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SOME MINIMAX THEOREMS

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1. Introduction.

Any real-valued function f defined on a product set $X \times Y$ satisfies the inequality

$$\sup_{y\in Y} \inf_{x\in X} f(x,y) \leqq \inf_{x\in X} \sup_{y\in Y} f(x,y) .$$

Von Neumann's minimax theorem [6] can be stated as follows: If X and Y are finite-dimensional simplices and f is bilinear on $X \times Y$, then equality holds above (moreover, inf and sup may be replaced by min and max). This note is related to generalizations of von Neumann's minimax theorem by Kneser [4], Fan [3] and Sion [5].

The minimax problem is approached from the viewpoint of a set F of lower semi-continuous functions on a compact space X, with the order relation $\leqq$ on F induced by the ordering of the real numbers. If for any $f,g \in F$ there exists $h \in F$ with $f \leqq h$ and $g \leqq h$, that is if F is directed by $\leqq$, then a minimax condition holds (Theorem 1 and corollary), the assumptions being analogous with those of the classical Dini theorem. The following weaker order condition is then introduced in Theorem 2: For any $f,g \in F$ there exists $h \in F$ with $f+g \leqq 2h$. This establishes a connection with the concept of a concavelike function due to Fan. Further, finite intersections of sets of the form $\{x \in X : f(x) \leqq \alpha\}$ are required to be connected in Theorem 2. The requirement is satisfied, in particular, when the quasi-convexity condition of Sion's minimax theorem holds.

An application of Theorem 2 to a function on a product set $X \times Y$ immediately yields the principal minimax theorem of the note, Theorem 3, together with two corollaries, one of which is the Kneser–Fan minimax theorem for concave-convex functions. Examples are given to show that if the assumptions of Theorem 3 are not satisfied, then the conclusion does not necessarily hold. Theorem 3 can be applied to some functions which fail to satisfy the assumptions of Fan's and Sion's general minimax theorems. These theorems are not implied by Theorem 3, however, as indicated by examples.

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2. Lower semi-continuous functions on a compact space.

The lemma below, to which we shall refer frequently, states a well-known fact. It is assumed throughout in this section that $X \neq \emptyset$ and $F \neq \emptyset$.

LEMMA. *Let X be a compact space, and let F be a set of lower semi-continuous real-valued functions on X. The following are equivalent.*

(i) *For any $\alpha \in \mathsf{R}$ and any finite non-empty subset G of F such that $\alpha < \min_{x\in X} \max_{f\in G} f(x)$, there exists $h \in F$ with $\alpha \leqq \min_{x \in X} h(x)$.*

(ii) $\sup_{f\in F} \min_{x\in X} f(x) = \min_{x\in X} \sup_{f\in F} f(x)$.

PROOF. For every non-empty set $G \subset F$, finite or not, the function defined by $x \mapsto \sup_{f\in G} f(x)$ is lower semi-continuous, allowing the value $+\infty$. Any lower semi-continuous function on the compact space X attains a minimum.

Suppose (i) holds. Choose any $\alpha \in \mathsf{R}$ with

$$\alpha < \min_{x\in X} \sup_{f\in F} f(x) ,$$

and define $A(f) = \{x \in X : f(x) \leqq \alpha\}$ for $f \in F$. Then $\bigcap_{f\in F} A(f) = \emptyset$, and since X is compact and the sets $A(f)$ are closed, there exists a finite set $G \subset F$ with $\bigcap_{f\in G} A(f) = \emptyset$. We have $\max_{f\in G} f(x) > \alpha$ for each $x \in X$. Choose $h \in F$ according to (i), then the inequalities

$$\alpha \leqq \min_{x\in X} h(x) \leqq \sup_{f\in F} \min_{x\in X} f(x)$$

imply (ii). The converse implication is trivial.

For two real-valued functions f and g defined on a set X, write $f \leqq g$ if $f(x) \leqq g(x)$ for all $x \in X$. A set F of real-valued functions on X is said to be directed with respect to the relation $\leqq$, if for any $f, g \in F$ there exists $h \in F$ with $f \leqq h$ and $g \leqq h$.

The assumptions of Theorem 1 and the corollary are analogous with the assumptions of the Dini theorem for continuous functions (see e.g. Bourbaki [2, Chapter X, § 4.1, Theorem 1 and Corollary]).

