p$.

Let the vertices of Υ in order along Υ be $X_1, \ldots, X_n$, where $X_1 = V_1$ and $X_n = V_p$, and of course $2 \leqq n \leqq p$. By Lemma 11, Γ contains no $(X_g X_h)$-edge with $1 \leqq g < h \leqq n$ and $h - g > 1$.

The chain Ξ^* has a vertex term V_r with $r \geqq p+1$ such that $V_1, E_1, V_2, E_2, \ldots, V_r$ includes all the edges of Ξ and of Ξ' and $V_r = V_{p+1}$, for $d_\Gamma(V_{p+1})$ is infinite. Let $\Xi^{***} = V_r, E_r, V_{r+1}, E_{r+1}, \ldots$. The vertices of

Ξ^{***} are precisely all the vertices of Γ other than $X_1,\ldots,X_n$, because Ξ^{***} includes every vertex having infinite degree in Γ and $X_1,\ldots,X_n$ are the only vertices having finite degree in Γ and Ξ^{***} includes no edge incident with any of $X_1,\ldots,X_n$. Let Γ_r denote the subgraph of Γ whose vertices and edges are $V_r, E_r, V_{1+1}, E_{r+1},\ldots$. Then Γ_r is locally infinite traceable because it includes every edge of Γ except $E_1, E_2,\ldots,E_{r-1}$. Let Ω denote the subgraph of Γ obtained by adding to Γ_r all those edges among $E_1,\ldots,E_{r-1}$ both of whose end-vertices have infinite degree in Γ (all vertices having infinite degree in Γ belong to Γ_r); possibly $\Omega=\Gamma_r$. By Lemma 9, Ω is locally infinite traceable. Clearly Ω includes every edge of Γ except all those with one of $X_1,\ldots,X_n$ as end-vertex.

None of $X_1,\ldots,X_{n-1}$ are joined to any vertex of Ω. For suppose the contrary (reduction ad absurdum). Then Γ contains an edge E joining X_k to V_s with $1 \leqq k \leqq n-1$ and $s \geqq r$. Clearly $E \in \Xi'$. From the definition of Δ we may assume that

$$\mathscr{E}(\Delta) = \{E_1,\ldots,E_{p-1},E_{p+1},\ldots,E_{r-1}\} - \mathscr{E}(\Upsilon)\,.$$

Therefore if in $\mathscr{E}(\Delta)$ E is replaced by the edges of Υ between X_k and X_n together with E_p, and, if $V_s \neq V_r$, also $E_r,\ldots,E_{s-1}$, then the result is the set of edges of an Eulerian subgraph of Γ, say Δ^+. In $\Gamma - \mathscr{E}(\Delta^+)$ the only edges incident with $X_1,\ldots,X_n$ are E and, if $k>1$, the edges of Υ between X_1 and X_k. Therefore $X_{k+1},\ldots,X_n$ are isolated in $\Gamma - \mathscr{E}(\Delta^+)$. By Lemma 8 this is again contrary to hypothesis. This contradiction proves that none of $X_1,\ldots,X_{n-1}$ are joined to any vertex of Ω.

Γ is traceable from X_1, therefore $d_\Gamma(X_1)$ is odd and $d_\Gamma(X_2),\ldots,d_\Gamma(X_{n-1})$ are all even. X_1 is not joined to any vertex other than X_2 by what has been proved above. Therefore Γ contains an odd number of (X_1X_2)-edges, and no further edges incident with X_1. If $n=2$ then an odd number of edges join X_2 to vertices of Ω since $d_\Gamma(X_2)$ is even. If $n>2$ then by what has been proved above X_2 is not joined to any vertex other than X_1 and X_3, also $d_\Gamma(X_2)$ is even, therefore Γ contains an odd number of (X_2X_3)-edges. By repetition of this argument we have that the only edges incident with $X_1,\ldots,X_n$ are an odd number of (X_iX_{i+1})-edges for $1 \leqq i \leqq n-1$ and an odd number of edges joining X_n to vertices of Ω. Therefore Γ belongs to category III also if $p>1$.

It has now been proved that if Γ is any infinite graph having the properties described in the first paragraph of Theorem 19 then Γ belongs to one of the three categories enumerated in Theorem 19. This completes the proof of Theorem 19.

From Theorems 18 and 19 there follows directly the characterisation of type 2 traceable graphs given in Theorem 20. Of course Theorem 20 can also be proved directly, without using Theorem 19.

THEOREM 20. *There are only three categories of graphs without isolated vertices which are type 2 traceable from a vertex* X_1, *namely*

I. *All locally infinite traceable graphs.*

II. *All those which consist of the infinite set of distinct vertices* $X_1, X_2, X_3, \ldots$ *together with an odd number of* (X_1X_2)*-edges and just one* (X_iX_{i+1})*-edge for* $i=2,3,4,\ldots$, *and no other edges.*

III. *All those which consist of the distinct vertices* $X_1,\ldots,X_n$, *where* $n \geqq 1$, *and an odd number of* (X_1X_2)*-edges if* $n>1$, *and just one* (X_iX_{i+1})*-edge for* $2 \leqq i \leqq n-1$ *if* $n>2$, *together with a locally infinite traceable graph not containing any of* $X_1,\ldots,X_n$, *and an odd number of edges joining* X_n *to the vertices of the locally infinite traceable graph, and no other edges.*

Next we will establish some properties of the above kinds of traceable graphs.

THEOREM 21. *If we delete from a graph which is semi-arbitrarily/type 1/type 2/type 3 traceable from a vertex any finite set of edges such that each vertex having finite degree in the graph is incident with an even number of edges of the set, then the remaining graph is semi-arbitrarily/type 1/type 2/type 3 traceable from that vertex.*

PROOF. For type 3 traceable graphs this follows directly from Lemma 9 and for type 2 traceable graphs from Theorems 19 and 20. It only remains to prove Theorem 21 for semi-arbitrarily traceable graphs and for type 1 traceable graphs.

Let Γ denote the graph and $\mathscr{E}$ a set of edges as described in the theorem. The graph Γ has infinitely many edges, and is traceable from V_1, say. Therefore in $\Gamma-\mathscr{E}$ the degree of V_1 is odd or infinite and the degrees of the vertices other than V_1 are even or infinite. Therefore by Lemmas 2 and 3, $\Gamma-\mathscr{E}$ has a chain starting with V_1 which includes all but at most a finite number of the edges of $\Gamma-\mathscr{E}$. Among all such chains of $\Gamma-\mathscr{E}$ let Ξ_0 be one such that the number of edges of $\Gamma-\mathscr{E}$ not contained in it is minimal.

Ξ_0 is an Euler chain of $\Gamma-\mathscr{E}$. For suppose not (reductio ad absurdum) and let Γ_0 denote the graph composed of the edges of $\Gamma-\mathscr{E}-\mathscr{E}(\Xi_0)$ together with their end-vertices.

Γ_0 contains no circuit. For if Θ is a circuit of Γ_0 then by Theorem 15/16 either $V_1 \in \Theta$ or one of the vertices of Θ has infinite degree in

$\Gamma - \mathscr{E}$, and in the latter case Ξ_0 contains a vertex of Θ. Since Ξ_0 contains a vertex of Θ, therefore Ξ_0 can be augmented into a chain of $\Gamma - \mathscr{E}$ starting with V_1 and including the edges of Θ, which is contrary to the definition of Ξ_0. Hence Γ_0 contains no circuit.

It follows that Γ_0 is the union of a system $\mathscr{S}$ of paths of length $\geqq 1$ as described in Lemma 4. Let the paths of $\mathscr{S}$ be $\Upsilon_1, \ldots, \Upsilon_a$, where $a \geqq 1$, and let the end-vertices of Υ_i be X_i and Y_i for $i = 1, \ldots, a$. Here $X_1, Y_1, \ldots, X_a, Y_a$ are all distinct and constitute all the vertices having odd degree in $\Gamma - \mathscr{E} - \mathscr{E}(\Xi_0)$. Now $X_1, Y_1, \ldots, X_a, Y_a$ all have infinite degree in Γ, for if $d_\Gamma(V_1)$ is odd then $d_{\Gamma-\mathscr{E}}(V_1)$ is odd and therefore $d_{\Gamma-\mathscr{E}-\mathscr{E}(\Xi_0)}(V_1)$ is even, and if W is any vertex of Γ other than V_1 and $d_\Gamma(W)$ is even, then $d_{\Gamma-\mathscr{E}}(W)$ is even and therefore $d_{\Gamma-\mathscr{E}-\mathscr{E}(\Xi_0)}(W)$ is even, consequently all vertices which have finite degree in Γ have even degree in $\Gamma - \mathscr{E} - \mathscr{E}(\Xi_0)$. Hence $X_1, Y_1, \ldots, X_a, Y_a$ all have infinite degree in $\Gamma - \mathscr{E}$.

By the same reasoning as in the proof of Theorem 12 with Γ replaced by $\Gamma - \mathscr{E}$, it follows that Ξ_0 can be augmented into a chain of $\Gamma - \mathscr{E}$ starting with V_1 which includes the edges of Υ_1, which is contrary to the definition of Ξ_0. This contradiction proves that Ξ_0 is an Euler chain of $\Gamma - \mathscr{E}$.

If Γ is semi-arbitrarily traceable from V_1 then by Theorem 15 every circuit of Γ contains a vertex of infinite degree. Hence every circuit of $\Gamma - \mathscr{E}$ contains a vertex of infinite degree, and we have shown that $\Gamma - \mathscr{E}$ has a 1-way infinite Euler chain starting with V_1. It follows by Theorem 15 that $\Gamma - \mathscr{E}$ is semi-arbitrarily traceable from V_1.

If Γ is type 1 traceable from V_1 then by Theorem 16 every circuit of Γ which does not include V_1 contains a vertex of infinite degree, and we have shown that $\Gamma - \mathscr{E}$ has a 1-way infinite Euler chain starting with V_1. It follows by Theorem 16 that $\Gamma - \mathscr{E}$ is type 1 traceable from V_1. Theorem 21 is now proved.

