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ON CERTAIN HOMOMORPHISM PROPERTIES OF GRAPHS I

IVAN TAFTEBERG JAKOBSEN

The present paper and its continuation "On certain homomorphism properties of graphs II" contain the first part of the results presented in my thesis ([4]). These two papers will in the sequel be referred to as I and II.

The first section gives the definitions and terminology used in both I and II, and section 2 is an introduction to both papers containing an account of earlier investigations concerning the problem considered. The results are then presented in the following sections.

In this paper it is proved that every finite graph with $n \geqq 8$ vertices and at least $5n-14$ edges which is not a 4-cockade composed of complete 7-graphs is homomorphic to a complete 8-graph with two edges deleted.

1. Definitions.

1) *General definitions.*

In this paper a graph is an undirected graph without loops and without multiple edges.

Let Γ be a graph. Then $V(\Gamma)$ denotes the set of vertices and $E(\Gamma)$ the set of edges. $n(\Gamma)$ denotes the number of vertices in Γ and $e(\Gamma)$ the number of edges. If S is a set, $|S|$ denotes the number of elements of S. An edge joining two vertices x and y is denoted by (x,y) or (y,x). The *complement* of Γ denoted by $\bar{\Gamma}$, is the graph whose set of vertices is $V(\Gamma)$ and whose set of edges is the set of all un-ordered pairs of distinct vertices not contained in $E(\Gamma)$. If $W \subseteqq V(\Gamma)$ then $\Gamma - W$ denotes the graph obtained from Γ by deleting all vertices belonging to W and all edges incident with at least one vertex of W. $\Gamma(W)$ denotes the *subgraph of Γ spanned by the set W* defined as the subgraph of Γ whose set of vertices is W and whose set of edges is the set of all edges of Γ having both endvertices in W. Any such subgraph is called a *spanned subgraph.*

Let Γ' be a subgraph of Γ. The graph $\Gamma - V(\Gamma')$ is also written $\Gamma - \Gamma'$. Let $\Gamma' - x$ denote $\Gamma' - \{x\}$ if $x \in V(\Gamma')$, otherwise Γ'. If $x,y \in V(\Gamma)$,

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$(x,y) \notin E(\Gamma)$ (resp. $(x,y) \in E(\Gamma)$), then $\Gamma \cup (x,y)$ (resp. $\Gamma - (x,y)$) denotes the graph obtained from Γ by adding (resp. deleting) the edge (x,y). For convenience if $(x,y) \notin E(\Gamma')$ we define $\Gamma' - (x,y) = \Gamma'$. If $x \notin V(\Gamma')$, $\Gamma' \cup x$ denotes $\Gamma(V(\Gamma') \cup \{x\})$. Let $v(x,\Gamma)$ denote the *valency* of x in Γ. The subgraph of Γ spanned by the set of vertices joined to x is called the *neighbour-configuration* of x in Γ.

A *regular* graph Γ is a graph in which the valency of each vertex is the same, and this number is called the *valency of* Γ.

A *path* Π is a graph with vertices $x_1, x_2, \ldots, x_\mu$, $\mu \geqq 2$, and edges (x_1,x_2), $(x_2,x_3), \ldots, (x_{\mu-1},x_\mu)$, where $x_1, \ldots, x_\mu$ are all distinct. The vertices x_1 and x_μ are called the *end-vertices* of the path and are said to be *joined by the path*. Let $x,y \in V(\Pi)$. Then $\Pi[x,y]$ denotes that subgraph of Π which is a path and has x and y as its two end-vertices. $\Pi[x,x]$ is defined as the graph consisting of the vertex x. The *length* of a path is the number of its edges.

If Γ_1 and Γ_2 are two disjoint subgraphs of Γ, a $(\Gamma_1)(\Gamma_2)$-*path* is a path contained in Γ with one end-vertex belonging to Γ_1, and the other to Γ_2 and which has nothing else in common with $\Gamma_1 \cup \Gamma_2$.

A subgraph $\Delta \subseteqq \Gamma'$ is said to be *joined* to a set W of vertices of Γ, if in Γ each vertex of W is joined by an edge to at least one vertex of Δ.

A *circuit* is a graph with vertices $x_1, x_2, \ldots, x_\mu$, $\mu \geqq 3$, and edges (x_1,x_2), $(x_2,x_3), \ldots, (x_{\mu-1},x_\mu), (x_\mu,x_1)$, where $x_1, \ldots, x_\mu$ are all distinct. The circuit is denoted by $((x_1,x_2,\ldots,x_\mu))$. If the circuit is a subgraph of a graph Γ it will sometimes be denoted by $\Gamma((x_1,x_2,\ldots,x_\mu))$ to emphasize this. The *length* of a circuit is the number of its edges. A circuit of length μ is called a μ-*circuit*.

A graph Γ is said to be λ-*fold connected*, $\lambda \geqq 2$, if $n(\Gamma) \geqq \lambda + 1$ and whenever $\leqq \lambda - 1$ vertices are deleted the remaining graph is connected.

Let Γ be a connected graph. A *cut-set* of Γ is a set $S \subseteqq V(\Gamma)$ such that $\Gamma - S$ is disconnected. A *minimal cut-set* of Γ is a cut-set of which no proper sub-set is a cut-set of Γ. A cut-set is said to *separate* Γ. Furthermore Γ is said to be *separated* by a subgraph Γ' if $\Gamma - \Gamma'$ is disconnected.

A set of vertices of a graph Γ is called *independent* if no two of them are joined by an edge, and a set of edges of Γ is called *independent* if no two of them have a vertex in common.

A finite graph with ν vertices in which each pair of distinct vertices is joined by an edge is called a *complete ν-graph* and denoted by $\langle \nu \rangle$. By convention $\langle 0 \rangle = \emptyset$.

A $\langle \nu \rangle$ with exactly one edge deleted is denoted by $\langle \nu - \rangle$ (for $\nu \geqq 2$).

A $\langle \nu \rangle$ with exactly two edges deleted is denoted by $\langle \nu = \rangle$ (for $\nu \geqq 3$). If the two deleted edges are independent, it is denoted by $\langle \nu = i \rangle$.

A $\langle \nu \rangle$ with exactly three edges deleted is denoted by $\langle \nu \equiv \rangle$ (for $\nu \geqq 3$). If the three deleted edges are independent, it is denoted by $\langle \nu \equiv i \rangle$.

A $\langle \nu \rangle$ with exactly four edges deleted is denoted by $\langle \nu \equiv \rangle$ (for $\nu \geqq 4$). If the four deleted edges are independent, it is denoted by $\langle \nu \equiv i \rangle$.

2) *Contractions.*

Let Γ and Δ be two graphs. A *contraction* is a mapping m from $V(\Gamma)$ *onto* $V(\Delta)$ such that (1) $\forall x \in V(\Delta): \Gamma(m^{-1}(x))$ is connected and (2) $\forall x,y \in V(\Delta)$: $(x,y) \in E(\Delta)$ if and only if Γ contains at least one edge joining a vertex of $m^{-1}(x)$ and a vertex of $m^{-1}(y)$.

Δ is said to be obtained from Γ by a contraction and Γ is said to be *contracted into* Δ if such a mapping exists and is applied on Γ.

Γ is said to be *homomorphic* to Δ, written $\Gamma \succ \Delta$, if Γ can be contracted into a graph containing Δ as a subgraph. Γ not homomorphic to Δ is written $\Gamma \not\succ \Delta$.

If Γ is contracted into a graph consisting of one vertex then Γ is said to be *contracted into one vertex* for short.

By the term *contraction of an edge* is meant the contraction of the graph consisting of the edge and its two end-vertices into one vertex.

3) *Projections.*

Let Γ be a connected, finite graph. Let Λ be a spanned proper subgraph of Γ. Let C denote a connected component of $\Gamma - \Lambda$, and let x be any vertex of Λ joined to C. Any contraction P from $\Gamma(V(\Lambda \cup C))$ into a graph Δ, defined by contracting $C \cup x$ into one vertex and keeping the other vertices of Λ fixed is called a *simple projection from C onto Λ* and Δ is denoted by $P\Lambda$. $V(\Delta) = V(\Lambda)$ and $\Delta \supseteqq \Lambda$.

For convenience the identical mapping on $V(\Lambda)$ is also considered to be a simple projection from any connected component of $\Gamma - \Lambda$ onto Λ.

Let $C_1, \ldots, C_n$, $n = 1$ possibly, be distinct connected components of $\Gamma - \Lambda$. Let P_i be a simple projection from C_i onto Λ, $i = 1, \ldots, n$. Here P_i may possibly be the identical mapping on $V(\Lambda)$. By applying the simple projections $P_1, \ldots, P_n$ successively a contraction P of $\Gamma(\Lambda \cup C_1 \cup \ldots \cup C_n)$ into a graph Δ is defined, where $V(\Delta) = V(\Lambda)$ and $\Delta \supseteqq \Lambda$. Any contraction P defined in such a way is called a *projection* from $\Gamma - \Lambda$ onto Λ, and P is said to be composed of the successive simple projections $P_1, \ldots, P_n$, written $P = P_n \circ P_{n-1} \circ \ldots \circ P_1$. Here $P = P_1$ possibly (i.e. a simple projection is a special case of a projection). Δ is denoted by $P\Lambda$. Each of the simple projections P_j, $1 \leqq j \leqq n$, is said to be a *part* of P. Note that P and $P\Lambda$ are independent of the order of the parts.

$P - P_j$, for $1 \leqq j \leqq n$, denotes for $n = 1$ the identical mapping on $V(\Lambda)$, and for $n \geqq 2$ the projection from $\Gamma - \Lambda$ onto Λ composed of all $P_1, \ldots, P_n$, except P_j.

If P' is the identical mapping on $V(\Lambda)$, then $P - P'$ is defined to be P, regardless of whether P' is a part of P or not.

Let $x, y \in V(\Lambda)$ and $(x,y) \notin E(\Lambda)$. If the projection P from $\Gamma - \Lambda$ onto Λ yields a graph $P\Lambda$ containing $\Lambda \cup (x,y)$ as a subgraph, then the *new edge* (x,y) is said to be *provided for* Λ *by* P from $\Gamma - \Lambda$.

REMARK. Let P be any projection from Γ—Λ onto Λ. It is possible that $P\Lambda = \Lambda$ even if P is not the identical mapping on $V(\Lambda)$. Clearly $\Gamma \succ P\Lambda$.

A similar concept of projection has been introduced by W. Mader in [6].

4) *Cockades.*

Let μ be an integer $\geqq 0$ and let X_1 and X_2 be two graphs each containing a $\langle\mu\rangle$ as a subgraph.

Any graph K being the union of $\varkappa$ ($\varkappa$ any integer $\geqq 1$) graphs $\Phi_1, \Phi_2, \ldots, \Phi_\varkappa$ such that for $2 \leqq k \leqq \varkappa$

$$(\Phi_1 \cup \Phi_2 \cup \ldots \cup \Phi_{k-1}) \cap \Phi_k = \langle\mu\rangle$$

and Φ_1 and Φ_k is isomorphic to either X_1 or X_2, is called an (X_1, X_2)-*cockade of strength* μ. K is said to be *composed* of $\Phi_1, \Phi_2, \ldots, \Phi_\varkappa$, successively.

The class of all (X_1, X_2)-cockades of strength μ is denoted by $\mathscr{K}^\mu(X_1, X_2)$.