THEOREM 1. *Let X be a compact space, and let F be a set of lower semi-continuous real-valued functions on X which is directed with respect to the relation $\leqq$. Then*

$$\sup_{f\in F} \min_{x\in X} f(x) = \min_{x\in X} \sup_{f\in F} f(x) .$$

PROOF. Condition (i) of the lemma is satisfied: If $G \subset F$ is finite, then there exists $h \in F$ with $f \leqq h$ for each $f \in G$.

COROLLARY. *Let X be a compact space, and let (f_n) be a sequence of lower semi-continuous real-valued functions on X with $f_n(x) \leqq f_{n+1}(x)$ for $n=1,2,\ldots,$ and $x \in X$. Then*

$$\lim_{n\to\infty} \min_{x\in X} f_n(x) = \min_{x\in X}(\lim_{n\to\infty} f_n(x)) .$$

In view of Theorem 1 it is natural to investigate whether a weaker condition involving the order relation $\leqq$, possibly together with some additional assumption, is sufficient to imply the minimax conclusion. An answer is provided in Theorem 2 by means of a slight generalization of Fan's concept of a concavelike function together with a generalization of the concept of quasi-convexity (for these concepts, see the following section).

If $f \leqq h$ and $g \leqq h$, then $f+g \leqq 2h$ (the average of f and g is majorized by h). Thus, (i) of the following theorem is implied by the order assumption of Theorem 1. The induction proof of statement (b) below is similar to the method used in Kneser [4] and Fan [3].

THEOREM 2. *Let X be a compact space, and let F be a set of lower semi-continuous real-valued functions on X satisfying*:

(i) *For any $f,g \in F$ there exists $h \in F$ such that $f+g \leqq 2h$.*

(ii) *Every finite intersection of sets of the form $\{x \in X : f(x) \leqq \alpha\}$ is connected, with $(f,\alpha) \in F \times \mathsf{R}$. Then*

$$\sup_{f\in F} \min_{x\in X} f(x) = \min_{x\in X} \sup_{f\in F} f(x) .$$

PROOF. First we shal prove:

(a) For any $\alpha \in \mathsf{R}$ and any $f,g \in F$ such that

$$\alpha < \min_{x\in X} \max(f(x),g(x)) ,$$

there exists $h \in F$ with $\alpha < \min_{x\in X} h(x)$.

Choose β with

$$\alpha < \beta < \min_{x\in X} \max(f(x),g(x)) .$$

Defining $\gamma = \min_{x\in X} f(x)$ and $\delta = \min_{x\in X} g(x)$, we may assume that $\max(\gamma,\delta) < \beta$. According to (i), choose $k \in F$ with $f+g \leqq 2k$. Let

$$A = \{x \in X : f(x) \leqq \beta\}, \quad B = \{x \in X : g(x) \leqq \beta\} ,$$
$$C = \{x \in X : k(x) \leqq \beta\} ,$$

and observe that A and B are non-empty closed sets with $A \cap B = \emptyset$. If $x \notin A \cup B$, we have $k(x) \geqq \frac{1}{2}(f(x)+g(x)) > \beta$, so $C \subset A \cup B$. By (ii), C is connected, and therefore either $C \subset A$ or $C \subset B$ holds. If $C \subset A$, set $f_1 = k$, $g_1 = g$, and if $C \subset B$, set $f_1 = f$, $g_1 = k$. Then