Theorem 22. *If a graph is type* 1/*type* 2 *traceable from the vertex* V_1 *then it is semi-arbitrarily/semi-arbitrarily and type* 2 *traceable from every vertex of infinite degree if and only if the degree of* V_1 *is infinite.*

Proof. Let Γ denote the graph. If $d_\Gamma(V_1)$ is odd then Γ is not traceable from any vertex other than V_1.

Suppose that $d_\Gamma(V_1)$ is infinite. Then it follows from Theorem 16/17 that every circuit of Γ contains a vertex having infinite degree in Γ. Let W be any vertex having infinite degree in Γ. By Lemma 6, Γ is traceable from W. It now follows from Theorem 15 that Γ is semi-arbitrarily traceable from W. Furthermore if Γ is type 2 traceable from V_1 then it now follows from Theorem 18 that Γ is also type 2 traceable from W. This proves Theorem 22.

THEOREM 23. *Suppose that the graph Γ is type* 1 *traceable from the vertex V_1 and has no isolated vertices.*

Then if Ξ is any closed chain of Γ, there exists an Euler chain of Γ starting with V_1 which has Ξ or a rotation of Ξ as a segment.

If Ξ is any open (W_1W_n)-chain of Γ then either Γ has an Euler chain starting with V_1 which has Ξ or Ξ^{rev} as a segment, or $d_\Gamma(W_1)$ and $d_\Gamma(W_n)$ are even, and the edges of $\Gamma-\mathscr{E}(\Xi)$ which are not accessible from V_1 in $\Gamma-\mathscr{E}(\Xi)$ together with their end-vertices form a (W_1W_n)-path, and Ξ' being the chain of this path starting with W_n, Γ has an Euler chain starting with V_1 which has a segment a rotation of the closed chain Ξ,Ξ' starting with a vertex of Ξ which does not belong to Ξ'.

PROOF. Let $\Xi=V'_1,E'_1,\ldots,V'_n$, where $V'_1=V'_n$. Γ has a finite chain Ξ_1 which contains exactly one of the vertices of Ξ, V'_a say, and is a $(V_1V'_a)$-chain. Let Ξ_2 denote the chain $V'_aE'_a,\ldots,E'_{n-1},V'_1,E'_1,\ldots,E'_{a-1},V'_a$. If $a=1$ then $\Xi=\Xi_2$, otherwise Ξ_2 is a rotation of Ξ. In any case Ξ_1,Ξ_2 is a finite chain of Γ starting with V_1. Since Γ is type 1 traceable from V_1, Γ has an Euler chain starting with V_1 which has Ξ_1,Ξ_2 as a segment. This proves the first part of Theorem 23.

Now suppose that Ξ is an open (W_1W_n)-chain of Γ. If $\Gamma-\mathscr{E}(\Xi)$ has a (V_1W_1)-chain, say Ξ_1, then Ξ_1,Ξ is a chain of Γ starting with V_1, therefore Γ has an Euler chain starting with V_1, which has Ξ_1,Ξ as a segment. Similarly if $\Gamma-\mathscr{E}(\Xi)$ has a (W_1W_n)-chain then Γ has an Euler chain starting with V_1 which has the chain Ξ_1,Ξ^{rev} as a segment.

Suppose finally that $\Gamma-\mathscr{E}(\Xi)$ has no (V_1W_1)-chain and no (V_1W_n)-chain. Then by Lemma 1 and Lemma 3 the connected component of $\Gamma-\mathscr{E}(\Xi)$ to which V_1 belongs contains all but at most a finite number of the edges of Γ, and by Lemma 7, W_1 and W_n belong to the same finite connected component of $\Gamma-\mathscr{E}(\Xi)$, say Γ'. The degree in Γ of each vertex of each finite connected component of $\Gamma-\mathscr{E}(\Xi)$ is finite and therefore even, therefore $d_{\Gamma'}(W_1)$ and $d_{\Gamma'}(W_n)$ are odd, and all vertices of all finite connected components of $\Gamma-\mathscr{E}(\Xi)$ other than W_1 and W_n have even degree in $\Gamma-\mathscr{E}(\Xi)$. Now by Theorem 16 every circuit of Γ which does not include V_1 contains a vertex having infinite degree in Γ, so by Lemma 4 it follows that Γ' consists of a (W_1W_n)-path Π, and the finite connected components of $\Gamma-\mathscr{E}(\Xi)$ other than Γ' (if any) are isolated vertices. As Ξ' is the chain of Π starting with W_n, clearly Ξ,Ξ' is a closed chain of Γ. Also Γ contains a chain Ξ_1 which starts with V_1 and ends with a vertex W_a of Ξ, where $1 < a < n$, and contains no other vertex of Ξ and no vertex of Ξ'. Let Ξ_2 be the roation of Ξ, Ξ' starting with W_a. Then Ξ_1, Ξ_2 is a finite chain of Γ starting with V_1, and therefore Γ has an Euler chain starting with V_1 which has Ξ_1, Ξ_2 as a segment. Theorem 23 is now proved.

7. Arbitrarily traceable infinite directed graphs.

DEFINITION 15. Γ being a directed graph, any finite sequence $V_1, E_1, V_2, E_2, \ldots, E_{n-1}, V_n$ $(n \geqq 1)$ or 1-way infinite sequence $V_1, E_1, V_2, E_2, \ldots$, where the V's are vertices and the E's are edges of Γ, for each i E_i is a $(V_i \to V_{i+1})$-edge, and $E_i \neq E_j$ when $i \neq j$, is called a *directed chain* of Γ and is said to *start with* V_1, and to be *directed away from* V_1, and to be a $(V_1 \to V_n)$*-chain*. If E_i is a $(V_{i+1} \to V_i)$-edge for each i instead of being a $(V_i \to V_{i+1})$-edge then we have a directed chain *directed towards* V_1 which *ends with* V_1 and a $(V_n \to V_1)$*-chain*. The chain is *closed* if $V_1 = V_n$ and *open* if $V_1 \neq V_n$. A directed chain of a directed graph which includes every edge of the graph is called a *directed Euler chain* of the graph. A directed graph with infinitely many edges is said to be *traceable away from the vertex* V_1 if it has a directed Euler chain starting with and directed away from V_1 (which is necessarily 1-way infinte), and it is said to be *traceable towards* V_1 if it has a directed Euler chain ending with and directed towards V_1. Any chain starting with the vertex V and directed away from V which includes the edge E will be called a $(V \to E)$*-chain*, and if a graph has a $(E \leftarrow V)$-chain then we say that E is *accessible away from* V in the graph. Analogously for an $(E \to V)$*-chain*, and if a graph has a $(E \to V)$-chain then we say that E is *accessible towards* V.

DEFINITION 16. A directed graph will be called *arbitrarily traceable away from the vertex* V_1/*towards* V_1 if it has infinitely many edges and a directed Euler chain starting/ending with V_1 and directed away from/towards V_1, and if every infinite chain of the graph starting/ending with V_1 and directed away from/towards V_1 is a directed Euler chain of the graph. It will be called *semi-arbitrarily traceable away from* V_1/*towards* V_1 if it has infinitely many edges and if every finite directed chain of the graph starting/ending with V_1 and directed away from V_1/towards V_1 is a starting/ending segment of at least one directed Euler chain of the graph starting/ending with V_1 and directed away from V_1/towards V_1.

Corresponding to Theorem 11 we have

THEOREM 24. *A directed graph with infinitely many edges is arbitrarily traceable away from/towards the vertex V_1 if and only if it is the union of a 1-way infinite directed path starting/ending with V_1 and directed away from/towards V_1, and a set (possibly empty) of isolated vertices.*

PROOF. The obvious adaptation of the proof of Theorem 11.

REMARK 8. Because of the obvious symmetry between the two directions, in what follows only the case of paths and chains directed away from V_1 will be considered. Any closed directed chain which contains V_1 may be regarded both as directed away from and towards V_1, and if the degree of V_1 is infinite, so may any directed Euler chain.

Corresponding to Lemma 1 we have for a directed graph

LEMMA 13. *If V is a vertex of a closed directed chain Ξ of the directed graph Γ and if either $d_{o\Gamma}(V) > d_{i\Gamma}(V)$ or $d_{o\Gamma}(V)$ is infinite, then Γ contains an edge directed away from V and not belonging to Ξ. If Ξ' is an open $(X \to Y)$-chain of the directed graph Γ and if either $d_{o\Gamma}(Y) \geqq d_{i\Gamma}(Y)$ or $d_{o\Gamma}(Y)$ is infinite, then Γ contains an edge directed away from Y and not belonging to Ξ'.*

PROOF. The obvious adaptation of the proof of Lemma 1.

The directed analogue of Lemma 2 is

LEMMA 14. *If V is a vertex of the directed graph Γ and either $d_{o\Gamma}(V) > d_{i\Gamma}(V)$ or $d_{o\Gamma}(V)$ is infinite, and for each vertex $Z \neq V$ of Γ either $d_{o\Gamma}(Z) \geqq d_{i\Gamma}(Z)$ or $d_{o\Gamma}(Z)$ is infinite, then every finite chain of Γ starting with and directed away from V is a starting segment of a 1-way infinite directed chain of Γ.*

PROOF. By Lemma 13 every finite chain starting with and directed away from V can be continued indefinitely.

The directed analogue of Lemma 3 is

LEMMA 15. *Let Δ be any directed graph with a 1-way infinite Euler chain directed away from its starting vertex, let Δ' be obtained by deleting from Δ any finite set of edges, and let V be any vertex of Δ. Then either the set of edges accessible away from V in Δ' is finite, or Δ'*

has a chain starting with and directed away from V which includes all but at most a finite number of the edges of Δ. The second alternative holds for at least two vertices, and for all but at most a finite number of the vertices of Δ having non-zero degree.

PROOF. The obvious adaptation of the proof of Lemma 3.

The directed analogue of Lemma 4 is

LEMMA 16. *Any directed graph with a finite non-zero number of edges is the union of a finite set of cycles and/or directed paths of length $\geqq 1$ such that no two members of the set have an edge in common, and the starting/ending vertex of any path of the set is never the ending/starting vertex of another path of the set, together with a set (possibly empty) of isolated vertices. The vertices of the graph which are not balanced are precisely the end-vertices of the paths of the set.*

PROOF. The obvious adaptation of the proof of Lemma 4.