REMARK. Let $K \in \mathscr{K}^\mu(X_1, X_2)$. K may be composed of just one graph isomorphic to either X_1 or X_2, that is $K \simeq X_1$, or $K \simeq X_2$ may be the case If $\mu = 0$, K is the union of a finite number of disjoint graphs each being isomorphic to X_1 or X_2.

A special case is $X_1 \simeq X_2 \simeq X$. Then an (X_1, X_2)-cockade of strength μ is simply called an X-cockade of strength μ, and $\mathscr{K}^\mu(X_1, X_2)$ is also denoted by $\mathscr{K}^\mu(X)$. In the case $X = \langle\nu\rangle$, $\nu \geqq \mu$, these graphs were first described by G. A. Dirac in [2].

There are two types of cockades that will be of importance here. The first type is the $\langle\nu\rangle$-cockades of strength μ, where $\mu \leqq \nu$ and ν is an integer $\geqq 1$. The class $\mathscr{K}^\mu(\langle\nu\rangle)$ is denoted by $\mathscr{K}_\nu^\mu$. The other type are the $(\langle\nu - 1\rangle, \langle\nu + 1 \equiv i\rangle)$-cockades of strength $\nu - 3$, where ν is an integer $\geqq 7$, and this class is denoted by $\mathscr{C}_{\nu-1}^{\nu-3}$.

Finally a definition concerning the notation of numbers: Let t be a real number $\geqq 0$. Then $[t]$ denotes the greatest integer less than or equal to t and $\{t\}$ denotes the least integer greater than or equal to t.

2. Introduction.

It is the object of the papers I and II to consider some special cases of a general extremal problem in the theory of graphs. Among the results is a theorem announced in [3], but not proved there.

The general extremal problem which is considered is the following:

PROBLEM. Let n and ν be integers such that $n \geqq \nu \geqq 4$. How many edges must a finite graph with n vertices contain in order to be homomorphic to a complete ν-graph or to a complete ν-graph with one or two edges deleted?

Investigations of this problem were initiated by G. A. Dirac in [2]. The results hitherto obtained have been of two kinds. The first kind are best possible results for small values of ν. It was proved by Dirac in [2] that:

THEOREM α. *For $\nu = 5$ and 6: Every finite graph with n vertices, $n \geqq \nu$, and at least $(\nu - 5/2)n - \frac{1}{2}(\nu - 1)(\nu - 3)$ edges which is not a member of $\mathscr{K}_{\nu-1}^{\nu-3}$ is homomorphic to a $\langle \nu - \rangle$, and*

THEOREM β. *For $\nu = 5$ and 6: Every finite graph with n vertices, $n \geqq \nu$, and at least $(\nu - 3)n - \frac{1}{2}(\nu - 1)(\nu - 4)$ edges which is not a member of $\mathscr{K}_{\nu-1}^{\nu-4}$ is homomorphic to a $\langle \nu = \rangle$.*

It is easy to see that both of the above theorems are true for $\nu = 4$ as well.

Furthermore it was proved by W. Mader in [6] that:

THEOREM γ. *For $3 \leqq \nu \leqq 7$: Every finite graph with n vertices, $n \geqq \nu$ and at least $(\nu - 2)n - \frac{1}{2}\nu(\nu - 3)$ edges is homomorphic to a $\langle \nu \rangle$.*

Theorem γ for $\nu = 3,4$ and 5 follows from Theorem β for $\nu = 4,5$ and 6, respectively. Theorem γ is best possible, because as observed by Dirac in [2] there exist for each $n \geqq \nu$ graphs with n vertices, and $(\nu - 2)n - \frac{1}{2}\nu(\nu - 3) - 1$ edges, which are not homomorphic to a $\langle \nu \rangle$, for example the members of $\mathscr{K}_{\nu-1}^{\nu-2}$.

The second kind of results are estimates of the number of edges required for arbitrary values of n and ν.

It was proved by W. Mader in [5] that there exists a mapping d_0 from the integers $\geqq 3$ to the real numbers such that every finite graph Γ with $n(\Gamma) \geqq \nu$ satisfying the condition

$$e(\Gamma) \geqq d_0(\nu) n(\Gamma) ,$$

where

$$d_0(\nu) = \operatorname{Inf}\{t \mid t \text{ a real number and every graph satisfying } e(\Gamma) \geqq tn(\Gamma) \text{ is homomorphic to a } \langle \nu \rangle\} ,$$

is homomorphic to a $\langle \nu \rangle$. Moreover it was proved (see [2], [5], and [6]) that the following lower and upper bounds hold for $d_0(\nu)$:

$$\nu - 2 \leqq d_0(\nu) \leqq 8\{\nu \log \nu / \log 2\} \quad (\nu \geqq 3) .$$

From this and Theorem γ it follows that $d_0(\nu) = \nu - 2$ for $3 \leqq \nu \leqq 7$.

The present author proved in [3] that Theorem β holds for $\nu = 7$ as well and claimed that it holds for $\nu = 8$. This will be proved in section 4 of this paper. These two results imply Theorem γ for $\nu = 6$ and 7.

In section 6 of the paper II it is proved that Theorem α holds for $\nu = 7$ as well if $\mathscr{K}_{\nu-1}^{\nu-3}$ is replaced by the class $\mathscr{C}_{\nu-1}^{\nu-3}$.

In section 7 some estimates of the required number of edges for arbitrary values of ν and certain (infinitely many) values of n are proved. These are, I think, mainly of interest because they show what cannot be hoped for regarding the establishing of analogues for higher values of ν of Theorems α, β, and γ.

Section 3 contains some theorems and auxiliary results to be used in section 4, while section 5 contains theorems and auxiliary results to be used in section 6.

3. Results to be used.

The following theorems (A) and (B) will be needed.

(A) Every finite graph with n vertices, $n \geqq 7$, and at least $4n - 9$ edges which is not a member of $\mathscr{K}_6{}^3$ is homomorphic to a $\langle 7 = \rangle$. (See Theorem 1 n [3]; this is Theorem β for $\nu = 7$.)

(B) *A Corollary to an extension of the Theorem of Menger* (see [1]). Let λ be a positive integer and Γ a λ-fold connected graph. Let $A, B \subseteqq V(\Gamma)$, $A \neq \emptyset$, $B \neq \emptyset$, $A \cap B = \emptyset$. A mapping $\mathcal{A}$ from A to the positive integers is defined such that if $|A| \geqq \lambda$ then $\mathcal{A}(a) = 1$ for all $a \in A$ and if $|A| < \lambda$ then $\sum_{a \in A} \mathcal{A}(a) = \lambda$.

If $\mathcal{B}$ is a mapping from B to the positive integers defined analogously then Γ contains a set of λ $(A)(B)$-paths $\mathcal{Y}$ such that

1. The intersection of any two of the members of $\mathcal{Y}$ has nothing in common with $\Gamma-(A\cup B)$.
2. $\forall a\in A$: a belongs to at most $\mathcal{A}(a)$ of the paths in $\mathcal{Y}$.
3. $\forall b\in B$: b belongs to at most $\mathcal{B}(b)$ of the paths in $\mathcal{Y}$.

Furthermore we need two lemmas:

Lemma 1. *Let $K \in \mathcal{K}_{\nu}{}^{\mu}$, where $\nu \geqq \mu \geqq 0$. Then*

$$e(K) = \tfrac{1}{2}(\nu+\mu-1)n(K)-\tfrac{1}{2}\nu\mu .$$

The proof is by induction over the number of $\langle\nu\rangle$-s of which K is composed and is contained in [2] for $\mu>0$. The lemma clearly holds for $\mu=0$ as well.

Lemma 2. *Let $K \in \mathcal{K}_{\nu-1}^{\nu-4}$ for $\nu \geqq 4$. Then the following holds.*

A. *If $K \neq \langle \nu-1\rangle$, then K contains at least six vertices of valency $\nu-2$, and among these there are six which in K span a graph consisting of two disjoint triangles.*

B. *$K \not> \langle \nu = \rangle$, but if $K \neq \langle\nu-1\rangle$ and an edge joining any two vertices not already joined by an edge is added to K and any edge of K deleted, then the resulting graph is homomorphic to a $\langle\nu=\rangle$.*

The lemma is proved as lemma 1 in [3] for $\nu\geqq 5$; it clearly holds for $\nu=4$ as well.

Remark. Let X be a graph and let $K \in \mathcal{K}^{\mu}(X)$. Let K be composed of $\Phi_1,\Phi_2,\ldots,\Phi_\varkappa$. Then every complete subgraph of K is a subgraph of some Φ_i.

4. Homomorphism Theorems for $\langle 8=\rangle$.

Theorem 1. *Let Γ be a graph containing a vertex x_0 of valency 9. Let the vertices joined to x_0 be denoted by $x_1,x_2,\ldots,x_9$, and let $\Gamma(x_1,x_2,\ldots,x_9)$ be denoted by Γ_9. If Γ satisfies the following conditions*

1) *Γ is 5-fold connected,*
2) *Γ is not separated by a $\langle 5\rangle$, $\langle 5-\rangle$ or $\langle 5=\rangle$,*
3) *$\forall\, x \in V(\Gamma)$: $v(x,\Gamma)\geqq 9$,*
4) *$\forall\, x_k \in V(\Gamma_9)$: $v(x_k,\Gamma_9)\geqq 5$,*

then $\Gamma > \langle 8=\rangle$.

PROOF OF THEOREM 1. If $\Gamma = \Gamma_9 \cup x_0$, then by 3), $\Gamma = \langle 10 \rangle$. Hence it may be assumed that $\Gamma - \Gamma_9 - x_0 \neq \emptyset$. Every connected component of $\Gamma - \Gamma_9 - x_0$ is by 1) joined to at least five vertices of Γ_9. By 4, $e(\Gamma_9) \geqq \frac{1}{2}5 \cdot 9$ that is $e(\Gamma_9) \geqq 23$.

It will first be proved that the following is a consequence of the assumptions of Theorem 1.

(1) There exists a projection P from $\Gamma - \Gamma_9 - x_0$ onto Γ_9 such that $e(P\Gamma_9) \geqq 26$. (Possibly P is the identical mapping on $V(\Gamma_9)$.)

PROOF OF (1). The proof will be by *reductio ad absurdum*.

Suppose that there does not exist any projection P from $\Gamma - \Gamma_9 - x_0$ onto Γ_9 such that $e(P\Gamma_9) \geqq 26$.

(1.1). *Γ_9 does not contain a $\langle 6 \rangle$ or $\langle 6- \rangle$.*

For suppose it does. The total number of edges of Γ_9 incident with any three vertices of Γ_9 is by 4) at least 12. Hence $e(\Gamma_9) \geqq 14 + 12 = 26$, contrary to hypothesis. This proves (1.1).

Obviously $e(\Gamma_9) \leqq 25$, hence at least four vertices of Γ_9 have valency 5 in Γ_9.

(1.2). *$\Gamma - \Gamma_9 - x_0$ has at least two connected components.*

For suppose not. Then it has exactly one, C say. Suppose that $v(x_1, \Gamma_9) = 5$ and (x_1, x_2), (x_1, x_3), $(x_1, x_4) \notin E(\Gamma_9)$. Then by 3) C is joined to each of x_1, x_2, x_3, x_4 because the valency of these vertices in $\Gamma_9 \cup x_0$ is at most 8. By contracting $C \cup x_1$ into one vertex a simple projection P is obtained such that $e(P\Gamma_9) \geqq e(\Gamma_9) + 3$, which is contrary to hypothesis, because $e(\Gamma_9) \geqq 23$. This proves (1.2).