$$\beta < \min_{x\in X}\max\big(f_1(x),g_1(x)\big)$$

in each case. Defining

$$\gamma_1 = \min_{x\in X} f_1(x), \quad \delta_1 = \min_{x\in X} g_1(x)\ ,$$

it can be verified that we have either $\gamma_1 > \frac{1}{2}(\gamma+\beta)$ and $\delta_1 = \delta$ (in the case $C \subset A$), or $\gamma_1 = \gamma$ and $\delta_1 > \frac{1}{2}(\beta+\delta)$ (in the case $C \subset B$). If $\max(\gamma_1,\delta_1) \geqq \beta$, say $\gamma_1 \geqq \beta$, then $h = f_1$ satisfies the conclusion of (a). Otherwise, suppose the procedure can be repeated for all $i = 1,2,3,\ldots$. We obtain functions $f_i, g_i \in F$ whose minimum values γ_i and δ_i satisfy $\max(\gamma_i,\delta_i) < \beta$ and either $\gamma_i > \frac{1}{2}(\gamma_{i-1}+\beta)$ and $\delta_i = \delta_{i-1}$, or $\gamma_i = \gamma_{i-1}$ and $\delta_i > \frac{1}{2}(\beta+\delta_{i-1})$. Clearly, at least one of the sequences (γ_i) and (δ_i) must converge to β. Suppose $\gamma_i \to \beta$ for $i \to \infty$. Since $\alpha < \beta$, there exists i such that $\alpha < \gamma_i$. Set $h = f_i$, then h satisfies the conclusion of (a). In order to use the lemma we now prove by induction:

(b) For any $\alpha \in \mathsf{R}$ and any set $\{f_1,\ldots,f_n\} \subset F$ such that

$$\alpha < \min_{x\in X}\max_{1\leqq i\leqq n} f_i(x)\ ,$$

there exists $h \in F$ with $\alpha < \min_{x\in X} h(x)$.

This is trivial for $n = 1$, and the case $n = 2$ is identical with (a). Suppose the assumptions of the theorem imply that (b) is true for sets containing less than n functions f_i. Let $\alpha \in \mathsf{R}$ and $\{f_1,\ldots,f_n\} \subset F$ be as in (b). Define $A = \{x \in X : f_n(x) \leqq \alpha\}$, a compact set. We may assume $A \neq \emptyset$, otherwise take $h = f_n$ in (b). The set of restrictions $\{f|A : f \in F\}$ satisfies the assumptions of the theorem with respect to A. In particular, intersections of the form

$$\textstyle\bigcap_{j=1}^m \{x \in A : g_j(x) \leqq \beta_j\} = A \cap \bigcap_{j=1}^m \{x \in X : g_j(x) \leqq \beta_j\}$$

are connected, with $(g_j,\beta_j) \in F \times \mathsf{R}$. We have

$$\alpha < \min_{x\in A}\max_{1\leqq i\leqq n-1} f_i(x)\ ,$$

so by the induction hypothesis there exists $k \in F$ with $\alpha < \min_{x\in A} k(x)$. Then

$$\alpha < \min_{x\in X}\max\big(k(x),f_n(x)\big)\ .$$

Now (a) implies the existence of $h \in F$ with $\alpha < \min_{x\in X} h(x)$. We conclude that (b) holds for all $n = 1,2,3,\ldots$, hence (i) of the lemma is satisfied.

The assumptions of Theorem 2 imply that X itself is connected. For let $f \in F$ and $x_0 \in X$ satisfy $f(x_0) = \min_{x\in X} f(x)$. Then

$$X = \textstyle\bigcup_{\alpha \geqq f(x_0)} \{x \in X : f(x) \leqq \alpha\}\ ,$$

i.e. X is a union of connected sets whose intersection contains the point x_0. Hence, in Theorem 3 of the next section we do not lose generality by assuming, for emphasis, that X is connected.

3. Main result and corollaries.

The usual object of minimax theorems is now considered: a real-valued function f defined on a product set $X \times Y$. We assume throughout that X and Y are non-empty. Setting

$$F = \{x \mapsto f(x,y) : \ y \in Y\},$$

Theorem 2 is equivalent with the following minimax theorem.

THEOREM 3. *Let X be a compact connected space, let Y be a set, and let $f\colon X \times Y \to \mathsf{R}$ be a function satisfying*:

(i) *For any $y_1, y_2 \in Y$ there exists $y_0 \in Y$ such that*

$$f(x,y_0) \geqq \tfrac{1}{2}(f(x,y_1)+f(x,y_2))$$

for all $x \in X$.

(ii) *Every finite intersection of sets of the form $\{x \in X : f(x,y) \leqq \alpha\}$, with $(y,\alpha) \in Y \times \mathsf{R}$, is closed and connected.*

Then

$$\sup_{y\in Y} \min_{x\in X} f(x,y) = \min_{x\in X} \sup_{y\in Y} f(x,y).$$

It is shown by Examples 1 and 2 that if any one of the assumptions of Theorem 3 is not satisfied, then the conclusion may fail to hold.