REMARK 9. From Lemma 16 there follows the known fact that if Γ is a directed graph with a finite number of edges then $\sum d_{o\Gamma}(V) = \sum d_{i\Gamma}(V)$, the summation being over all the vertices V of Γ.

The directed analogue of Theorem 12 is part (A) of

THEOREM 25. (A) *Suppose that the directed graph Γ has an infinite Euler chain starting with and directed away from the vertex V_1. Then if Ξ is any $(V_1 \to V_n)$-chain of length $\geqq 1$, the edges of $\Gamma - \mathscr{E}(\Xi)$ which are not accessible away from V_n in $\Gamma - \mathscr{E}(\Xi)$ (if any) constitute (with their two end-vertices) an Eulerian subgraph, Γ_Ξ say, of $\Gamma - \mathscr{E}(\Xi)$, and each vertex of Γ_Ξ is in Γ incident only with a subset of $\mathscr{E}(\Gamma_\Xi) \cup \mathscr{E}(\Xi)$ and is not accessible away from V_n in $\Gamma - \mathscr{E}(\Xi)$, and Γ has a directed chain Ξ^* such that Ξ^* has Ξ as a starting segment and includes every edge of $\Gamma - \mathscr{E}(\Xi)$ which is accessible away from V_n in $\Gamma - \mathscr{E}(\Xi)$; each connected component of Γ_Ξ (if any) contains a vertex of Ξ, and Γ_Ξ does not contain V_n or any vertex of Ξ^* which occurs beyond Ξ.*

(B) *If $\mathscr{F}$ is any finite subset of $\mathscr{E}(\Gamma)$ such that for each vertex having finite degree in Γ the number of edges of $\mathscr{F}$ directed away from it is equal to the number of edges of $\mathscr{F}$ directed towards it, then the edges of $\Gamma - \mathscr{F}$ which are not accessible away from V_1 in $\Gamma - \mathscr{F}$ (if any) constitute (with their two-end vertices) an Eulerian subgraph, $\Gamma_{\mathscr{F}}$ say, of $\Gamma - \mathscr{F}$, each vertex* *of $\Gamma_{\mathscr{F}}$ is in Γ incident only with a subset of $\mathscr{F} \cup \mathscr{E}(\Gamma_{\mathscr{F}})$, and is not accessible away from V_1 in $\Gamma - \mathscr{F}$, furthermore $\Gamma - \mathscr{F}$ has a chain starting with and directed away from V_1 which includes every edge of $\Gamma - \mathscr{F}$ except those which are not accessible away from V_1 in $\Gamma - \mathscr{F}$.*

PROOF OF (A). From Lemmas 14 and 15 it follows along the lines of the first part of the proof of Theorem 12 that Γ has a directed chain Ξ_0 which has Ξ as a starting segment and includes all but at most a finite number of the edges of Γ. If Ξ_0 is a directed Euler chain of Γ then Theorem 24 holds with $\Xi^* = \Xi_0$. In what follows suppose that Ξ_0 is not a directed Euler chain of Γ.

If $\Gamma - \mathscr{E}(\Xi_0)$ is Eulerian, then let $\Xi' = \Xi_0$. Otherwise $\Gamma - \mathscr{E}(\Xi_0)$ contains a finite set $\mathscr{S}$ of directed paths and/or cycles as described in Lemma 16, and $\mathscr{S}$ contains at least one directed path because $\Gamma - \mathscr{E}(\Xi_0)$ is not Eulerian. Let the directed paths of $\mathscr{S}$ be $\Upsilon_1, \ldots, \Upsilon_a$, where $a \geqq 1$, and for $i = 1, \ldots, a$ let Υ_i be a $(X_i \to Y_i)$-path where

$$\{X_1, \ldots, X_a\} \cap \{Y_1, \ldots, Y_a\} = \emptyset .$$

Clearly $d_\Gamma(X_1), d_\Gamma(Y_1), \ldots, d_\Gamma(X_a), d_\Gamma(Y_a)$ are all infinite. It follows as in the proof of Theorem 12 that Γ has a directed chain Ξ' such that Ξ' has Ξ as a starting segment and

$$\mathscr{E}(\Xi') = \mathscr{E}(\Xi_0) \cup \mathscr{E}(\Upsilon_1) \cup \ldots \cup \mathscr{E}(\Upsilon_a) ,$$

so that $\Gamma - \mathscr{E}(\Xi')$ is Eulerian.

The proof of Theorem 25 (A) can now be completed by an obvious adaptation of the last part of the proof of Theorem 12.

PROOF OF (B). If $\Gamma - \mathscr{F}$ has an Euler chain starting with and directed away from V_1 then (B) holds with $\Gamma_{\mathscr{F}} = \emptyset$. In the remainder of the proof it will be assumed that $\Gamma - \mathscr{F}$ has no such Euler chain.

$\Gamma - \mathscr{F}$ satisfies the conditions of Lemma 14, therefore by Lemma 15 $\Gamma - \mathscr{F}$ has a directed chain starting with and directed away from V_1 which includes all but a finite number of the edges of $\Gamma - \mathscr{F}$, say Ξ_0.

If $\Gamma - \mathscr{F} - \mathscr{E}(\Xi_0)$ is Eulerian then let $\Xi' = \Xi_0$, if $\Gamma - \mathscr{F} - \mathscr{E}(\Xi_0)$ is not Eulerian, then as in the corresponding part of the proof of (A), $\Gamma - \mathscr{F}$ has a chain Ξ' starting with and directed away from V_1, such that $\Gamma - \mathscr{F} - \mathscr{E}(\Xi')$ is Eulerian.

As in the proof of Theorem 12, Ξ' can be augmented if necessary into a chain Ξ^* of $\Gamma - \mathscr{F}$ starting with and directed away from V_1 which includes every edge of Ξ' and of all those connected components of $\Gamma - \mathscr{F} - \mathscr{E}(\Xi')$ which have a vertex in common with Ξ'. The connected components with edges of $\Gamma - \mathscr{F} - \mathscr{E}(\Xi^*)$ are Eulerian and have no vertex in common with Ξ^*. Their union is $\Gamma_{\mathscr{F}}$. Clearly $\mathscr{F} \cap \mathscr{E}(\Xi^*) \cup \mathscr{E}(\Gamma_{\mathscr{F}}) = \mathscr{E}(\Gamma)$. Theorem 25 is now proved.

In connection with Theorem 13 we observe that it is not true that if Γ is a directed graph with infinitely many edges and if for any finite chain Ξ of Γ starting with and directed away from the vertex V_1, Γ has a directed chain which has Ξ as a starting segment and includes all but at most a finite number of the edges of Γ, and every edge is accessible away from V_1, then Γ is traceable away from V_1. A simple counterexample is a graph which consists of a 1-way infinite path starting with and directed away from the vertex V_1 together with one or more edges directed away from V_1 which join V_1 to vertices of the path but do not belong to the path. But it is true if in addition $d_{o\Gamma}(V_1) = d_{i\Gamma}(V_1) + 1$ and the other vertices are balanced.

Corresponding to Theorem 14 we have

THEOREM 26. *A directed graph Γ is semi-arbitrarily traceable away from the vertex V_1 if and only if $\mathscr{E}(\Gamma)$ is enumerably infinite, and if Ξ being any $(V_1 \to V_n)$-chain of Γ, in $\Gamma - \mathscr{E}(\Xi)$ every edge is accessible away from V_n.*

PROOF. The obvious adaptation of the proof of Theorem 14.

The directed analogue of Theorem 15 is

THEOREM 27. *A directed graph is semi-arbitrarily traceable away from the vertex V_1 if and only if it has infinitely many edges and is traceable away from V_1, and every cycle of the graph contains at least one vertex having infinite degree in the graph.*

PROOF. The obvious adaptation of the proof of Theorem 15, using Theorem 25 in place of Theorem 12.

Corresponding to Corollary 4 we have

COROLLARY 7. *The only locally finite directed graphs which are semi-arbitrarily traceable away from a vertex are those which consist of a 1-way infinite directed path starting with and directed away from the vertex, and a set (possibly empty) of isolated vertices.*

Criteria for infinite directed graphs to be traceable have been established by C. St. J. A. Nash-Williams [5]. With the help of these Theorem 27 can be re-formulated, replacing the condition that the directed graph is traceable away from the vertex V_1 by the equivalent structural properties. This form of Theorem 27 will not be given here because of the length of the required definitions.

For the directed analogues of the other kinds of arbitrary traceability discussed in the previous section we formulate

DEFINITION 17. Let $\Xi = W_1, E_1, \ldots, E_{n-1}, W_n$ be a closed directed chain. Then the closed directed chain $W_a, E_a, \ldots, E_{n-1}, W_1, E_1, \ldots, E_{a-1}, W_a$, where $2 \leqq a \leqq n-1$, will be called a *rotation of* Ξ.

If $\Xi = W_1, E_1, \ldots, W_n$ and $\Xi' = W'_1, E'_1, \ldots, W'_m$ are chains directed away from W_1 and W'_1 respectively and have no edge in common, and if $W_n = W'_1$, then $W_1, E_1, \ldots, W_n, E'_1, W'_2, \ldots, W'_m$ is a chain directed away from W_1, it will be denoted by Ξ, Ξ'.

DEFINITION 18. Let Γ be a directed graph with infinitely many edges and V_1 a vertex of Γ. We call Γ

a) *type* I *traceable away from* V_1 if for any finite chain Ξ of Γ starting with and directed away from V_1, Γ has an Euler chain starting with and directed away from V_1 which has Ξ as a segment,

b) *type* II *traceable away from* V_1 if for any finite chain Ξ of Γ other than an isolated vertex, Γ has an Euler chain starting with and directed away from V_1 which has Ξ as a segment.

Clearly if a directed graph is type II traceable away from V_1 then it is type I traceable away from V_1, but not conversely.

The directed analogue of Theorem 16 is

THEOREM 28. *A directed graph is type* I *traceable away from the vertex* V_1 *if and only if it has infinitely many edges and is traceable away from* V_1, *and every cycle of the graph which does not include* V_1 *contains a vertex having infinite degree in the graph.*

PROOF. The obvious adaptation of the proof of Theorem 16, using Theorem 25 in place of Theorem 12.