(1.3). *Every connected component of $\Gamma - \Gamma_9 - x_0$ is joined to three vertices of Γ_9 such that one of them is not joined to either of the others.*

Suppose on the contrary that C_1 is a connected component of $\Gamma - \Gamma_9 - x_0$ such that any three vertices of Γ_9 joined to C_1 span a $\langle 3- \rangle$ or a $\langle 3 \rangle$ in Γ_9. Then any five vertices of Γ_9 joined to C_1 span a graph containing a $\langle 5 = i \rangle$ as a subgraph and any six vertices of Γ_9 joined to C_1 span a graph containing a $\langle 6 \equiv i \rangle$ as a subgraph. From this it follows by 2) that it may without loss of generality be assumed that C_1 is joined to $x_1, x_2, \ldots, x_6$

and that $\Gamma_9(x_1,\ldots,x_6) \geqq \langle 6 \equiv i\rangle$. The total number of edges of Γ_9 incident with x_7,x_8,x_9 is by 4) at least 12, hence $e(\Gamma_9) \geqq 24$. By (1.1), $\Gamma_9(x_1,\ldots,x_6) \leqq \langle 6 = \rangle$. By (1.2), $\Gamma - \Gamma_9 - x_0$ contains another connected component, C_2 say. By 2) and (1.1), C_2 is joined to two vertices of Γ_9 which are not joined by an edge in Γ_9. Therefore there exist simple projections P_1 and P_2 from C_1, C_2 onto Γ_9 respectively such that $e((P_1 \circ P_2)\Gamma_9) \geqq 24+2=26$, contrary to hypothesis. This contradiction proves (1.3).

Now let C_1 be a connected component of $\Gamma - \Gamma_9 - x_0$. Assume without loss of generality that C_1 is joined to $x_1,\ldots,x_5$ and by (1.3) that $(x_1,x_2),(x_2,x_3) \notin E(\Gamma_9)$. Let C_2 be another connected component of $\Gamma - \Gamma_9 - x_0$. By (1.3), C_2 is joined to $x_p,x_q,x_r \in V(\Gamma_9)$ such that (x_p,x_q), $(x_q,x_r) \notin E(\Gamma_9)$. Now there always exist simple projections P_1 and P_2 from C_1, C_2 onto Γ_9 respectively such that $e((P_1 \circ P_2)\Gamma_9) \geqq e(\Gamma_9)+3 \geqq 26$, unless $x_p = x_1, x_q = x_2, x_r = x_3$ (or, analogously, $x_p = x_3, x_q = x_2, x_r = x_1$) and $\Gamma_9(x_1,\ldots,x_5) = \langle 5 = \rangle$, hence by hypothesis this must be the case. But then by 2), C_1 is joined to at least one more vertex of Γ_9, say to x_6. Again there exist simple projections P_1 and P_2 from C_1, C_2 onto Γ_9 respectively such that $e((P_1 \circ P_2)\Gamma_9) \geqq 26$, unless x_6 is joined to each of $x_1,\ldots,x_5$. In this case, however, by contracting $C_1 \cup x_2$ into one vertex a simple projection P from C_1 onto Γ_9 is obtained such that $e(P\Gamma_9) \geqq e(\langle 6 = \rangle)+12+2=27$, contrary to hypothesis. This completes the proof of (1).

(2) If there exists a projection P from $\Gamma - \Gamma_9 - x_0$ onto Γ_9 (possibly P is the identical mapping on $V(\Gamma_9)$) such that $e(P\Gamma_9) \geqq 27$, then $\Gamma \succ \langle 8 = \rangle$.

Proof of (2). As $e(P\Gamma_9) \geqq 27 = 4\cdot 9 - 9$, it follows that if $P\Gamma_9 \notin \mathscr{K}_6{}^3$ then $P\Gamma_9 \succ \langle 7 = \rangle$ by (A), hence $\Gamma \succ \langle 8 = \rangle$. Assume now that $P\Gamma_9 \in \mathscr{K}_6{}^3$ is composed of two $\langle 6\rangle$-s. Assume without loss of generality

$$P\Gamma_9(x_1,x_2,x_3,x_4,x_5,x_6) = \langle 6\rangle, \quad P\Gamma_9(x_4,x_5,x_6,x_7,x_8,x_9) = \langle 6\rangle .$$

By 4) no edge incident with a vertex of valency 5 in $P\Gamma_9$ can have been provided by P, hence at most the edges $(x_4,x_5),(x_5,x_6),(x_6,x_4)$ can have been provided by P.

Let C be a connected component of $\Gamma - \Gamma_9 - x_0$. Assume that C is joined to a vertex different from x_4,x_5,x_6 in both $\langle 6\rangle$-s of $P\Gamma_9$; say C is joined to x_2 and x_7. By contracting each of $C \cup x_2$, $\Gamma(x_1,x_4)$, and $\Gamma(x_3,x_6)$ into one vertex $\Gamma - x_0$ is contracted into a graph containing a $\langle 7 = \rangle$ as a subgraph, all the vertices of which are joined to x_0, hence $\Gamma \succ \langle 8 = \rangle$ in this case. Assume now that C is not joined to vertices different from

x_4, x_5, x_6 in *both* $\langle 6\rangle$-s of $P\Gamma_9$. Then C is joined to at least five vertices of one of the $\langle 6\rangle$-s of $P\Gamma_9$, say to five of the vertices $x_1, x_2, \ldots, x_6$. By contracting each of $\Gamma(x_7, x_4)$, $\Gamma(x_9, x_6)$ and C into one vertex $\Gamma - x_0$ is contracted into a graph containing a $\langle 7-\rangle$ as a subgraph six vertices of which are joined to x_0, hence again $\Gamma \succ \langle 8=\rangle$. This proves (2).

(3) If there exists a projection P from $\Gamma - \Gamma_9 - x_0$ onto Γ_9 such that $P\Gamma_9 \supset \langle 6\rangle$, then $\Gamma \succ \langle 8=\rangle$.

Proof of (3): By 4) the total number of edges of Γ_9 incident with any three vertices of Γ_9 is at least 12. Hence $e(P\Gamma_9) \geqq 12+15=27$ and $\Gamma \succ \langle 8=\rangle$ by (2). This proves (3).

In the rest of the proof of Theorem 1 let P be a projection from $\Gamma - \Gamma_9 - x_0$ onto Γ_9 such that $e(P\Gamma_9)=26$.

(i) $P\Gamma_9$ contains a vertex of valency 5 in $P\Gamma_9$. For otherwise $e(P\Gamma_9) \geqq 27$.

(ii) If there is a vertex of valency 5 in $P\Gamma_9$ whose neighbour-configuration in $P\Gamma_9$ contains a vertex of valency $\leqq 2$ in the neighbour-configuration, then $\Gamma \succ \langle 8=\rangle$.

Proof of (ii): Let the notation be chosen so that x_1 is joined to exactly $x_2, x_3, \ldots, x_6$ in $P\Gamma_9$ and

$$v(x_2, P\Gamma_9(x_2, \ldots, x_6)) \leqq 2\ .$$

By contracting $P\Gamma_9(x_1, x_2)$ into one vertex $P\Gamma_9$ is contracted into a graph Δ such that $n(\Delta)=8$, $e(\Delta) \geqq 26-5+2=23=4\cdot 8-9$. $\Delta \notin \mathscr{K}_6^3$ because a member of $\mathscr{K}_6^3$ cannot have 8 vertices, hence by (A), $\Delta \succ \langle 7=\rangle$ and consequently $\Gamma \succ \langle 8=\rangle$. This proves (ii).

Because of (ii) it may be assumed from now on that there are no vertices of the kind described in (ii). Furthermore, if any vertex of $P\Gamma_9$ has a $\langle 5\rangle$ as neighbour-configuration in $P\Gamma_9$, then $P\Gamma_9 \supset \langle 6\rangle$ and by (3), $\Gamma \succ (8=\rangle$. Hence in what follows it will be assumed that this is not the case. These assumptions imply that

(iii) Every vertex of valency 5 in $P\Gamma_9$ has one of the two neighbour-configurations $\langle 5=i\rangle$ or $\langle 5-\rangle$ in $P\Gamma_9$.

These two possibilities will be analysed in turn.

(A) *Assume that there is a vertex in $P\Gamma_9$ whose neighbour-configuration in $P\Gamma_9$ is a $\langle 5=i\rangle$.*

Assume without loss of generality that x_1 is joined to exactly $x_2, x_3, \ldots, x_6$ that $P\Gamma_9(x_2, \ldots, x_6) = \langle 5 = i \rangle$, and that $(x_2, x_3), (x_5, x_6) \notin E(P\Gamma_9)$.

The total number of edges of $P\Gamma_9$ incident with x_7, x_8, x_9 is $26 - 13 = 13$, hence by 4) there are two alternatives, α) and β):

α) $v(x_k, P\Gamma_9) = 5$ for $k = 7, 8, 9$ and $P\Gamma_9(x_7, x_8, x_9)$ is a $\langle 3 - \rangle$.

β) Two vertices of x_7, x_8, x_9 have valency 5 in $P\Gamma_9$, one vertex has valency 6 in $P\Gamma_9$ and $P\Gamma_9(x_7, x_8, x_9)$ is a $\langle 3 \rangle$.

Alternative α). Assume without loss of generality $(x_7, x_8), (x_8, x_9) \in E(P\Gamma_9)$ and $(x_7, x_9) \notin E(P\Gamma_9)$. The vertex x_7 has valency 5 in $P\Gamma_9$, hence by (iii) its neighbour-configuration in $P\Gamma_9$ contains a $\langle 5 = i \rangle$ as a subgraph.

If x_7 were joined to x_2, x_3, x_5, x_6 in $P\Gamma_9$, then x_8 would have to be joined to x_2, x_3, x_5, x_6 in $P\Gamma_9$ as well, contrary to the fact that x_8 has valency 5 in $P\Gamma_9$. Hence x_7 is not joined to all of x_2, x_3, x_5, x_6 in $P\Gamma_9$ and by an analogous argument x_9 is not joined to all of x_2, x_3, x_5, x_6 in $P\Gamma_9$. Therefore x_7 and x_9 are both joined to x_4 in $P\Gamma_9$. Furthermore x_7 is joined to three more vertices among x_2, x_3, x_5, x_6 in $P\Gamma_9$. Assume without loss of generality x_7 joined to x_2, x_3, x_5 in $P\Gamma_9$. The neighbour-configuration of x_7 in $P\Gamma_9$ contains a $\langle 5 = i \rangle$ as a subgraph by (iii), hence x_8 is joined to x_2 and x_3 in $P\Gamma_9$. Furthermore the neighbour-configuration of x_8 in $P\Gamma_9$ contains a $\langle 5 = i \rangle$ as a subgraph by (iii), hence x_9 is joined to x_2 and x_3 in $P\Gamma_9$. Now $(x_8, x_6) \notin E(P\Gamma_9)$, because otherwise by (iii) applied to x_8, $(x_7, x_6) \in E(P\Gamma_9)$ contrary to $v(x_7, P\Gamma_9) = 5$. But $v(x_6, P\Gamma_9) \geqq 5$, hence necessarily $(x_6, x_9) \in E(P\Gamma_9)$. By (iii) the neighbour-configuration of x_9 in $P\Gamma_9$ contains a $\langle 5 = i \rangle$ as a subgraph, hence $(x_8, x_4) \in E(P\Gamma_9)$. There are no further edges in $P\Gamma_9$.