Corresponding to Corollary 5 we have

COROLLARY 8. *If a directed graph is type* I *traceable away from a vertex whose degree is* 1 *or infinite, then the graph is semi-arbitrarily traceable away from that vertex.*

Proof. The obvious adaptation of the proof of Corollary 5 using Theorems 27 and 28 in place of Theorems 15 and 16, respectively.

Corresponding to Corollary 6 we have

Corollary 9. *There are only two kinds of locally finite directed graphs which are type* I *traceable away from a vertex* V_1, *apart from isolated vertices*:

1. *a* 1*-way infinite directed path starting with and directed away from* V_1,
2. *the union of a* 1*-way infinite directed path starting with and directed away from* V_1 *and a finite directed graph arbitrarily traceable from* V_1 *to* V_1, *such that the two graphs have no edge in common and every cycle of their union contains* V_1.

Proof. The obvious adaptation of the proof of Corollary 6, using Theorem 28 in place of Theorem 16 and Harary's theorem for directed graphs in place of Ore's Theorem.

The directed analogue of Lemma 6 is

Lemma 17. *If a directed graph is traceable away from a vertex of infinite degree, then it is traceable away from, and also towards, every vertex of infinite degree.*

Proof. The obvious adaptation of the proof of Lemma 6.

Corresponding to Lemma 7 we have

Lemma 18. *Let* Γ *be a directed graph with an infinite Euler chain starting with and directed away from the vertex* V_1, *and let* Ξ *be an open* $(W_1 \to W_n)$*-chain of* Γ. *If* $\Gamma - \mathscr{E}(\Xi)$ *has no* $(V_1 \to W_1)$*-chain then the number of edges not accessible away from* V_1 *in* $\Gamma - \mathscr{E}(\Xi)$ *is finite, and* $\Gamma - \mathscr{E}(\Xi)$ *has a* $(W_n \to W_1)$*-chain.*

Proof. Obviously $W_1 \neq V_1$, so $d_{i\Gamma}(W_1) = d_{o\Gamma}(W_1)$. If Ξ' is any $(V_1 \to V_m)$-chain of $\Gamma - \mathscr{E}(\Xi)$ then $\Gamma - \mathscr{E}(\Xi)$ contains an edge not belonging to Ξ' and directed away from V_m, for if $V_m = V_1$ then either

$$d_{o\Gamma-\mathscr{E}(\Xi)-\mathscr{E}(\Xi')}(V_m) > d_{i\Gamma-\mathscr{E}(\Xi)-\mathscr{E}(\Xi')}(V_m)$$

or $d_{o\Gamma-\mathscr{E}(\Xi)-\mathscr{E}(\Xi')}(V_m)$ is infinite, and if $V_m \neq V_1$ then

$$d_{o\Gamma-\mathscr{E}(\Xi)-\mathscr{E}(\Xi')}(V_m) > d_{i\Gamma-\mathscr{E}(\Xi)-\mathscr{E}(\Xi')}(V_m)$$

because $V_m \neq W_1$. Therefore $\Gamma - \mathscr{E}(\Xi)$ has an infinite chain starting with and directed away from V_1. Consequently by Lemma 15, $\Gamma - \mathscr{E}(\Xi)$ has a chain starting with and directed away from V_1 which includes all but at most a finite number of the edges of $\Gamma - \mathscr{E}(\Xi)$.

$\Gamma - \mathscr{E}(\Xi)$ contains an $(X_1 \to W_1)$-edge E'_1, where $X_1 \neq V_1$. For $d_{i\Gamma-\mathscr{E}(\Xi)}(W_1) \geqq 1$ because $d_{i\Gamma}(W_1) = d_{o\Gamma}(W_1)$ and Ξ is an open $(W_1 \to W_n)$-chain, hence $\Gamma - \mathscr{E}(\Xi)$ contains an $(X_1 \to W_1)$-edge. Note that $X_1 \neq V_1$ because $\Gamma - \mathscr{E}(\Xi)$ has no $(V_1 \to W_1)$-chain.

If $X_1 \neq W_n$ then $\Gamma - \mathscr{E}(\Xi) - E'_1$ contains an $(X_2 \to X_1)$-edge E'_2, where $X_2 \neq V_1$. For $d_{i\Gamma-\mathscr{E}(\Xi)-E'_1}(X_1) \geqq 1$ because $d_{i\Gamma}(X_1) = d_{o\Gamma}(X_1)$ since $X_1 \neq V_1$, and $X_1 \neq W_n$, and Ξ is an open $(W_1 \to W_n)$-chain, hence $\Gamma - \mathscr{E}(\Xi) - E'_1$ contains an $(X_2 \to X_1)$-edge E'_2. Note that $X_2 \neq V_1$ because $\Gamma - \mathscr{E}(\Xi)$ has no $(V_1 \to W_1)$-chain. If $X_1 = W_n$ then $\Gamma - \mathscr{E}(\Xi)$ has a $(W_n \to W_1)$-chain.

By repetition of the above arguments we have that if $X_i \neq W_n$ then $\Gamma - \mathscr{E}(\Xi) - E'_1 - \ldots - E'_i$ contains an $(X_{i+1} \to X_i)$-edge E'_{i+1}, where $X_{i+1} \neq V_1$, for $i = 2, 3, \ldots$.

$E'_1, E'_2, E'_3, \ldots$ are not accessible away from V_1 in $\Gamma - \mathscr{E}(\Xi)$ because $\Gamma - \mathscr{E}(\Xi)$ has no $(V_1 \to W_1)$-chain. It was shown above that the number of edges not accessible away from V_1 in $\Gamma - \mathscr{E}(\Xi)$ is finite. Therefore for some integer j $(j \geqq 1)$, $X_j = W_n$. Then $W_n, E'_j, X_{j-1}, E'_{j-1}, \ldots, X_1, E'_1, W_1$ is a $(W_n \to W_1)$-chain of $\Gamma - \mathscr{E}(\Xi)$. Lemma 18 is now proved.

Corresponding to Theorem 18 we have

THEOREM 29. *A directed graph is type* II *traceable away from the vertex V_1 if and only if it has infinitely many edges and is traceable from V_1, every cycle of the graph which does not include V_1 contains a vertex having infinite degree in the graph, and if the edges of any closed directed chain of the graph are deleted then in the remaining graph every vertex which is not an isolated vertex of the original graph is accessible away from V_1.*

The proof of this theorem will use

LEMMA 19. *If Γ is a directed graph, $\mathscr{E}$ is a subset of the set of edges of Γ, and the vertex B of Γ is accessible away from the vertex A in Γ but not in $\Gamma - \mathscr{E}$, then there exists an ending-vertex of an edge of $\mathscr{E}$ which is accessible away from A in Γ but not in $\Gamma - \mathscr{E}$.*

PROOF. If B is the ending-vertex of an edge of $\mathscr{E}$ then there is nothing to prove. In what follows suppose that B is not the ending vertex of any of the edges of $\mathscr{E}$. Every $(A \to B)$-path in Γ includes an edge of $\mathscr{E}$.

Let Π be an $(A \to B)$-path in Γ and let C denote the last vertex incident with an edge of Π belonging to $\mathscr{E}$ when Π is followed from A to B. Clearly $C \neq A, B$ and C is accessible away from A in Γ.

C is not accessible away from A in $\Gamma - \mathscr{E}$. For suppose on the contrary (reductio ad absurdum) that $\Gamma - \mathscr{E}$ contains an $(A \to C)$-path Π'. When Π' is followed from A to C, let D be the first vertex encountered belonging to the part of Π starting with C and ending with B; $C = D$ possibly. Then the union of the part of Π' going from A to D and the part of Π going from D to B is an $(A \to B)$-path contained in $\Gamma - \mathscr{E}$, which is contrary to hypothesis. Therefore C is not accessible away from A in $\Gamma - \mathscr{E}$. Obviously C is an ending-vertex of an edge of $\mathscr{E}$. Therefore C has the required properties. This proves Lemma 19.

PROOF OF THEOREM 29. This an adaptation of the proof of Theorem 18, using Theorem 28 in place of Theorem 16. Suppose that Γ is a directed graph which satisfies the conditions of Theorem 29. Then Γ is type I traceable away from V_1 by Theorem 28.

Ξ being any closed directed chain of Γ of length >0, Γ has an Euler chain starting with and directed away from V_1 which has Ξ as a segment, by an obvious adaptation of the corresponding part of the proof of Theorem 18, using Theorem 28 in place of Theorem 16.

Let Ξ be any open $(W_1 \to W_n)$-chain of Γ. In $\Gamma - \mathscr{E}(\Xi)$ there is a $(V_1 \to W_1)$-chain. For if not, then by Lemma 17 $\Gamma - \mathscr{E}(\Xi)$ has a $(W_n \to W_1)$-chain, Ξ' say; clearly Ξ, Ξ' is a closed directed chain of Γ and in $\Gamma - \mathscr{E}(\Xi, \Xi')$ W_1 is not accessible away from V_1, which is contrary to hypothesis. Since $\Gamma - \mathscr{E}(\Xi)$ has a $(V_1 \to W_1)$-chain and Γ is type I traceable away from V_1, it follows at once that Γ has an Euler chain starting with and directed away from V_1 which has Ξ as a segment. This proves that if Γ satisfies the conditions of the theorem then Γ is type II traceable away from V_1.

Next we will prove that if the directed graph Γ is type II traceable away from the vertex V_1 then it satisfies the conditions of the theorem. From Definition 18 Γ has infinitely many edges and is traceable away from V_1. Now Γ is type I traceable away from V_1, therefore by Theorem 28 every cycle of Γ which does not include V_1 contains a vertex having infinite degree in Γ.

Suppose (reductio ad absurdum) that Ξ is a closed directed chain of Γ and in $\Gamma - \mathscr{E}(\Xi)$ a vertex Y having non-zero degree in Γ is not accessible away from V_1. Now Y is accessible away from V_1 in Γ since Γ is traceable away from V_1. Therefore by Lemma 19 a vertex W of Ξ is not accessible away from V_1 in $\Gamma - \mathscr{E}(\Xi)$. It follows that Γ has no Euler chain starting with and directed away from V_1 which has as a segment the rotation of Ξ starting and ending with W (or Ξ itself if Ξ starts and ends with W). Therefore Γ is not type II traceable away from V_1. But this is contrary to hypothesis. This contradiction proves that Γ satisfies the conditions of the theorem. Theorem 29 is now proved.