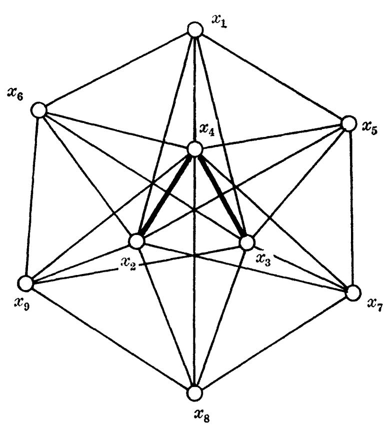

Figure 1.

$P\Gamma_9$ may be described as follows: It consists of a 6-circuit, $((x_1,x_5,x_7,x_8,x_9,x_6))$, and a path of length 2, spanned by the vertices x_2,x_3,x_4, disjoint from it, such that each vertex of the 6-circuit is joined to all three vertices of the path of length 2. (See Figure 1).

No edge incident with a vertex of valency 5 in $P\Gamma_9$ can have been provided by P, hence at most (x_3,x_4) and (x_2,x_4) can have been provided by P (heavily drawn on figure). In particular, $((x_1,x_5,x_7,x_8,x_9,x_6)) \subseteqq \Gamma_9$.

Let C be a connected component of $\Gamma-\Gamma_9-x_0$. Assume first that C is joined to 2 non-consecutive vertices of the 6-circuit $((x_1,x_5,x_7,x_8,x_9,x_6))$. Then it may be supposed by symmetry that C is joined to x_9 and x_1 or to x_9 and x_5. By contracting each of $\Gamma(x_8,x_4)$, $\Gamma(x_7,x_3)$ and $C\cup x_9$ into one vertex, $\Gamma-x_0$ is contracted into a graph containing a $\langle 7=\rangle$ as a subgraph, all the vertices of which are joined to x_0, hence $\Gamma\succ\langle 8=\rangle$ in this case. Assume secondly that no connected component of $\Gamma-\Gamma_9-x_0$ is joined to two non-consecutive vertices of the abovementioned 6-circuit. Then $\Gamma-\Gamma_9-x_0$ has at least three connected components because each of x_1,x_5,x_7,x_8,x_9,x_6 is joined to $\Gamma-\Gamma_9-x_0$ (the minimal valency of a vertex in Γ being 9); and every component is joined to x_2,x_3,x_4 and two consecutive vertices of $((x_1,x_5,x_7,x_8,x_9,x_6))$. Let C_1 and C_2 be two components of $\Gamma-\Gamma_9-x_0$. By contracting each of $C_1\cup x_4$, $C_2\cup x_2$, and $\Gamma(x_5,x_7,x_8)$ into one vertex $\Gamma-x_0$ is contracted into a graph containing a $\langle 7=\rangle$ as a subgraph all the vertices of which are joined to x_0, hence $\Gamma\succ\langle 8=\rangle$ in this case also.

Alternative β). Assume without loss of generality $v(x_7,P\Gamma_9)=6$, $v(x_k,P\Gamma_9)=5$ for $k=8,9$. x_7 is then joined to four of x_2,x_3,x_4,x_5,x_6 in $P\Gamma_9$, hence without loss of generality it may be assumed that x_7 is joined in $P\Gamma_9$ either to

$$\text{a)}\quad x_2,x_3,x_5,x_6 \qquad \text{or to} \qquad \text{b)}\quad x_2,x_3,x_4,x_5\,.$$

Case a). We have $v(x_k,P\Gamma_9)\geqq 6$ for $k=2,3,5,6$ by (iii) because x_2,x_3,x_5,x_6 are all joined to each of x_1,x_4,x_7 and (x_1,x_7), $(x_4,x_7)\notin E(P\Gamma_9)$. Hence x_2,x_3,x_5,x_6 are each joined to at least one of x_8,x_9 in $P\Gamma_9$. If either x_8 or x_9 is joined to x_4 in $P\Gamma_9$, then by contracting each of $P\Gamma_9(x_2,x_7)$ and $P\Gamma_9(x_8,x_9)$ into one vertex, $P\Gamma_9$ is contracted into a $\langle 7=\rangle$, therefore $\Gamma\succ\langle 8=\rangle$ in this case. Now (x_8,x_4), $(x_9,x_4)\notin E(P\Gamma_9)$ may be assumed. Then without loss of generality suppose that x_9 is joined to exactly x_7,x_8,x_6,x_5,x_3 in $P\Gamma_9$. Then necessarily $(x_2,x_8)\in E(P\Gamma_9)$. By (iii) the neighbour-configuration of x_9 contains a $\langle 5=i\rangle$ as a subgraph, hence x_8 is joined to x_5 and x_6 in $P\Gamma_9$. There are no further edges in $P\Gamma_9$. By contracting each of $P\Gamma_9(x_7,x_3)$ and $P\Gamma_9(x_5,x_9)$ into one vertex $P\Gamma_9$ is contracted into a $\langle 7=\rangle$, hence $\Gamma\succ\langle 8=\rangle$ in this case also.

Case b). The vertex x_6 is joined to x_8 or x_9 in $P\Gamma_9$, because $v(x_6, P\Gamma_9) \geqq 5$. Furthermore x_6 is joined to both x_8 and x_9 in $P\Gamma_9$, because if e.g. x_6 is joined to x_9 and not to x_8 then $v(x_6, P\Gamma_9) = 5$ and by (iii) the neighbour-configuration of x_6 in $P\Gamma_9$ contains a $\langle 5 = i \rangle$ as a subgraph so that x_9 is joined to x_2, x_3, x_4 in $P\Gamma_9$ ((x_2, x_3) and $(x_1, x_9) \notin E(P\Gamma_9)$), but this is contrary to the fact that $v(x_9, P\Gamma_9) = 5$. Hence x_6 is joined to both x_8 and x_9 in $P\Gamma_9$.

We have $v(x_k, P\Gamma_9) \geqq 6$ for $k = 2, 3$ by (iii) because x_2, x_3 are both joined to each of x_1, x_6, x_7, and $(x_1, x_7), (x_6, x_7) \notin E(P\Gamma_9)$. Hence x_2 and x_3 are each joined to at least one of x_8, x_9, in $P\Gamma_9$. If one of x_8, x_9, say x_8, is joined to both x_2 and x_3 in $P\Gamma_9$, then by contracting each of $P\Gamma_9(x_8, x_2)$ and $P\Gamma_9(x_7, x_9)$ into one vertex $P\Gamma_9$ is contracted into a $\langle 7 = \rangle$, hence $\Gamma \succ \langle 8 = \rangle$ in this case. Therefore it may now be assumed without loss of generality that (x_2, x_8), $(x_3, x_9) \in E(P\Gamma_9)$ and (x_8, x_3), $(x_9, x_2) \notin E(P\Gamma_9)$. For $k = 8, 9$ we have $v(x_k, P\Gamma_9) = 5$, hence by (iii) the neighbour-configuration of x_k in $P\Gamma_9$ contains a $\langle 5 = i \rangle$ as a subgraph. Therefore (x_9, x_5), $(x_8, x_5) \notin E(P\Gamma_9)$, because (x_5, x_6), $(x_6, x_7) \notin E(P\Gamma_9)$. Then necessarily (x_8, x_4), $(x_9, x_4) \in E(P\Gamma_9)$. There are no further edges in $P\Gamma_9$.

$P\Gamma_9$ may be described as follows: It consists of a 4-circuit $\sum = ((x_3, x_7, x_2, x_6))$, and a set $S = \{x_1, x_5, x_8, x_9\}$, of four vertices not on $\sum$, each having valency 5 in $P\Gamma_9$ and each joined to three vertices of $\sum$ so that no two vertices of S are joined to the same three vertices of $\sum$, and finally a ninth vertex, x_4, joined to all the others. Furthermore any two vertices of S are joined by an edge if and only if they are joined to the same pair of non-adjacent vertices of $\sum$. (See Figure 2).

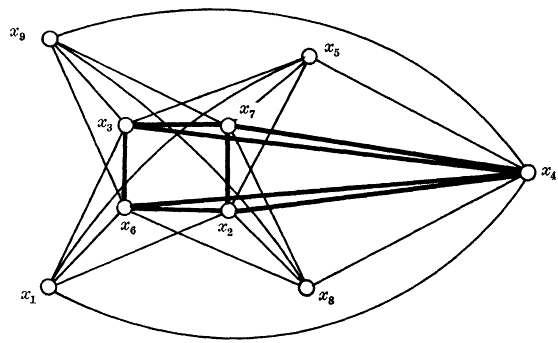

Figure 2.

By 4) no edge incident with a vertex of valency 5 in $P\Gamma_9$ can have been provided by P, hence at most the edges of $\sum$ and the edges joining x_4 to the vertices of $\sum$ can have been provided by P (heavily drawn on figure).

1. *Assume that at least one edge of* $\sum$ *has been provided by* P.

Assume without loss of generality that (x_7, x_2) has been provided by P. The valency of x_7 and x_2 in Γ_9 is then 5 by 4), hence no other edge incident with x_7 or x_2 can have been provided by P. Then (x_7, x_2) has been provided for Γ_9 by a simple projection denoted by P_1 from a connected component C of $\Gamma - \Gamma_9 - x_0$ onto Γ_9 as part of P, and no other edge of $P\Gamma_9$ not in Γ_9 has been provided by P_1 for Γ_9. C is of course joined to x_2 and x_7. Assume that C is not joined to any of x_1, x_3, x_6 or x_9. Then by 1), C is joined to exactly x_2, x_4, x_5, x_7, x_8. But $(x_4, x_2), (x_4, x_7) \in E(\Gamma_9)$ because as mentioned above these edges cannot have been provided by P; furthermore $(x_2, x_7) \notin E(\Gamma_9)$, hence $\Gamma_9(x_2, x_4, x_5, x_7, x_8) = \langle 5 = \rangle$ contrary to 2). This contradiction shows that C must be joined to one of x_1, x_3, x_6, x_9. It may then be assumed that C is joined to x_3 or x_9. Let P_2 denote the simple projection from C onto Γ_9 obtained by contracting $C \cup x_2$ into one vertex. $P' = (P - P_1) \circ P_2$ is then a projection from $\Gamma - \Gamma_9 - x_0$ onto Γ_9 such that $P'\Gamma_9 \supseteqq P\Gamma_9 \cup (x_2, x_3)$ or $P'\Gamma_9 \supseteqq P\Gamma_9 \cup (x_2, x_9)$ and therefore $e(P'\Gamma_9) \geqq 27$; hence by (2) $\Gamma \succ \langle 8 = \rangle$. (Note that $P = P_1$ possibly). So if at least one edge of $\sum$ has been provided by P then $\Gamma \succ \langle 8 = \rangle$.

At most three edges can have been provided for Γ_9 by P because $e(\Gamma_9) \geqq 23$. Hence the alternative to the assumption that at least one edge of $\sum$ has been provided by P is:

2. *The only edges provided by* P, *if any, are three or fewer of* (x_4, x_2), (x_4, x_3), (x_4, x_6), (x_4, x_7).

In what follows assume without loss of generality that $(x_4, x_6) \in E(\Gamma_9)$ and at most (x_4, x_2), (x_4, x_3), (x_4, x_7) have been provided by P.

Let C be a connected component of $\Gamma - \Gamma_9 - x_0$. Assume that no edge of $P\Gamma_9$ not in Γ_9 has been provided for Γ_9 by a simple projection from C onto Γ_9 as a part of P. It may then be assumed that P does not contain a part which is a simple projection from C onto Γ_9. C is joined to at least five vertices of Γ_9 and $P\Gamma_9 \not\supseteq \langle 5 \rangle$, hence there clearly exists a simple projection P_1 from C onto Γ_9 which provides at least one new edge for Γ_9 not contained in $P\Gamma_9$. It follows that $P_1 \circ P$ is a projection from $\Gamma - \Gamma_9 - x_0$ onto Γ_9 such that $e((P_1 \circ P)\Gamma_9) \geqq 27$, hence by (2), $\Gamma \succ \langle 8 = \rangle$ in this case. Consequently it may be assumed that at least one edge of $P\Gamma_9$ not in Γ_9 has been provided for Γ_9 by a simple projection from C onto Γ_9 as a part of P. Let this simple projection be denoted by P_1.

Now let the following three cases (a), (b) and (c) be considered.

(a) *C is joined to x_6 and x_7.* Let P_2 denote the simple projection from C onto Γ_9 obtained by contracting $C \cup x_7$ into one vertex, and let $P^* = P_2 \circ (P - P_1)$. Clearly $(x_6, x_7) \in E(P^*\Gamma_9)$. Also $(x_4, x_7) \in E(P^*\Gamma_9)$. In fact this is clearly true if $(x_4, x_7) \in E(\Gamma_9)$ and also if (x_4, x_7) is provided for Γ_9 by $P - P_1$. The remaining possibility is that (x_4, x_7) is provided for Γ_9 by P_1. If this is so then obviously x_4 is joined to C and consequently $(x_4, x_7) \in E(P_2\Gamma_9)$, hence $(x_4, x_7) \in E(P^*\Gamma_9)$. By contracting each of $P^*\Gamma_9(x_2, x_5)$ and $P^*\Gamma_9(x_1, x_3)$ into one vertex $P^*\Gamma_9$ is contracted into a $\langle 7 = \rangle$. Hence $\Gamma \succ \langle 8 = \rangle$ in this case.

(b) *C is joined to x_2 and one of x_3 and x_9.* Let P_2 denote the simple projection from C onto Γ_9 obtained by contracting $C \cup x_2$ into one vertex, and let $P^* = P_2 \circ (P - P_1)$. Clearly (x_2, x_3) or $(x_2, x_9) \in E(P^*\Gamma_9)$. Also $(x_4, x_2) \in E(P^*\Gamma_9)$ for the same reasons that $(x_4, x_7) \in E(P^*\Gamma_9)$ in case (a) above. By contracting each of $P^*\Gamma_9(x_7, x_8)$ and $P^*\Gamma_9(x_3, x_9)$ into one vertex, $P^*\Gamma_9$ is contracted into a $\langle 7 = \rangle$. Hence $\Gamma \succ \langle 8 = \rangle$ in this case.

(c) *C is joined to x_3 and x_8.* By symmetry this is analogous to case (b).

Each of the vertices of Γ_9 except possibly x_4 is joined to one or more connected components of $\Gamma - \Gamma_9 - x_0$ because the valency of these vertices in $\Gamma_9 \cup x_0$ is at most 7 and their valency in Γ is at least 9. This is in particular the case for x_6. Let C from now on be a connected component of $\Gamma - \Gamma_9 - x_0$ joined to x_6.

If one of the cases (a), (b) or (c) holds for C, then $\Gamma \succ \langle 8 = \rangle$ is proved above. Assume consequently that none of (a), (b), (c) holds for C. Furthermore it may be assumed that at least one edge of $P\Gamma_9$ not in Γ_9 has been provided for Γ_9 by a simple projection, denoted by P_1, from C onto Γ_9 as a part of P, because otherwise $\Gamma \succ \langle 8 = \rangle$ as proved above for any connected component of $\Gamma - \Gamma_9 - x_0$. In this case C is not joined to x_7, and not to both of x_2 and x_3. Therefore exactly one edge of $P\Gamma_9$ has been provided by P_1 as a part of P, namely either (x_4, x_2) or (x_4, x_3), by symmetry it may be assumed to be (x_4, x_2). Then C is of course joined to x_4 and x_2. Since cases (a) and (b) do not hold it follows that C is joined to none of x_3, x_7, x_9. But C is joined to at least five vertices of Γ_9, hence it is joined to at least two of the vertices x_1, x_5, x_8. If C is not joined to x_5, then Γ is separated by $\Gamma_9(x_1, x_6, x_2, x_8, x_4) = \langle 5 = \rangle$, which is contrary to 2). If C is not joined to x_8, then Γ is separated by $\Gamma_9(x_1, x_6, x_2, x_5, x_4) = \langle 5 = \rangle$, which is contrary to 2). Hence C is joined to a least x_6, x_2, x_4, x_5, x_8. Let P_2 denote the simple projection from C onto Γ_9 obtained by contracting $C \cup x_5$ into one vertex. Then $P^* = P_2 \circ (P - P_1)$ is a projection from $\Gamma - \Gamma_9 - x_0$ onto

Γ_9 such that $P^*\Gamma_9 \geqq P\Gamma_9 \cup (x_5, x_6) \cup (x_5, x_8) - (x_2, x_4)$ and therefore $e(P^*\Gamma_9) \geqq 27$ hence by (2) $\Gamma \succ \langle 8 = \rangle$.

This shows that $\Gamma \succ \langle 8 = \rangle$ also if (a), (b) and (c) do not hold for C. It has then been proved that $\Gamma \succ \langle 8 = \rangle$ also in the alternative case β). Hence $\Gamma \succ \langle 8 = \rangle$ if possibility (A) holds. The alternative to (A) is by (iii)

(**B**) *Every vertex of valency* 5 *in* $P\Gamma_9$ *has a* $\langle 5 - \rangle$ *as neighbour-configuration in* $P\Gamma_9$.

Assume without loss of generality that x_1 is joined to exactly $x_2, \ldots, x_6$ in $P\Gamma_9$, that $P\Gamma_9(x_2, \ldots, x_6) = \langle 5 - \rangle$, and that $(x_2, x_3) \notin E(P\Gamma_9)$. The total number of edges of $P\Gamma_9$ incident with x_7, x_8, x_9 is then $26 - 14 = 12$, hence by 4) necessarily $P\Gamma_9(x_7, x_8, x_9) = \langle 3 \rangle$ and each of x_7, x_8, x_9 has valency 5 in $P\Gamma_9$ and is consequently joined to exactly 3 of $x_2, \ldots, x_6$ in $P\Gamma_9$.

At least one of x_7, x_8, x_9 *is joined to both* x_2 *and* x_3 *in* $P\Gamma_9$. For x_2 is joined to at least one of x_7, x_8, x_9 in $P\Gamma_9$ because of 4). Assume without loss of generality that it is joined to x_7. By (**B**) the neighbour-configuration of x_7 is a $\langle 5 - \rangle$, hence x_2 is joined to at least two of x_7, x_8, x_9. Similarly x_3 is joined to at least two of x_7, x_8, x_9. But then clearly at least one of x_7, x_8, x_9 is joined to both x_2 and x_3.

Assume without loss of generality in what follows that x_7 is joined to both x_2 and x_3. Assume forthermore without loss of generality that x_7 is joined to x_4. By (**B**) the neighbour-configuration of x_7 in $P\Gamma_9$ is a $\langle 5 - \rangle$, hence x_8 and x_9 are both joined to x_2, x_3, x_4. There are no further edges in $P\Gamma_9$. So $P\Gamma_9$ consists of two $\langle 6 - \rangle$ -s ($P\Gamma_9(x_1, x_2, \ldots, x_6)$ and $P\Gamma_9(x_2, x_3, x_4, x_7, x_8, x_9)$) having a $\langle 3 - \rangle$ ($P\Gamma_9(x_2, x_3, x_4)$) in common. (See Figure 3).

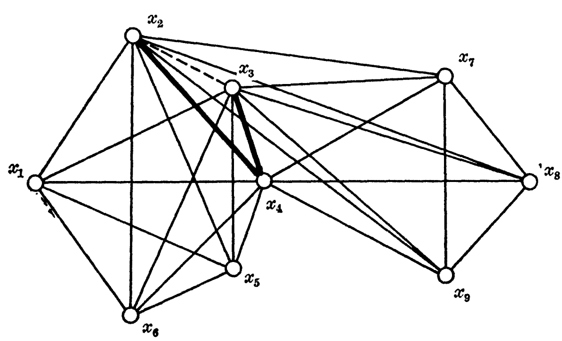

Figure 3.

Let $\Gamma_9(x_1,x_2,\ldots,x_6)$ be denoted by Λ_1 and $\Gamma_9(x_2,x_3,x_4,x_7,x_8,x_9)$ by Λ_2. No edge incident with a vertex of valency 5 in $P\Gamma_9$ can have been provided by P, hence at most the edges (x_2,x_4), (x_3,x_4) can have been provided by P (heavily drawn on figure).

Let C be a connected component of $\Gamma-\Gamma_9-x_0$. Assume that C is joined to at least five vertices of either Λ_1 or Λ_2, say Λ_1. By contracting each of $\Gamma(x_7,x_2)$, $\Gamma(x_8,x_3)$, and C into one vertex $\Gamma-x_0$ is contracted into a graph containing a $\langle 7-\rangle$ as a subgraph, six vertices of which are joined to x_0, hence $\Gamma\succ\langle 8=\rangle$ in this case. The alternative is that C is joined to a vertex of Λ_1 and a vertex of Λ_2, both different from x_2,x_3,x_4. Assume without loss of generality that C is joined to x_5, and x_9. By contracting each of $C\cup x_9$, $\Gamma(x_7,x_2)$, and $\Gamma(x_8,x_3)$ into one vertex $\Gamma-x_0$ is contracted into a graph containing a $\langle 7=\rangle$ as a subgraph, all the vertices of which are joined to x_0, hence $\Gamma\succ\langle 8=\rangle$ in this case also.

Thus $\Gamma\succ\langle 8=\rangle$ also when (**B**) holds, and (**B**) is the alternative to (**A**). Hence $\Gamma\succ\langle 8=\rangle$ always, and the proof of Theorem 1 is completed.

Theorem 2. *Let Γ be a finite graph with n vertices and e edges. If $n\geqq 8$, $e\geqq 5n-14$, and $\Gamma\notin\mathscr{K}_7{}^4$, then $\Gamma\succ\langle 8=\rangle$.*

Proof. By induction over n. The theorem is trivially true for $n=8$.

Induction hypothesis: Assume the theorem is true for all graphs with m vertices satisfying the conditions, where $8\leqq m\leqq n-1$.

Let Γ be any graph with n vertices and e edges satisfying the conditions of the theorem.

It is sufficient to consider the case $e=5n-14$ in the rest of the proof. For assume that $e>5n-14$ and that the theorem holds for all graphs having exactly $5n-14$ edges. By deleting edges from Γ a graph Γ^* may be obtained such that $e(\Gamma^*)=5n-14$. If $\Gamma^*\notin\mathscr{K}_7{}^4$ then $\Gamma^*\succ\langle 8=\rangle$ by the last assumption and therefore $\Gamma\succ\langle 8=\rangle$. If $\Gamma^*\in\mathscr{K}_7{}^4$ then, because $n>8$ and by Lemma 2 B, again $\Gamma\succ\langle 8=\rangle$. This proves the assertion. Assume therefore in the sequel that $e=5n-14$.