An alternative form of Theorem 29 and the directed analogue of Theorem 18′ is

THEOREM 29′. *A directed graph without isolated vertices is type* II *traceable away from the vertex V_1 if and only if it has infinitely many edges and is traceable away from V_1, every cycle of the graph which does not include V_1 contains a vertex having infinite degree in the graph, and if the edges of any Eulerian connected subgraph are deleted from the graph then the remaining graph never has an isolated vertex.*

This follows directly from the following analogue of Lemma 8:

LEMMA 20. *If a (finite or infinite) directed graph is traceable away from the vertex V_1, and if it has an Eulerian subgraph such that when its edges are deleted from the graph then in the remaining graph not every vertex is accessible from V_1, then the graph has an Eulerian connected subgraph such that when its edges are deleted from the graph then the remaining graph has an isolated vertex.*

PROOF. If the graph has a closed Euler chain then deleting its edges leaves only isolated vertices. Suppose that Γ is a directed graph without an isolated vertex which has an open or infinite Euler chain starting with and directed away from the vertex V_1, and that Φ is an Eulerian subgraph of Γ such that in $\Gamma - \mathscr{E}(\Phi)$ not every vertex is accessible away from V_1. If $\Gamma - \mathscr{E}(\Phi)$ has an isolated vertex W, then W is an isolated vertex of the graph obtained by deleting from Γ the edges of the connected component of Φ to which W belongs. Lemma 20 is therefore true for Γ if there is an isolated vertex in $\Gamma - \mathscr{E}(\Phi)$. In what follows we will assume that this is not the case.

Suppose first that $\mathscr{E}(\Gamma)$ is finite. Then clearly $\Gamma - \mathscr{E}(\Phi)$ is $(V_1 \to V_2)$-odd, and therefore it has a $(V_1 \to V_2)$-chain. Let Ξ be a longest $(V_1 \to V_2)$-chain of $\Gamma - \mathscr{E}(\Phi)$. Ξ does not contain every vertex of Γ because not every vertex of Γ is accessible away from V_1 in $\Gamma - \mathscr{E}(\Phi)$. By our previous assumption the vertices of Γ which do not belong to Ξ are not isolated in $\Gamma - \mathscr{E}(\Phi)$. Therefore $\Gamma - \mathscr{E}(\Phi) - \mathscr{E}(\Xi)$ is an Eulerian graph with edges.

Let Λ be a connected component with edges of $\Gamma - \mathscr{E}(\Phi) - \mathscr{E}(\Xi)$. Then Λ is Eulerian, and has no vertex in common with Ξ because Ξ is a longest $(V_1 \to V_2)$-chain of $\Gamma - \mathscr{E}(\Phi)$. Therefore the vertices of Λ are in Γ incident only with a subset of $\mathscr{E}(\Lambda \cup \Phi)$. The union Ψ say, of Λ and all those connected components of Φ which have a vertex in common with Λ, is an Eulerian connected subgraph of Γ and in $\Gamma - \mathscr{E}(\Psi)$ the vertices of Λ are isolated.

Suppose next that $\mathscr{E}(\Gamma)$ is infinite. From Theorem 25 (B) with $\mathscr{F} = \mathscr{E}(\Phi)$, and the above assumptions, it follows that $\Gamma - \mathscr{E}(\Phi)$ has a chain starting with and directed away from V_1 which contains every vertex and every edge of $\Gamma - \mathscr{E}(\Phi)$ except the vertices and edges of a non-empty Eulerian subgraph Δ of $\Gamma - \mathscr{E}(\Phi)$. Let Λ denote a connected component of Δ. Each vertex of Λ is in Γ incident only with a subset of $\mathscr{E}(\Lambda \cup \Phi)$. The union, Ψ say, of Λ and all those connected components of Φ which have a vertex in common with Λ is Eulerian and connected, and in $\Gamma - \mathscr{E}(\Psi)$ the vertices of Λ are isolated. This completes the proof of Lemma 20.

The directed analogue of Definition 14 is

Definition 19. A directed graph will be called *locally infinite traceable* if it is traceable away from a vertex and each of its vertices has infinite degree.

Remark 10. By an argument analogous to that of Remark 4 a directed graph without isolated vertices is locally infinite traceable if and only if it is enumerably infinite and remains strongly connected when any finite set of edges is deleted from it, and any such graph is type II and semi-arbitrarily traceable away from and towards each vertex.

Theorem 30. (A) *Every locally infinite traceable directed graph is type* II *and semi-arbitrarily traceable away from, and also towards, each of its vertices.*

(B) *If a directed graph has no isolated vertices and is traceable away from and also towards the vertex V_1, and if deleting the edges of any Eulerian connected subgraph never results in a graph with an isolated vertex, then the graph is locally infinite traceable.*

Proof of (A). This can be proved as indicated in Remark 10, and otherwise thus: By Lemma 17 every locally infinite traceable directed graph is traceable away from, and also towards, each of its vertices.

Therefore by Theorem 29′ every such graph is type II traceable away from each of its vertices, and analogously it is also type II traceable towards each of its vertices. Hence by Corollary 8 every such graph is semi-arbitrarily traceable away from each of its vertices, and analogously it is also semi-arbitrarily traceable towards each of its vertices. This proves (A).

PROOF OF (B). Let Γ be the graph. $d_\Gamma(V_1)$ is infinite, because if $d_\Gamma(V_1)$ is finite then Γ has a closed Euler chain Ξ and so $\Gamma - \mathscr{E}(\Xi)$ consists of isolated vertices, which is contrary to hypothesis. Suppose (reductio ad absurdum) that Γ has a vertex W with finite degree. Obviously $W \neq V_1$. Let Ξ^* be an Euler chain of Γ starting with and directed away from V_1. Since $d_\Gamma(V_1)$ is infinite and $d_\Gamma(W)$ is finite, Ξ^* has as a starting segment a closed directed chain Ξ_0 which includes every edge of Γ incident with W. Then in $\Gamma - \mathscr{E}(\Xi_0)$ the vertex W is isolated, which is contrary to hypothesis. This contradiction shows that Γ is locally infinite traceable. Theorem 30 is now proved.

The directed analogue of Lemma 9 is

LEMMA 21. *If a directed graph has no isolated vertices and is traceable away from the vertex V_1, then if any finite set of directed edges whose end-vertices all have infinite degree in the graph are deleted or added, the resulting graph always has an Euler chain starting with and directed away from V_1 which includes every vertex.*

PROOF. The obvious adaptation of the proof of Lemma 9.

REMARK 11. It can be shown that if any enumerably infinite set of directed edges whose end-vertices all have infinite degree in the original graph are added, then the resulting graph always has an Euler chain starting with and directed away from V_1 which includes every vertex. This result will not be required in the sequel though.

The directed analogue of Lemma 10 is

LEMMA 22. *If Ω is a locally infinite traceable directed graph and Ω' is any graph obtained from Ω by adding a vertex A not belonging to Ω and a positive number β of edges directed away from A towards vertices of Ω, and $\beta - 1$ edges directed towards A away from the vertices of Ω then Ω' is semi-arbitrarily and type* II *traceable away from A and is not traceable away from or towards any vertex other than A.*

PROOF. The obvious adaptation of the proof of Lemma 10, using Lemma 17 in place of Lemma 8.

REMARK 12. The directed analogue of Remark 5 holds.

Lemma 11 has no valid directed analogue. In fact there exist locally finite directed graphs which are type II traceable away from the vertex X_1 and which contain cycles of length >2. For example let a be an integer >2 and let the directed graph Γ consist of the infinite sequence of distinct vertices $X_1, X_2, X_3, \ldots$ together with one $(X_a \to X_1)$-edge, E say, two $(X_{i-1} \to X_i)$-edges for $i=2,\ldots,a$ and one $(X_{i-1} \to X_i)$-edge for $i=a+1, a+2, a+3, \ldots$. Obviously Γ contains a cycle of length a and is locally finite and traceable away from X_1, and every cycle of Γ includes E and therefore X_1. Since every cycle of Γ includes E, it folllows from König's Theorem and the structure of Γ that every Eulerian subgraph of Γ is a cycle consisting of E and an $(X_1 \to X_a)$-path whose vertices in order are $X_1, X_2, \ldots, X_a$. If the edges of such a cycle are deleted then the remaining graph is a directed path whose vertices in order are $X_1, X_2, X_3, \ldots$, which contains no isolated vertices. Therefore by Theorem 29′ we have that Γ is type II traceable away from X_1. Because Lemma 11 has no valid directed analogue, therefore a directed analogue of Lemma 12 is irrelevant here.

In consequence there is no close directed analogy to Theorem 19. We have only the following

THEOREM 31. *There is only one category of finite directed graphs without isolated vertices which are traceable away from a vertex* X_1 *and are such that if the edges of any Eulerian connected subgraph are deleted then the remaining graph has no isolated vertex, namely all those having the following structure*:

(i) *Their vertices comprise the set* $X_1, \ldots, X_n$ *where* $n \geqq 2$.

(ii) *For* $g=1,\ldots,n$ *and* $h=2,\ldots,n$ *they contain no* $(X_g \to X_h)$*-edge with* $h-g>1$.

(iii) *If* $\varkappa_{gh}$ *denotes the number of* $(X_g \leftarrow X_h)$*-edges with* $g<h$ *then for* $k=1,\ldots,n-1$ *the total number of* $(X_k \to X_{k+1})$*-edges is*

$$1+\sum_{i=1}^{k}\sum_{j=k+1}^{n}\varkappa_{ij}$$

There are only three categories of infinite directed graphs with the above properties, namely:

1. *All locally infinite traceable directed graphs containing* X_1.
2. *All directed graphs having the following structure*:

(i) *Their vertices comprise the infinite sequence* $X_1, X_2, X_3, \ldots$ *of distinct vertices.*

(ii) *For* $g,h = 1,2,3,\ldots$ *they contain no* $(X_g \to X_h)$*-edge with* $h-g>1$.