(1) If $\exists x\in V(\Gamma): v(x,\Gamma)\leqq 4$, then $\Gamma\succ\langle 8=\rangle$.

Proof of (1). As $n(\Gamma-x)=n-1\geqq 8$, and $e(\Gamma-x)\geqq e-4=5(n-1)-13$ it follows by lemma 1 and the induction hypothesis that $\Gamma-x\succ\langle 8=\rangle$.

(2) Let Γ' be a graph with n' vertices and e' edges, where $6\leqq n'<n$. If $e'\geqq 5n'-14$, then either $\Gamma'\succ\langle 8=\rangle$, or $\Gamma'\in\mathscr{K}_7{}^4$.

Proof of (2). Clearly $e' \geqq 5n'-14$ implies that $n' \neq 6$. If $n'=7$, then $e' \geqq 21$, hence $\Gamma' = \langle 7 \rangle \in \mathscr{K}_7^4$. If $n' \geqq 8$, then by the induction hypothesis $\Gamma' \succ \langle 8= \rangle$ or $\Gamma' \in \mathscr{K}_7^4$. This proves (2).

(3) If Γ is disconnected or has a cut-set S such that $|S| \leqq 4$ or such that $|S|=5$ and $\Gamma(S) = \langle 5 \rangle$, $\langle 5- \rangle$, or $\langle 5= \rangle$, then $\Gamma \succ \langle 8= \rangle$.

Proof of (3). If Γ has a cut-set, then it has a minimal cut-set. In the sequel let S denote a minimal cut-set of Γ if Γ is connected and $\emptyset$ if Γ is disconnected.

Let $\Gamma = \Gamma_1 \cup \Gamma_2$, Γ_1 and Γ_2 being spanned subgraphs of Γ such that $V(\Gamma_1 \cap \Gamma_2) = S$ and $\Gamma_1 - S \neq \emptyset$ and $\Gamma_2 - S \neq \emptyset$. Let $|S| = \sigma$, where $0 \leqq \sigma \leqq 5$, and if $\sigma \geqq 1$ let $S = \{s_1, s_2, \ldots, s_\sigma\}$. Let $|E(\Gamma(S))| = p$ and

$$|V(\Gamma_i)| = n_i, \quad |E(\Gamma_i)| = e_i \quad \text{for} \quad i = 1, 2 .$$

Then $n = n_1 + n_2 - \sigma$ and $e = e_1 + e_2 - p$. If $S \neq \emptyset$, then let Γ_i' be a connected component of $\Gamma - S$ contained in $\Gamma_i - S$ for $i = 1, 2$. Then Γ_i' is joined by edges to every vertex of S, because S is a minimal cut-set.

Let P_1 denote the simple projection from Γ_1' onto Γ_2 obtained by contracting $\Gamma_1' \cup s_1$ into one vertex, and let P_2 denote the simple projection from Γ_2' onto Γ_1 obtained by contracting $\Gamma_2' \cup s_1$ into one vertex.

If $n_i \leqq 5$ for $i = 1$ or 2 then every vertex of $\Gamma_i - S$ has valency $\leqq 4$ in Γ, therefore by (1), $\Gamma \succ \langle 8= \rangle$ in this case. Hence it may be assumed from now on that

(3.1) $$n_i \geqq 6, \quad i = 1, 2 .$$

Now

$$e_1 + e_2 = e + p = 5n - 14 + p = 5(n_1 + n_2) - 5\sigma - 14 + p .$$

By the symmetry between Γ_1 and Γ_2 it may be assumed that

$$e_1 \geqq 5n_1 - \tfrac{1}{2}(5\sigma + 14 - p) .$$

i) *Suppose that* $\sigma \leqq 2$.
Then $p \leqq 1$ and $e_1 \geqq 5n_1 - 12$, hence by (3.1), (2) and Lemma 1, $\Gamma_1 \succ \langle 8= \rangle$.

ii) *Suppose that* $\sigma = 3$.
1) *Let* $p \geqq 2$. Then $e_1 \geqq 5n_1 - 27/2$. By (3.1), (2) and Lemma 1, $\Gamma_1 \succ \langle 8= \rangle$.

2) *Let* $p \leqq 1$. Then $e_1 \geqq 5n_1 - 29/2$. Assume without loss of generality $(s_1, s_2), (s_1, s_3) \notin E(\Gamma)$. Consider $P_2\Gamma_1 = \Gamma_1 \cup (s_1, s_2) \cup (s_1, s_3)$. Then $n(P_2\Gamma_1) = n_1 \geqq 6$ by (3.1) and $e(P_2\Gamma_1) = e_1 + 2 \geqq 5n_1 - 25/2$. By Lemma 1 and (2), $P_2\Gamma_1 \succ \langle 8= \rangle$.

This shows that it remains to assume that Γ is connected and has no cut-set with fewer than 4 vertices.

iii) *Suppose that* $\sigma = 4$.

1) *Let* $p = 6$. Then $e_1 \geqq 5n_1 - 14$. By (3.1) and (2), $\Gamma \succ \langle 8 = \rangle$ except when $\Gamma_1 \in \mathscr{K}_7{}^4$. If $\Gamma_1 \in \mathscr{K}_7{}^4$, then $e_1 = 5n_1 - 14$ by Lemma 1 and consequently $e_2 \geqq 5n_2 - 14$. By (3.1) and (2), $\Gamma_2 \succ \langle 8 = \rangle$ except when $\Gamma_2 \in \mathscr{K}_7{}^4$. But if this is so then $\Gamma \in \mathscr{K}_7{}^4$, contrary to hypothesis. Hence $\Gamma \succ \langle 8 = \rangle$ in this case.

2) *Let* $p = 5$. Then $e_1 \geqq 5n_1 - 29/2$. Assume without loss of generality that $(s_1, s_2) \notin E(\Gamma)$. Consider $P_2\Gamma_1 = \Gamma_1 \cup (s_1, s_2)$. Now $n(P_2\Gamma_1) = n_1 \geqq 6$ by (3.1) and $e(P_2\Gamma_1) = e_1 + 1 > 5n_1 - 14$. By (2) and Lemma 1, $P_2\Gamma_1 \succ \langle 8 = \rangle$.

3) *Let* $p = 4$. Then $e_1 \geqq 5n_1 - 15$. Assume without loss of generality that $(s_1, s_2) \notin E(\Gamma)$ and s_3 is incident with the other missing edge, denoted by ε, and s_1 is not incident with ε. Consider $P_2\Gamma_1 = \Gamma_1 \cup (s_1, s_2)$. Then $n(P_2\Gamma_1) = n_1 \geqq 6$ by (3.1) and $e(P_2\Gamma_1) = e_1 + 1 \geqq 5n_1 - 14$. By (2), $P_2\Gamma_1 \succ \langle 8 = \rangle$ except when $P_2\Gamma_1 \in \mathscr{K}_7{}^4$. Futhermore $P_2\Gamma_1 \neq \langle 7 \rangle$ because $\varepsilon \notin E(P_2\Gamma_1)$. Assume therefore that $P_2\Gamma_1 \in \mathscr{K}_7{}^4$. By contracting $\Gamma_2' \cup s_3$ into one vertex we get $P_2\Gamma_1 - (s_1, s_2) \cup \varepsilon$, which is homomorphic to $\langle 8 =)$ by Lemma 2B since $\varepsilon \notin E(P_2\Gamma_1)$.

4) *Let* $p = 3$. Then $e_1 \geqq 5n_1 - 31/2$. Now $\Gamma(S)$ contains at least one vertex of valency $\leqq 1$ in $\Gamma(S)$. Assume without loss of generality that s_1 is such a vertex and $(s_1, s_2), (s_1, s_3) \notin E(\Gamma)$. Consider $P_2\Gamma_1 \supseteqq \Gamma_1 \cup (s_1, s_2) \cup (s_1, s_3)$. Then $n(P_2\Gamma_1) = n_1 \geqq 6$ by (3.1) and $e(P_2\Gamma_1) \geqq e_1 + 2 \geqq 5n_1 - 27/2 > 5n_1 - 14$. By Lemma 1 and (2), $P_2\Gamma_1 \succ \langle 8 = \rangle$.

5) *Let* $p = 2$. Then $e_1 \geqq 5n_1 - 16$. Assume without loss of generality that $(s_1, s_2), (s_1, s_3) \notin E(\Gamma) (s_1, s_4) \in E(\Gamma)$ and that s_2 is incident with a third missing edge denoted by ε. Consider $P_2\Gamma_1 = \Gamma_1 \cup (s_1, s_2) \cup (s_1, s_3)$. Then $n(P_2\Gamma_1) = n_1 \geqq 6$ by (3.1) and $e(P_2\Gamma_1) = e_1 + 2 \geqq 5n_1 - 14$. By (2), $P_2\Gamma_1 \succ \langle 8 = \rangle$ except when $P_2\Gamma_1 \in \mathscr{K}_7{}^4$. Furthermore $P_2\Gamma_1 \neq \langle 7 \rangle$ because $\varepsilon \notin E(P_2\Gamma_1)$. Assume therefore that $P_2\Gamma_1 \in \mathscr{K}_7{}^4$. By contracting $\Gamma_2' \cup s_2$ into one vertex we get $P_2\Gamma_1 - (s_1, s_3) \cup \varepsilon$, which is homomorphic to $\langle 8 = \rangle$ by Lemma 2B since $\varepsilon \notin E(P_2\Gamma_1)$.

6) *Let* $p = 1$. Then $e_1 \geqq 5n_1 - 33/2$. Assume without loss of generality that (s_3, s_4) is the only edge of $\Gamma(S)$. Consider $P_2\Gamma_1 = \Gamma_1 \cup (s_1, s_2) \cup (s_1, s_3) \cup (s_1, s_4)$. Then $n(P_2\Gamma_1) = n_1 \geqq 6$ by (3.1) and $e(P_2\Gamma_1) = e_1 + 3 \geqq 5n_1 - 27/2 > 5n_1 - 14$. By Lemma 1 and (2), $P_2\Gamma_1 \succ \langle 8 = \rangle$.

7) *Let* $p=0$. Then $e_1 \geqq 5n_1-17$. Consider $P_2\Gamma_1=\Gamma_1 \cup (s_1,s_2) \cup (s_1,s_3) \cup (s_1,s_4)$. Then $n(P_2\Gamma_1)=n_1 \geqq 6$ by (3.1) and $e(P_2\Gamma_1)=e_1+3 \geqq 5n_1-14$. By (2), $P_2\Gamma_1 \succ \langle 8= \rangle$ except when $P_2\Gamma_1 \in \mathscr{K}_7{}^4$. Suppose that $P_2\Gamma_1 \in \mathscr{K}_7{}^4$. Then by Lemma 1, $e(P_2\Gamma_1)=5n_1-14$. By the remark just after Lemma 2, $P_2\Gamma_1$ in this case contains a $\langle 7 \rangle$, Λ say, to which s_1 and s_4 both belong. $s_2 \notin V(\Lambda)$ because $(s_2,s_4) \notin E(P_2\Gamma_1)$. Also $s_3 \notin V(\Lambda)$ because $(s_3,s_4) \notin E(P_2\Gamma_1)$. Now $\Gamma-s_1-s_4$ is 2-fold connected because Γ has no cut-set with fewer than 4 vertices, hence by (B) contains 2 disjoint $(\Lambda-s_1-s_4)$ $(\Gamma_2' \cup s_2 \cup s_3)$-paths, say Π_1 and Π_2. Clearly Π_1, Π_2 have s_2 and s_3 as one pair of end-vertices. (See Figure 4).

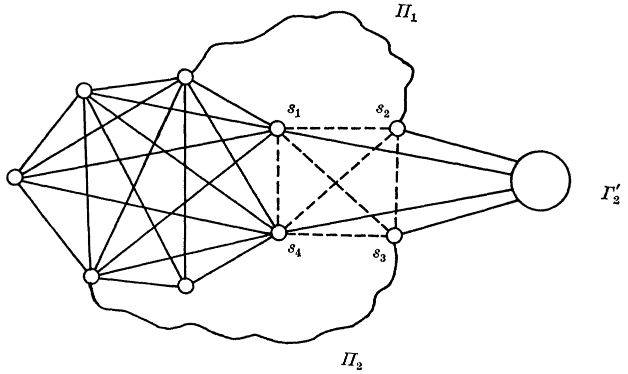

Figure 4.

The graph $\Gamma'=\Gamma_2 \cup (s_1,s_2) \cup (s_2,s_4) \cup (s_1,s_3) \cup (s_3,s_4) \cup (s_2,s_3)$ is obtained from Γ by contracting each of Π_1 and Π_2 into one vertex. $e_1=e(P_2\Gamma_1)-3=5n_1-17$, hence $e_2 \geqq 5n_2-17$. Therefore $e(\Gamma')=e_2+5 \geqq 5n_2-12$. Also $n(\Gamma')=n_2 \geqq 6$ by (3.1). By Lemma 1 and (2) therefore $\Gamma' \succ \langle 8= \rangle$.

iv) *Suppose that* $\sigma=5$.

1) *Let* $\Gamma(S)=\langle 5 \rangle$. Then $p=10$ and $e_1 \geqq 5n_1-29/2$, hence $e_1 \geqq 5n_1-14$. By (3.1) and (2) $\Gamma \succ \langle 8= \rangle$ except when $\Gamma_1 \in \mathscr{K}_7{}^4$. If $\Gamma_1 \in \mathscr{K}_7{}^4$, then (by the remark just after Lemma 2), $\Gamma(S)$ is contained in a $\langle 7 \rangle \subseteqq \Gamma_1$. By contracting Γ_2' into one vertex Γ is contracted into a graph containing a $\langle 8= \rangle$ as a subgraph.

2) *Let* $\Gamma(S)=\langle 5- \rangle$. Then $p=9$ and $e_1 \geqq 5n_1-15$. Assume without loss of generality $(s_1,s_2) \notin E(\Gamma)$. Consider $P_2\Gamma_1=\Gamma_1 \cup (s_1,s_2)$. Then $n(P_2\Gamma_1)=n_1 \geqq 6$ by (3.1) and $e(P_2\Gamma_1)=e_1+1 \geqq 5n_2-14$. By (2), $P_2\Gamma_1 \succ \langle 8= \rangle$ except when $P_2\Gamma_1 \in \mathscr{K}_7{}^4$. If $P_2\Gamma_1 \in \mathscr{K}_7{}^4$, then by Lemma 1, $e_1=5n_1-15$ and consequently $e_2=5n_2-15$. Consider $P_1\Gamma_2=\Gamma_2 \cup (s_1,s_2)$. Then $n(P_1\Gamma_2)=$

$n_2 \geqq 6$ by (3.1) and $e(P_1\Gamma_2) = e_2 + 1 = 5n_2 - 14$. By (2), $P_1\Gamma_2 \succ \langle 8= \rangle$ except when $P_1\Gamma_2 \in \mathscr{K}_7{}^4$. Suppose that $P_1\Gamma_2 \in \mathscr{K}_7{}^4$. Now $P_2\Gamma_1(S)$ is contained in a $\langle 7 \rangle \subseteqq P_2\Gamma_1$ and $P_1\Gamma_2(S)$ is contained in a $\langle 7 \rangle \subseteqq P_1\Gamma_2$. Let the former $\langle 7 \rangle$ be denoted by Λ' and the latter by Λ''. Then $\Gamma \supseteqq \Lambda' \cup \Lambda'' - (s_1, s_2) \succ \langle 8= \rangle$.

3) *Let* $\Gamma(S) = \langle 5= \rangle$. Then $p = 8$ and $e_1 \geqq 5n_1 - 31/2$, hence $e_1 \geqq 5n_1 - 15$. From here the argumentation is identical with that of case iii), 3).

As $\sigma \leqq 5$ all possibilities are exhausted. This completes the proof of (3).

(4) If Γ is 5-fold connected and Γ is not separated by a $\langle 5 \rangle$, $\langle 5- \rangle$, or $\langle 5= \rangle$, then $\Gamma \succ \langle 8= \rangle$.

PROOF OF (4). Assume Γ has the properties stated in (4). The proof will be by the steps (4.1)–(4.8).

(4.1) $$\text{If} \quad \Gamma \supseteqq \langle 7 \rangle, \quad \text{then} \quad \Gamma \succ \langle 8= \rangle .$$

PROOF OF (4.1). Let Λ be a $\langle 7 \rangle \subseteqq \Gamma$. Then $\Gamma - \Lambda \neq \emptyset$, because $n > 8$. Let C be a connected component of $\Gamma - \Lambda$. As Γ is 5-fold connected C is joined to at least 5 of the vertices of Λ, and by contracting C into one vertex Γ is contracted into a graph containing an $\langle 8= \rangle$ as a subgraph. This proves (4.1).

Let x_0 be a vertex of minimal valency in Γ. Let $v(x_0, \Gamma) = j$, say. Then $j \geqq 5$ because Γ is 5-fold connected. If $j \geqq 10$ then $e \geqq 5n$ contrary to the assumption that $e = 5n - 14$. Hence

(4.2) $$5 \leqq j \leqq 9 .$$

Let the vertices joined to x_0 be denoted by $x_1, x_2, \ldots, x_j$, and let $\Gamma(x_1, x_2, \ldots, x_j)$ be denoted by Γ_j.

(4.3) $$\text{If} \quad \Gamma_j = \langle j \rangle, \quad \text{then} \quad \Gamma \succ \langle 8= \rangle .$$

PROOF OF (4.3). Clearly $j \geqq 6$ implies $\Gamma_j \cup x_0 = \langle 7 \rangle$, hence by (4.1). $\Gamma \succ \langle 8= \rangle$. Now $j = 5$ implies $e(\Gamma - x_0) = 5(n-1) - 14$. By (2) with $n' = n - 1 \geqq 8$, $\Gamma - x_0 \succ \langle 8= \rangle$ except when $\Gamma - x_0 \in \mathscr{K}_7{}^4$. But if $\Gamma - x_0 \in \mathscr{K}_7{}^4$, then $\Gamma \supseteqq \langle 7 \rangle$ and by (4.1), $\Gamma \succ \langle 8= \rangle$.

(4.4) $$\text{If} \quad \exists x_i \in V(\Gamma_j) : v(x_i, \Gamma_j) \leqq 3, \quad \text{then} \quad \Gamma \succ \langle 8= \rangle .$$

PROOF OF (4.4). Assume without loss of generality $i = 1$. By contracting $\Gamma(x_0, x_1)$ into one vertex Γ is contracted into Γ', say. Then $n(\Gamma') = n - 1 \geqq 8$ and $e(\Gamma') \geqq e - j + (j-4) \geqq 5(n-1) - 13$. By Lemma 1 and (2), $\Gamma' \succ \langle 8= \rangle$.

(4.5) If $\forall x_k \in V(\Gamma_j) : v(x_k, \Gamma_j) \geqq 4$ and
$\exists\, x_i \in V(\Gamma_j) : v(x_i, \Gamma_j) = 4,$ then $\Gamma \succ \langle 8= \rangle .$

PROOF OF (4.5). Clearly $j=5$ implies $\Gamma_j = \langle j \rangle$, hence because of (4.3), $j \geqq 6$ may be assumed.

Assume without loss of generality that $i=1$ and x_1 is joined to $x_2, \ldots, x_5$. By contracting $\Gamma(x_0, x_1)$ into one vertex Γ is contracted into

$$(\Gamma - x_0) \cup \bigcup_{k=6}^{j} (x_1, x_k) = \Gamma' .$$

Then $n(\Gamma') = n-1 \geqq 8$ and $e(\Gamma') = e - j + j - 5 = 5(n-1) - 14$. By (2), $\Gamma' \succ \langle 8= \rangle$ except when $\Gamma' \in \mathscr{K}_7{}^4$. Assume therefore that $\Gamma' \in \mathscr{K}_7{}^4$. Then $\Gamma' \neq \langle 7 \rangle$, because $n-1 \geqq 8$. Futhermore $j=6$. For suppose on the contrary that $j \geqq 7$. By the contraction of $\Gamma(x_0, x_1)$ only the vertices $x_2, \ldots, x_5$ have their valency decreased, since the valency of x_1 is not decreased because $j > 5$ and $(x_1, x_j) \notin E(\Gamma)$. Each of the valencies of $x_2, \ldots, x_5$ decreases by 1. Hence the minimal valency of Γ' is $\geqq j-1 \geqq 6$ and at most four vertices have valency 6, contrary to Lemma 2A. Hence $j=6$.

In Γ the vertex x_1 is joined to $x_2, \ldots, x_5$ and not to x_6. Hence $\Gamma' = (\Gamma - x_0) \cup (x_1, x_6)$. Assume firstly that Γ' contains a $\langle 7 \rangle$ to which $x_1, \ldots, x_6$ all belong; then x_0 is joined in Γ to six vertices of a $\langle 7- \rangle \subseteqq \Gamma - x_0$, hence $\Gamma \supseteqq \langle 8= \rangle$. Assume next that this is not the case. Then $\exists\, x_q, x_r \in V(\Gamma_j)$, $2 \leqq q < r \leqq 5$, such that $(x_q, x_r) \notin E(\Gamma)$, otherwise $x_1, \ldots, x_6$ would span a complete graph in Γ' and consequently (remark just after Lemma 2) Γ' would contain a $\langle 7 \rangle$ to which $x_1, \ldots, x_6$ all belong. By contracting $\Gamma(x_0, x_q)$ into one vertex Γ is contracted into a graph containing

$$(\Gamma - x_0) \cup (x_q, x_r) = \Gamma' - (x_1, x_6) \cup (x_q, x_r) \succ \langle 8= \rangle$$

by Lemma 2B, because $(x_q, x_r) \notin E(\Gamma')$. This proves (4.5).

As a consequence of (4.4) and (4.5) it may be assumed from now on that

(4.6) $$\forall x_k \in V(\Gamma_j) : v(x_k, \Gamma_j) \geqq 5 .$$

Clearly $j=6$ implies $\Gamma_j = \langle j \rangle$, hence by (4.3), $\Gamma \succ \langle 8= \rangle$ in this case. It may therefore be assumed that

(4.7) $$j \geqq 7 .$$

Assume $\Gamma - \Gamma_j - x_0 = \emptyset$. Then $\Gamma = \Gamma_j \cup x_0$, hence $n = j+1$. Now j is the minimal valency of Γ, hence $\Gamma = \langle j+1 \rangle \supseteqq \langle 8 \rangle$ by (4.7). Consequently

(4.8) $$\Gamma - \Gamma_j - x_0 \neq \emptyset$$

may be assumed from now on. Every connected component of $\Gamma - \Gamma_j - x_0$ is joined to at least 5 vertices of Γ_j because Γ is 5-fold connected.