(iii) *If* $\varkappa_{gh}$ *denotes the number of* $(X_g \leftarrow X_h)$*-edges with* $g<h$ *then for* $g=1,2,3,\ldots$ *the sum* $\sum_{h>g} \varkappa_{gh}$ *is finite, and for* $k=1,2,3,\ldots$ *the total number of* $(X_k \to X_{k+1})$*-edges is*

$$1+\sum_{i=1}^{k} \sum_{j=k+1}^{\infty} \varkappa_{ij} .$$

3. *All directed graphs having the following structure:*

(i) *They contain a finite number of (distinct) vertices* $X_1, \ldots, X_n$ *having finite degree, where* $n \geqq 1$, *and a locally infinite traceable directed graph* Ω *not including any of* $X_1, \ldots, X_n$.

(ii) *In case* $n>1$ *they contain no* $(X_g \to X_h)$*-edge with* $h-g>1$ *and no edge directed away from any of* $X_1, \ldots, X_{n-1}$ *towards any vertex of* Ω.

(iii) *If* δ_g *denotes the total number of edges in the graph directed towards* X_g *away from the vertices of* Ω *for* $g=1,2,3,\ldots$, *and if, for* $n>g\geqq 1$, $\varkappa_{gh}$ *denotes the number of* $(X_g \leftarrow X_h)$*-edges for* $h=g+1,\ldots,n$, *and* β *denotes the total number of edges directed away from* X_n *towards vertices of* Ω, *then* $\delta_g, \varkappa_{gh}$ *and* β *are all finite for* $g=1,\ldots,n$ *and* $h=2,\ldots,n$, *and if* $n=1$ *then* $\beta = 1+\delta_1$, *while if* $n>1$ *then for* $k=1,\ldots,n-1$ *the total number of* $(X_k \to X_{k+1})$*-edges is*

$$1+\sum_{i=1}^{k} \delta_i + \sum_{i=1}^{k} \sum_{j=k+1}^{n} \varkappa_{ij} ,$$

and $\beta = 1+\delta_1+\ldots+\delta_n$.

If we delete from any of these graphs any finite set of edges such that for each vertex having finite degree in the graph the number of edges in the set directed towards it is equal to the number of edges in the set directed away from it, then the remaining graph is in the same category as the original one.

Proof. To prove the part of the theorem concerning finite directed graphs we first verify that each graph in the category described has the properties stated in the theorem. Let Γ denote such a graph. It may easily be verified that Γ has no isolated vertex and has an $(X_1 \to X_n)$-Euler chain. Let Λ denote a graph which remains after deleting from Γ the edges of an Eulerian subgraph of Γ. Obviously Λ has the same vertices as Γ and contains no $(X_g \to X_h)$-edge with $h-g>1$. Also Λ is $(X_1 \to X_n)$-odd because Γ is. Therefore if $\varkappa'_{gh}$ denotes the number of $(X_g \leftarrow X_h)$-edges in Λ for $1 \leqq g<h\leqq n$ then the total number of $(X_1 \to X_2)$-edges in Λ is

$$1+\sum_{j=2}^{n} \varkappa'_{1j} .$$

If $n > 2$ then, since X_2 is balanced in Λ, the total number of $(X_2 \to X_3)$-edges in Λ is

$$1 + \sum_{j=2}^{n} \varkappa'_{1j} + \sum_{j=3}^{n} \varkappa'_{2j} - \varkappa'_{12} = 1 + \sum_{i=1}^{2} \sum_{j=3}^{n} \varkappa'_{ij} ,$$

and so on, for $k = 1, \ldots, n-1$ the total number of $(X_k \to X_{k+1})$-edges in Λ is

$$1 + \sum_{i=1}^{k} \sum_{j=k+1}^{n} \varkappa'_{ij} .$$

Therefore Λ is in the same category. Hence Λ does not have an isolated vertex. We have now proved that each graph in the category described has the properties stated in the first and the last paragraph of Theorem 31.

Next we show that if Γ is any finite directed graph having the properties stated in the first paragraph of the theorem then Γ belongs to the category described. Γ has more than one vertex, also Γ does not have a closed directed Euler chain obviously. Therefore Γ has an open Euler chain starting with and directed away from X_1, let Y denote the ending-vertex of this chain. Then Γ contains an $(X_1 \to Y)$-path.

Every $(X_1 \to Y)$-path of Γ contains every vertex of Γ. For suppose not (reductio ad absurdum) and let Υ be an $(X_1 \to Y)$-path of Γ which does not contain the vertex W of Γ. Then $\Gamma - \mathscr{E}(\Upsilon)$ is clearly Eulerian, therefore by König's Theorem it is the union of a finite set of cycles no two of which have an edge in common. The union of all cycles of this set containing W is an Eulerian connected subgraph of Γ, and if we delete all its edges from Γ then in the remaining graph W is isolated, which is contrary to hypothesis. This contradiction proves that every $(X_1 \to Y)$-path of Γ contains every vertex of Γ.

Let Π be an $(X_1 \to Y)$-path of Γ, and let the vertices of Π, in order along Π starting from X_1, be $X_1, \ldots, X_n$, where $n \geqq 2$ and $X_n = Y$. Then the vertices of Γ comprise the set $\{X_1, \ldots, X_n\}$ and Γ contains no $(X_g \to X_h)$-edge with $h - g > 1$. Γ is $(X_1 \to X_n)$-odd because it has an $(X_1 \to X_n)$-Euler chain. From this (iii) follows as for Λ above. Thus Γ is in the category described. The part of Theorem 31 concerning finite graphs is now proved.

To prove the part of the theorem concerning infinite graphs we first verify that every graph in each of the categories 1, 2 and 3 has no isolated vertex and is traceable away from X_1 and has the property stated at the end of Theorem 31.

If a directed graph is locally infinite traceable then it has no isolated vertex, and by Lemma 21 if any finite number of edges are deleted from it then the remaining graph is locally infinite traceable.

Let Γ be a directed graph in category 2. It may easily be verified that Γ has no isolated vertex and is locally finite, and that Γ is traceable away from X_1. Let Λ denote the graph which remains after deleting from Γ a set of edges as described in the last paragraph of Theorem 31. Obviously Λ has the same vertices as Γ and is locally finite and contains no $(X_g \to X_h)$-edge with $h-g>1$. Also $d_{o\Lambda}(X_1)=1+d_{i\Lambda}(X_1)$ and $X_2, X_3, X_4, \ldots$ are all balanced in Λ. Therefore if $\varkappa'_{gh}$ denotes the number of $(X_g \leftarrow X_h)$-edges in Λ for $1 \leqq g<h$ then the total number of $(X_1 \to X_2)$-edges in Λ is

$$1+\sum_{j=2}^{\infty} \varkappa'_{1j} ,$$

the total number of $(X_2 \to X_3)$-edges in Λ is

$$1+\sum_{j=2}^{\infty} \varkappa'_{1j}+\sum_{j=3}^{\infty} \varkappa'_{2j}-\varkappa'_{12} = 1+\sum_{i=1}^{2} \sum_{j=3}^{\infty} \varkappa'_{ij} ,$$

and so on, for $k=1,2,3,\ldots$ the total number of $(X_k \to X_{k+1})$-edges in Λ is

$$1+\sum_{i=1}^{k} \sum_{j=k+1}^{\infty} \varkappa'_{ij} .$$

Therefore Λ is in category 2. Hence Λ does not have an isolated vertex. We have now proved that each graph in category 2 has the properties stated in the first and last paragraph of Theorem 31.

Let Γ be a directed graph in category 3. Obviously Γ has no isolated vertex. It follows from Lemmas 21 and 22 that Γ is traceable away from X_1. Let Λ denote a graph remaining after deleting from Γ a set $\mathscr{E}$ of edges as described at the end of Theorem 27, and let Ω^- denote $\Omega-(\mathscr{E} \cap \Omega)$. By Lemma 21 Ω^- is a locally infinite traceable directed graph. Obviously Λ has the same vertices as Γ and $X_1, \ldots, X_n$ have finite degree in Λ, and if $n>1$ then Λ contains no $(X_g \to X_h)$-edge with $h-g>1$ and no edge directed away from any of $X_1, \ldots, X_{n-1}$ towards any vertex of Ω^-. Furthermore $d_{o\Lambda}(X_1)=1+d_{i\Lambda}(X_1)$, and if $n>1$ then $X_2, \ldots, X_n$ are balanced in Λ. Therefore if δ'_g denotes the total number of edges in Λ directed towards X_g away from the vertices of Ω^- for $g=1,\ldots,n$, and if $n>g\geqq 1$ then $\varkappa'_{gh}$ denotes the number of $(X_g \leftarrow X_h)$-edges in Λ for $h=g+1,\ldots,n$, and β' denotes the total number of edges directed away from X_n towards vertices of Ω^-, then δ'_g, $\varkappa'_{gh}$ and β' are all finite, and if $n=1$ then $\beta'=1+\delta'_1$, while if $n>1$ then the total number of $(X_1 \to X_2)$-edges in Λ is

$$1+\delta'_1+\sum_{j=2}^{n} \varkappa'_{1j} ,$$

the total number of $(X_2 \to X_3)$-edges in Λ is, if $n>2$,

$$1+\delta'_1+\sum_{j=2}^{n} \varkappa'_{1j}+\delta'_2+\sum_{j=3}^{n} \varkappa'_{2j}-\varkappa'_{12} = 1+\sum_{i=1}^{2} \delta'_i+\sum_{i=1}^{2} \sum_{j=3}^{n} \varkappa'_{ij}$$

and so on, for $k=1,\ldots,n-1$ the total number of $(X_k \rightarrow X_{k+1})$-edges in Λ is

$$1+\sum_{i=1}^{k} \delta'_i+\sum_{i=1}^{k} \sum_{j=k+1}^{n} \varkappa'_{ij} .$$

Therefore Λ is in category 3.

It has now been proved that each graph in each of the categories 1, 2 and 3 has no isolated vertex and is traceable away from X_1 and has the property stated in the last paragraph of Theorem 31, so à fortiori it has all the properties stated in the first paragraph of Theorem 31.

To prove Theorem 31 it remains to show that if Γ is any infinite directed graph without isolated vertices having the properties stated in the first paragraph of the theorem then Γ belongs to one of the categories 1, 2, 3. Clearly $\mathscr{E}(\Gamma)$ is enumerably infinite.