By (4.2) and (4.7), $7 \leqq j \leqq 9$. Each of the cases $j=7$, 8 and 9 will be considered separately.

i) *Suppose that* $j=7$.

By (4.8) there exists a connected component C of $\Gamma - \Gamma_7 - x_0$. *Assume* C *is joined to* $x_q, x_p \in V(\Gamma_7)$ *such that* $(x_q, x_p) \notin E(\Gamma)$. By contracting $C \cup x_q$ into one vertex a projection P from C onto Γ_7 is obtained such that $e(P\Gamma_7) \geqq e(\Gamma_7) + 1 \geqq \frac{1}{2}5\cdot 7 + 1$ that is $e(P\Gamma_7) \geqq 19 = 4\cdot 7 - 9$. Now $P\Gamma_7 \notin \mathscr{K}_6{}^3$, because a member of $\mathscr{K}_6{}^3$ cannot have 7 vertices, hence by (A), $P\Gamma_7 \succ \langle 7 = \rangle$, consequently $\Gamma \succ \langle 8 = \rangle$ in this case. *The alternative is that the* $\geqq 5$ *vertices of* Γ_7 *to which* C *is joined span a complete graph.* But a $\langle 5 \rangle$ does not separate Γ, hence C is joined to at least six vertices of Γ_7 spanning a $\langle 6 \rangle$ and then $\Gamma \succ \langle 8 - \rangle$. Therefore $\Gamma \succ \langle 8 = \rangle$ if $j=7$.

ii) *Suppose that* $j=8$.

Assume there exists a projection P from $\Gamma - \Gamma_8 - x_0$ onto Γ_8 (possibly P is the identical mapping on $V(\Gamma_8)$) such that $e(P\Gamma_8) \geqq 23$. Now $P\Gamma_8 \notin \mathscr{K}_6{}^3$ because a member of $\mathscr{K}_6{}^3$ cannot have 8 vertices. Hence by (A), $P\Gamma_8 \succ \langle 7 = \rangle$ and consequently $\Gamma \succ \langle 8 = \rangle$. It may then be assumed that

(4.9) There exists no projection P from $\Gamma - \Gamma_8 - x_0$ onto Γ_8 (including the identical mapping on $V(\Gamma_8)$) such that $e(P\Gamma_8) \geqq 23$.

As a consequence $e(\Gamma_8) < 23$. By (4.6), $e(\Gamma_8) \geqq \frac{1}{2}5\cdot 8 = 20$, so there are 3 possibilities: $e(\Gamma_8) = 22$, 21, and 20.

1) *Let* $e(\Gamma_8) = 22$.

(4.9′) By (4.9) no new edge can be provided for Γ_8 by any projection from $\Gamma - \Gamma_8 - x_0$ onto Γ_8.

By (4.8) there exists a connected component C of $\Gamma - \Gamma_8 - x_0$. A $\langle 5 \rangle$ does not separate Γ and C cannot by (4.9′) be joined to two vertices of Γ_8 not joined by an edge, hence C is joined to the vertices of a $\langle 6 \rangle \subseteqq \Gamma_8$. But then $\Gamma \succ \langle 8 - \rangle$.

2) *Let* $e(\Gamma_8) = 21$.

(4.9″) By (4.9) at most one new edge can be provided for Γ_8 by any projection from $\Gamma - \Gamma_8 - x_0$ onto Γ_8.

By (4.8) there exists a connected component C of $\Gamma - \Gamma_8 - x_0$. C is joined to at least 5 vertices of Γ_8 and by (4.9″) any such 5 vertices span a graph containing a $\langle 5 = i \rangle$ as a subgraph. But a $\langle 5 = \rangle$, $\langle 5 - \rangle$, or $\langle 5 \rangle$ does not separate Γ, hence C is joined to at least 6 vertices of Γ_8 and by (4.9″) these vertices span a graph containing a $\langle 6\equiv i\rangle$ as a subgraph.

Assume without loss of generality that C is joined to $x_1,x_2,\ldots,x_6$. It follows from $e(\Gamma_8)=21$ and (4.6) that $\Gamma(x_1,\ldots,x_6)=\langle 6\equiv i\rangle$, $(x_7,x_8)\in E(\Gamma)$, and $v(x_7,\Gamma)=v(x_8,\Gamma)=5$.

Since $\Gamma(x_1,\ldots,x_6)=\langle 6\equiv i\rangle$ by (4.6) each of $x_1,x_2,\ldots,x_6$ is joined to $\Gamma(x_7,x_8)$. Assume without loss of generality that $(x_1,x_2)\notin E(\Gamma)$. By contracting each of $C\cup x_1$ and $\Gamma(x_7,x_8)$ into one vertex $\Gamma-x_0$ is contracted into a graph containing a $\langle 7=\rangle$ as a subgraph all the vertices of which are joined to x_0, hence $\Gamma\succ\langle 8=\rangle$.

3) *Let* $e(\Gamma_8)=20$.

(4.9‴) By (4.9) at most two new edges can be provided for Γ_8 by any projection from $\Gamma-\Gamma_8-x_0$ onto Γ_8.

By (4.6) and 3) every vertex of Γ_8 has valency 5 in Γ_8, i.e. every vertex of $\bar{\Gamma}_8$ has valency 2 in $\bar{\Gamma}_8$. There are consequently three possibilities:

A) $\bar{\Gamma}_8$ consists of a 3-circuit and a 5-circuit, disjoint from each other.

B) $\bar{\Gamma}_8$ concists of 2 disjoint 4-circuits.

C) $\bar{\Gamma}_8$ consists of an 8-circuit.

A) *Suppose $\bar{\Gamma}_8$ consists of a 3-circuit and a 5-circuit, disjoint from each other.*

Assume without loss of generality $\bar{\Gamma}_8((x_1,x_2,x_3))$ to be the 3-circuit and $\bar{\Gamma}_8((x_4,x_5,x_6,x_7,x_8))$ to be the 5-circuit. Three subcases are distinguished:

Subcase A1. *A connected component C of $\Gamma-\Gamma_8-x_0$ is joined to 3 consecutive vertices of the 5-circuit of $\bar{\Gamma}_8$.* Assume without loss of generality that C is joined to x_4,x_5,x_6. Then by contracting each of $C\cup x_5$ and $\Gamma(x_1,x_8)$ into one vertex $\Gamma-x_0$ is contracted into a graph containing a $\langle 7=\rangle$ as a subgraph, all the vertices of which are joined to x_0, hence $\Gamma\succ\langle 8=\rangle$ in subcase A1.

Subcase A2. *An edge from the 3-circuit of $\bar{\Gamma}_8$ together with an edge from the 5-circuit of $\bar{\Gamma}_8$ can be provided for Γ_8 by a projection P from $\Gamma-\Gamma_8-x_0$ onto Γ_8.* Assume without loss of generality that (x_1,x_2) and (x_7,x_8) can be provided. Then by contracting $P\Gamma_8(x_3,x_4)$ into one vertex $P\Gamma_8$ is contracted into a $\langle 7=\rangle$, all the vertices of which are joined to x_0, hence $\Gamma\succ\langle 8=\rangle$ in subcase A2.

Subcase A3. *Neither* A1 *nor* A2 *holds*. Then $\Gamma-\Gamma_8-x_0$ has at least two connected components, because if not, the only existing connected component (there is one by (4.8)) would be joined to every vertex of Γ_8 since the minimal valency in Γ is 8, and then A1 would hold. Furthermore every connected component of $\Gamma-\Gamma_8-x_0$ is joined to at least two vertices of the 3-circuit of $\bar{\Gamma}_8$. No connected component of $\Gamma-\Gamma_8-x_0$ is then joined to two consecutive vertices of the 5-circuit of $\bar{\Gamma}_8$. For suppose the contrary. It has just been proved that there exists another connected component of $\Gamma-\Gamma_8-x_0$ which is joined to at least two vertices of the 3-circuit of $\bar{\Gamma}_8$, hence an edge from each circuit of $\bar{\Gamma}_8$ can be provided for Γ_8 in an obvious way, and then A2 would hold. It follows that every connected component of $\Gamma-\Gamma_8-x_0$ is joined to at most two of the vertices of the 5-circuit of $\bar{\Gamma}_8$ and hence to all three vertices of the 3-circuit of $\bar{\Gamma}_8$. There are at least two connected components of $\Gamma-\Gamma_8-x_0$, hence by suitable contractions all three of the edges $(x_1,x_2),(x_2,x_3),(x_1,x_3)$ can be provided for Γ_8, contrary to (4.9‴). Therefore A3 cannot hold.

The subcases A1, A2 and A3 clearly exhaust all the possibilities, hence $\Gamma \succ \langle 8=\rangle$ in case A).

B) $\bar{\Gamma}_8$ *consists of two disjoint* 4-*circuits*.

Assume without loss of generality $\bar{\Gamma}_8((x_1,x_2,x_3,x_4))$ and $\bar{\Gamma}_8((x_5,x_6,x_7,x_8))$ to be the 4-circuits of $\bar{\Gamma}_8$. Let C be a connected component of $\Gamma-\Gamma_8-x_0)$. C is joined to 3 consecutive vertices of at least one of the 4-circuits. Assume without loss of generality that C is joined to x_1,x_2,x_3. By contracting each of $C\cup x_2$ and $\Gamma(x_4,x_5)$ into one vertex $\Gamma-x_0$ is contracted into a graph containing a $\langle 7=\rangle$ as a subgraph, all the vertices of which are joined to x_0, hence $\Gamma \succ \langle 8=\rangle$ in case B).

C) $\bar{\Gamma}_8$ *consists of an* 8-*circuit*.

Assume without loss of generality the 8-circuit to be $\bar{\Gamma}_8((x_1,x_2,\ldots,x_8))$. Let C be a connected component of $\Gamma-\Gamma_8-x_0$. C is joined to at least 5 vertices of Γ_8 and Γ is not separated by a $\langle 5=\rangle$ hence (easily verified) C is joined to 3 consecutive vertices of the 8-circuit of $\bar{\Gamma}_8$, say to x_1,x_2,x_3. By contracting each of $C\cup x_2$ and $\Gamma(x_5,x_8)$ into one vertex $\Gamma-x_0$ is contracted into a graph containing a $\langle 7=\rangle$ as a subgraph all the vertices of which are joined to x_0, hence $\Gamma \succ \langle 8=\rangle$ in case C).

A), B), and C) deal with all possibilities when $e(\Gamma_8)=20$. So if $j=8$, then $\Gamma \succ \langle 8=\rangle$.

iii) *Suppose that* $j=9$.

In Theorem 1 the conditions 1) and 2) are the assumptions of (4), condition 3) is precisely iii), and condition 4) is (4.6). Hence all the conditions of Theorem 1 are satisfied and it follows that $\Gamma \succ \langle 8 = \rangle$ if $j=9$. This completes the proof of (4).

(3) and (4) exhaust all possibilities, therefore $\Gamma \succ \langle 8 = \rangle$. Thus under the induction hypothesis Theorem 2 is true for all graphs with n vertices satisfying its conditions. It is true when the number of vertices is 8, therefore it is true generally.

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UNIVERSITY OF AARHUS, DENMARK