One of the following three possibilities holds: (a) $d_\Gamma(X_1)$ is infinite, (b) Γ is locally finite, (c) $d_\Gamma(X_1)$ is finite and Γ has a vertex of infinite degree.

If (a) holds then Γ is traceable away from and towards X_1, so by Theorem 30 (B), Γ is in category 1.

Suppose that (b) holds. Then the set of vertices of Γ is enumerably infinite. Let $\Xi^* = V_1, E_1, V_2, E_2, \ldots$ be an Euler chain of Γ directed away from V_1, where $V_1 = X_1$. For $j=2,3,4,\ldots$ let X_j denote the vertex of Γ with smallest suffix in Ξ^* different from each $X_1,\ldots,X_{j-1}$. Clearly $X_2 = V_2$, and $X_1, X_2, X_3, \ldots$ is an infinite sequence of distinct vertices constituting the set of vertices of Γ, and furthermore for each $j \geqq 2$, Γ contains an $(X_i \rightarrow X_j)$-edge with $1 \leqq i \leqq j-1$.

Γ contains no $(X_g \rightarrow X_h)$-edge with $h > g+1 \geqq 2$. For suppose that this is not true (reductio ad absurdum) and let a be the smallest positive integer such that Γ contains a $(X_a \rightarrow X_b)$-edge where $b > a+1$. Then let c be so large that the chain $\Xi = V_1, E_1, V_2, E_2, \ldots, V_c$ contains every edge of Γ incident with every one of $X_1, \ldots, X_b$, and let Γ_c denote the graph whose vertices and edges are $V_1, E_1, V_2, E_2, \ldots, V_c$. In $\Gamma - \mathscr{E}(\Xi)$ the vertices $X_1, \ldots, X_b$ are isolated, therefore Ξ is open, that is $V_1 \neq V_c$, and Γ_c is $(V_1 \rightarrow V_c)$-odd. The graph Γ_c contains a $(V_1 \rightarrow V_c)$-path.

Every $(V_1 \rightarrow V_c)$-path contained in Γ_c includes each of $X_1, \ldots, X_b$. For suppose not (reductio ad absurdum) and let Υ be a $(V_1 \rightarrow V_c)$-path in Γ_c which does not contain X_d for some d such that $2 \leqq d \leqq b$. Now $\Gamma_c - \mathscr{E}(\Upsilon)$ is Eulerian, therefore by König's Theorem it is the union of a finite set of cycles no two of which have an edge in common. The union of all cycles of this set containing X_d is an Eulerian connected subgraph of Γ, and if we delete all its edges from Γ then in the remaining graph X_d is isolated, which is contrary to hypothesis. This contradiction proves that every $(V_1 \rightarrow V_c)$-path contained in Γ_c includes each of $X_1, \ldots, X_b$.

Let Π denote a $(V_1 \to V_c)$-path contained in Γ_c. Then $X_2 \in \Pi$, and Γ contains an $(X_1 \to X_2)$-edge, namely E_1. Therefore $X_3, \ldots, X_b$ all lie between X_2 and V_c on Π. Hence Γ contains no $(X_1 \to X_d)$-edge with $3 \leqq d \leqq b$. Therefore $a \geqq 2$ and $b \geqq 4$. Consequently every edge of Γ directed away from X_1 is an $(X_1 \to X_2)$-edge. It follows that Γ contains an $(X_2 \to X_3)$-edge. Therefore $X_4, \ldots, X_b$ all lie between X_3 and V_c on Π. Hence Γ contains no $(X_2 \to X_f)$-edge with $4 \leqq f \leqq b$. Therefore $a \geqq 3$ and $b \geqq 5$. This reasoning can be repeated indefinitely, showing that $a \geqq 4$, $a \geqq 5$, $a \geqq 6, \ldots$, whereas a is finite. This contradiction proves that Γ contains no $(X_g \to X_h)$-edge with $h > g+1 \geqq 2$.

Γ is traceable away from X_1 and locally finite. Therefore $d_{o\Gamma}(X_1) = 1 + d_{i\Gamma}(X_1)$ and $X_2, X_3, X_4, \ldots$ are balanced, and $\sum_{h>g} \varkappa_{gh}$ is finite for $g = 1, 2, 3, \ldots$. Furthermore the number of $(X_1 \to X_2)$-edges in Γ is

$$1 + \sum_{j=2}^{\infty} \varkappa_{1j} ,$$

therefore the number of $(X_2 \to X_3)$-edges in Γ is

$$1 + \sum_{j=2}^{\infty} \varkappa_{1j} + \sum_{j=3}^{\infty} \varkappa_{2j} - \varkappa_{12} = 1 + \sum_{i=1}^{2} \sum_{j=3}^{\infty} \varkappa_{ij} ,$$

and so on. Thus Γ is in category 2 if (b) holds.

Suppose that (c) holds, and let Ξ^*, p, Ξ, and Ξ^{**} be defined analogously to the part of the proof of Theorem 19 where (iii) is assumed. By the obvious directed analogy of the reasoning there, using Lemma 21 in place of Lemma 9, we have that $V_1, \ldots, V_p$ are the only vertices of Γ having finite degree in Γ and that if $p = 1$ then Γ is in category 3 with $n = 1$.

In what follows $p > 1$ will be assumed.

By the obvious directed analogy of the corresponding part of the above mentioned section of the proof of Theorem 19, using Lemma 20 in place of Lemma 8, we define Ξ' and have that $V_1 \neq V_p$ and that if Γ_p is the graph whose vertices and edges are $V_1, E_1, \ldots, V_p$, then Γ_p is $(V_1 \to V_p)$-odd and contains a $(V_1 \to V_p)$-path Υ which includes all of $V_1, \ldots, V_p$. Again let the vertices of Υ in order along Υ be $X_1, \ldots, X_n$, where $X_1 = V_1$ and $X_n = V_p$ and $2 \leqq n \leqq p$.

Γ contains no $(X_g \to X_h)$-edge with $h - g > 1$. For let $\Gamma_p - \mathscr{E}(\Upsilon) = \Phi$, then Φ is Eulerian, and therefore $\mathscr{E}(\Phi) \cup \mathscr{E}(\Xi')$ is the set of edges of an Eulerian subgraph of Γ, say Λ, since $\mathscr{E}(\Phi) \subset \mathscr{E}(\Xi)$ and $\mathscr{E}(\Xi) \cap \mathscr{E}(\Xi') = \emptyset$. The set $\mathscr{E}(\Phi) \cup \mathscr{E}(\Xi')$ contains every edge of Γ incident with each of $X_1, \ldots, X_n$ except E_p and the edges of Υ. Suppose that E is an $(X_k \to X_l)$-edge of Γ with $l - k > 1$ (reductio ad absurdum). Then if E is replaced in $\mathscr{E}(\Lambda)$ by the set of edges of Υ lying between X_k and X_l on Υ, the result is the set of edges of an Eulerian subgraph of Γ, say Λ^+. In $\Gamma - \mathscr{E}(\Lambda^+)$

the only edges incident with $X_1,\ldots,X_n$ are E_p, E, and the edges of Υ lying between X_1 and X_k on Υ (if $k>1$) and those lying between X_l and X_n on Υ (if $l<n$). Therefore $X_{k+1},\ldots,X_{l-1}$ are isolated in $\Gamma-\mathscr{E}(\Lambda^+)$. By Lemma 20 this is contrary to hypothesis. This contradiction proves that Γ contains no $(X_g \to X_h)$-edge with $h-g>1$.

Let r, Ξ^{***}, Γ_r and Ω be defined analogously to the corresponding part of the proof of Theorem 19. By Lemma 21, Ω is a locally infinite traceable directed graph, and also Ω includes every vertex of Γ except $X_1,\ldots,X_n$ and every edge of Γ except all those which have one of X_1, $\ldots,X_n$ as end-vertex. Γ contains no edge directed away from any of $X_1,\ldots,X_{n-1}$ towards any vertex of Ω by a reasoning analogous to the corresponding part of the proof of Theorem 19, using Lemma 20 in place of Lemma 8. Hence Γ satisfies (ii) of category 3.

Since $X_1,\ldots,X_n$ have finite degree in Γ, therefore all of $\delta_g, \varkappa_{gh}, \beta$ are finite. Because Γ is traceable away from X_1 we have that $d_{o\Gamma}(X_1)= 1+d_{i\Gamma}(X_1)$ and $X_2,\ldots,X_{n-1}$ are balanced in Γ. Therefore by (ii) of category 3 the total number of $(X_1 \to X_2)$-edges in Γ is

$$1+\delta_1+\textstyle\sum_{j=2}^{n} \varkappa_{1j}\,,$$

consequently the total number of $(X_2 \to X_3)$-edges in Γ is

$$1+\delta_1+\textstyle\sum_{j=2}^{n} \varkappa_{1j}+\delta_2+\sum_{j=3}^{n} \varkappa_{2j}-\varkappa_{12} \;=\; 1+\sum_{i=1}^{2} \delta_i+\sum_{i=1}^{2} \sum_{j=3}^{n} \varkappa_{ij}\,,$$

and so on, for $k=1,\ldots,n-1$ the total number of $(X_k \to X_{k+1})$-edges in Γ is

$$1+\textstyle\sum_{i=1}^{k} \delta_i+\sum_{i=1}^{k} \sum_{j=k+1}^{n} \varkappa_{ij}\,,$$

and $\beta=1+\delta_1+\ldots+\delta_n$. Thus Γ is in category 3 also if $p>1$. Hence if (c) holds then Γ is in category 3. This completes the proof of Theorem 31.

For Theorem 20 there is again no very close directed analogy, we have only the following

THEOREM 32. *There are only three categories of directed graphs without isolated vertices which are type* II *traceable away from a vertex* X_1:
1. *All locally infinite traceable directed graphs containing* X_1.
2. *All directed graphs having the following structure*:

(i) *Their vertices comprise the infinite sequence* $X_1, X_2, X_3, \ldots$ *of distinct vertices, all having finite degree.*

(ii) *They contain no* $(X_g \to X_h)$*-edge with* $1<h<g$ *or with* $h-g>1$.

(iii) *If* $\varkappa_i$ *denotes the number of* $(X_i \to X_1)$*-edges for* $i=2,3,4,\ldots$ *then* $\sum_i \varkappa_i$ *is finite, and for* $j=2,3,4,\ldots$ *the number of* $(X_{j-1} \to X_j)$*-edges is* $1+\varkappa_j+\varkappa_{j+1}+\varkappa_{j+2}+\ldots.$

3. *All directed graphs having the following structure*:

(i) *The contain a finite number of vertices* $X_1, X_2, \ldots, X_n$ *having finite degree, where* $n \geqq 1$ *and* $X_1, \ldots, X_n$ *are distinct, and a locally infinite traceable directed graph* Ω *not containing any of* $X_1, \ldots, X_n$.

(ii) *In case* $n > 1$ *they contain no* $(X_g \to X_h)$*-edge with* $1 < h < g \leqq n$ *or with* $h - g > 1$, *and no edge directed away from any of* $X_1, \ldots, X_{n-1}$ *towards any vertex of* Ω.

(iii) *If* δ_i *denotes the total number of edges directed towards* X_i *away from the vertices of* Ω *for* $i = 1, \ldots, n$, *and, in case* $n > 1$, $\varkappa_i$ *denotes the number of* $(X_i \to X_1)$*-edges, and* β *denotes the total number of edges directed away from* X_n *towards vertices of* Ω, *then* $\delta_1, \ldots, \delta_n, \varkappa_2, \ldots, \varkappa_n, \beta$ *are all finite, and if* $n = 1$ *then* $\beta = 1 + \delta_1$, *while if* $n > 1$ *then for* $j = 2, \ldots, n$ *the number of* $(X_{j-1} \to X_j)$*-edges is*

$$1 + \delta_1 + \ldots + \delta_{j-1} + \varkappa_j + \ldots + \varkappa_n$$

and $\beta = 1 + \delta_1 + \ldots + \delta_n$.

If we delete from any of these graphs any finite set of edges such that for each vertex having finite degree in the graph the numbeer of edges in the set directed away from it is equal to the number of edges in the set directed towards it, then the remaining graph is in the same category as the original one.

PROOF. Theorem 32 follows directly from Theorems 29′ and 31.

Of course Theorem 32 can also be proved directly, using the preceding part of the paper, without assuming Theorem 31.

The directed analogue of Theorem 21 is

THEOREM 33. *If we delete from a directed graph which is semi-arbitrarily/type* I/*type* II *traceable away from a vertex any finite set of edges such that for each vertex having finite degree in the graph the number of edges of the set directed towards it is equal to the number of edges of the set directed away from it, then the remaining graph is semi-arbitrarily/type I/type* II *traceable away from that vertex.*

PROOF. For type II traceable directed graphs this follows directly from Theorem 32. For semi-arbitrarily and for type I traceable directed graphs the proof is the directed analogue of the proof of Theorem 17 for semi-arbitrarily traceable and type 1 traceable undirected graphs, using Lemmas 14, 15, and 16 in place of Lemmas 2, 3 and 4, and using Theorems 27 and 28 in place of Theorems 15 and 16.

The directed analogue of Theorem 22 is

THEOREM 34. *If a directed graph is type* I/*type* II *traceable away from the vertex* V_1 *then it is semi-arbitrarily*/*semi-arbitrarily and type* II *traceable away from and towards every vertex of infinite degree in the graph if and only if the degree of* V_1 *is infinite.*

PROOF. Let Γ denote the graph. If $d_\Gamma(V_1)$ is finite then $d_{o\Gamma}(V_1) = d_{i\Gamma}(V_1) + 1$, and therefore Γ is not traceable from any vertex other than V_1.

Suppose that $d_\Gamma(V_1)$ is infinite. Then it follows from Theorems 28/29 that every cycle of Γ contains a vertex having infinite degree in Γ. Let W be any vertex having infinite degree in Γ.

By Lemma 17 Γ is traceable away from W and towards W. It now follows from Theorem 27 that Γ is semi-arbitrarily traceable away from W and towards W. Furthermore if Γ is type II traceable away from V_1 then it now follows from Theorem 29′ that Γ is also type II traceable away from W and towards W. Theorem 34 is now proved.

The directed analogue of Theorem 23 is

THEOREM 35. *Suppose that the directed graph* Γ *is type* I *traceable away from the vertex* V_1 *and has no isolated vertices.*

Then if Ξ *is any closed directed chain of* Γ, *there exists an Euler chain of* Γ *starting with and directed away from* V_1 *which has* Ξ *or a rotation of* Ξ *as a segment.*

If Ξ *is any open* $(W_1 \to W_n)$*-chain of* Γ *then either* Γ *has an Euler chain starting with and directed away from* V_1 *which has* Ξ *as a segment, or* $W_1 \neq V_1$ *and* $W_n \neq V_1$ *and* W_1 *and* W_n *both have finite degree in* Γ, *and the edges of* $\Gamma - \mathscr{E}(\Xi)$ *which are not accessible away from* V_1 *in* $\Gamma - \mathscr{E}(\Xi)$ *together with their end-vertices form a* $(W_n \to W_1)$*-path, and* Ξ' *being the chain of this path starting with* W_n, Γ *has an Euler chain starting with and directed away from* V_1 *which has as a segment a rotation of the closed directed chain* Ξ, Ξ' *starting with a vertex of* Ξ *which does not belong to* Ξ'.

PROOF. The proof of the first part of Theorem 35 is the obvious directed analogue of the proof of the first part of Theorem 23.

To prove the second part, suppose that Ξ is an open $(W_1 \to W_n)$-chain of Γ. If $\Gamma - \mathscr{E}(\Xi)$ has a $(V_1 \to W_1)$-chain, Ξ_1 say, then Ξ_1, Ξ is a chain of Γ starting with and directed away from V_1, therefore Γ has an Euler chain starting with and directed away from V_1 which has Ξ_1, Ξ as a segment.

Suppose that $\Gamma - \mathscr{E}(\Xi)$ has no $(V_1 \to W_1)$-chain. Then of course Γ has no Euler chain starting with and directed away from V_1 which has Ξ as a segment. By Lemma 18 the number of edges not accessible away from V_1 in $\Gamma - \mathscr{E}(\Xi)$ is finite, therefore by Lemma 15 $\Gamma - \mathscr{E}(\Xi)$ has a chain starting with and directed away from V_1 which includes all but a finite number of the edges of Γ. Among all the chains of $\Gamma - \mathscr{E}(\Xi)$ starting with and directed away from V_1 let Ξ_0 be one with the property that the number of edges of $\Gamma - \mathscr{E}(\Xi)$ not contained in it is minimal. The chain Ξ_0 is infinite and includes every vertex having infinite degree in Γ.

By Lemma 18, $\Gamma - \mathscr{E}(\Xi)$ has a $(W_n \to W_1)$-chain, say Ξ'. No vertex of Ξ' belongs to Ξ_0 because $\Gamma - \mathscr{E}(\Xi)$ has no $(V_1 \to W_1)$-chain. Therefore every vertex of Ξ' is different from V_1 and has finite degree in Γ, because Ξ_0 includes V_1 and every vertex having infinite degree in Γ. It follows from this and Theorem 28 that Ξ' is simple.

Let $\Gamma - \mathscr{E}(\Xi) - \mathscr{E}(\Xi') - \mathscr{E}(\Xi_0) = \Lambda$. In $\Gamma - \mathscr{E}(\Xi_0)$ every vertex which has finite degree in Γ is balanced, therefore, since no two of Ξ, Ξ', Ξ_0 have an edge in common and Ξ, Ξ' is a closed directed chain, every vertex which has finite degree in Γ is balanced in Λ. Also $\mathscr{E}(\Lambda)$ is finite.

Λ is Eulerian. For suppose not (reductio ad absurdum), then $\mathscr{E}(\Lambda) \neq \emptyset$ and so by Lemma 16, Λ contains a directed path Υ of length $\geqq 1$, neither of whose end-vertices is balanced in Λ. The two end-vertices of Υ have infinite degree in $\Gamma - \mathscr{E}(\Xi)$ because every vertex having finite degree in Γ is balanced in Λ. By the directed analogue of the reasoning used in the proof of Theorem 12, Ξ_0 can be augmented into a chain of $\Gamma - \mathscr{E}(\Xi)$ which includes the edges of Υ, which is contrary to the definition of Ξ_0. This contradiction proves that Λ is Eulerian.

$\mathscr{E}(\Lambda) = \emptyset$. For suppose not (reductio ad absurdum), then, since Λ is Eulerian, by König's Theorem Λ is the union of a finite number of cycles and possibly isolated vertices. No cycle of Λ contains V_1, because if a cycle of Λ contained V_1 then Ξ_0 could be augmented by prefixing the vertices and edges of this cycle, which would be contrary to the definition of Ξ_0. Therefore by Theorem 28 every cycle of Λ includes a vertex having infinite degree in Γ. Each such vertex is included in Ξ_0. Consequently Ξ_0 can be augmented into a chain of $\Gamma - \mathscr{E}(\Xi)$ which includes all the edges of Λ. But this is contrary to the definition of Ξ_0. This contradiction proves that $\mathscr{E}(\Lambda) = \emptyset$.

Therefore $\mathscr{E}(\Gamma) - \mathscr{E}(\Xi) - \mathscr{E}(\Xi_0) = \mathscr{E}(\Xi')$. Consequently in $\Gamma - \mathscr{E}(\Xi)$ every edge not belonging to Ξ' is accessible away from V_1. In $\Gamma - \mathscr{E}(\Xi)$ no vertex of Ξ' is accessible away from V_1 because $\Gamma - \mathscr{E}(\Xi)$ has no $(V_1 \to W_1)$-chain. Therefore the edges of $\Gamma - \mathscr{E}(\Xi)$ which are not accessible away from V_1 in $\Gamma - \mathscr{E}(\Xi)$ are precisely the edges of Ξ'. It was shown above that Ξ' is simple, so its edges together with all their end-vertices form a $(W_n \to W_1)$-path.

Ξ, Ξ' is a closed directed chain of Γ, and no vertex of Ξ' is accessible away from V_1 in $\Gamma - \mathscr{E}(\Xi)$. Therefore by the first part of Theorem 35, Γ has an Euler chain starting with and directed away from V_1 which has as a segment a rotation of Ξ, Ξ' starting with a vertex of Ξ which does not belong to Ξ'. Theorem 35 is now proved.

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