If Y is a convex subset of a vector space, and for each $x \in X$ the function $y \mapsto f(x,y)$ is concave on Y, then (i) of Theorem 3 is satisfied with $y_0 = \frac{1}{2}(y_1+y_2)$. The following concepts were introduced by Fan in [3] in order to prove minimax theorems involving no vector space structure: Let X and Y be arbitrary sets. A function $f\colon X \times Y \to \mathsf{R}$ is *convexlike* on X, if for any $x_1, x_2 \in X$ and $\alpha \in \mathsf{R}$ with $0 < \alpha < 1$, there exists $x_0 \in X$ such that

$$f(x_0,y) \leqq \alpha f(x_1,y) + (1-\alpha) f(x_2,y)$$

for all $y \in Y$. Similarly, f is *concavelike* on Y, if for any $y_1, y_2 \in Y$ and $\alpha \in \mathsf{R}$ with $0 < \alpha < 1$, there exists $y_0 \in Y$ such that

$$f(x,y_0) \geqq \alpha f(x,y_1) + (1-\alpha) f(x,y_2)$$

for all $x \in X$. In case, say, Y is convex and the functions $y \mapsto f(x,y)$ are concave, then f is concavelike on Y. Clearly, if f is concavelike on Y, then (i) of Theorem 3 is satisfied. The converse is not obvious in view of the fact that no continuity is involved.

The concepts above are used in the following minimax theorem, henceforth referred to as *Fan's theorem* [3, Theorem 2]. Let X be a compact space, Y a set, and $f\colon X\times Y\to \mathsf{R}$ a function satisfying: (i) f is concavelike on Y; (ii) f is convexlike on X, and for each $y\in Y$ the function $x\mapsto f(x,y)$ is lower semi-continuous on X. Then

$$\sup_y \min_x f = \min_x \sup_y f .$$

Another generalization of the concepts of convex and concave functions has been used in minimax theorems. Let X be a subset of a vector space. Recall that a function $\varphi\colon X\to\mathsf{R}$ is *quasi-convex* (resp. *quasi-concave*) on X, if $\{x\in X : \varphi(x)\leqq\alpha\}$ (resp. $\{x\in X : \varphi(x)\geqq\alpha\}$) is a convex set for each $\alpha\in\mathsf{R}$ (clearly X has to be convex).

Sion's theorem [5, Corollary 3.3 of Theorem 3.4] is as follows. Let X and Y be convex subsets of topological vector spaces, with X compact, and let $f\colon X\times Y\to\mathsf{R}$ satisfy: (i) For each $x\in X$ the function $y\mapsto f(x,y)$ is upper semi-continuous and quasi-concave on Y; (ii) for each $y\in Y$ the function $x\mapsto f(x,y)$ is lower semi-continuous and quasi-convex on X. Then

$$\sup_y \min_x f = \min_x \sup_y f .$$

(In the proof, Sion applies a theorem of Knaster, Kuratowski and Mazurkiewicz based on Sperner's lemma).

We observe that property (ii) of Theorem 3 is an extension of property (ii) of Sion's theorem. The relationship between Fan's and Sion's theorems and Theorem 3 is illustrated in Example 3: For each of the theorems, there exist functions satisfying the assumptions of that theorem and violating the assumptions of the other two theorems.

The following corollary depends on a simple property of the weak topology on a locally convex topological vector space.

COROLLARY 1. *Let X be a weakly compact and convex subset of a Hausdorff locally convex space E, let Y be a set, and let $f\colon X\times Y\to\mathsf{R}$ be a function satisfying:*

(i) *For any $y_1,y_2\in Y$ there exists $y_0\in Y$ such that $f(x,y_0)\geqq\frac{1}{2}(f(x,y_1)+f(x,y_2))$ for all $x\in X$.*

(ii) *For each $y\in Y$ the function $x\mapsto f(x,y)$ is lower semi-continuous and quasi-convex on X.*

Then

$$\sup_{y\in Y}\min_{x\in X} f(x,y) = \min_{x\in X}\sup_{y\in Y} f(x,y) .$$

PROOF. If C is any intersection of sets of the form $\{x \in X : f(x,y) \leqq \alpha\}$, then C is closed in X and convex. Since E is Hausdorff and locally convex, C is actually weakly closed.

The corollary below is the Kneser–Fan minimax theorem for concave-convex functions, [4], [3]. It is implied by Theorem 3. We present an alternative proof of the result based only on separation in R^n of disjoint convex sets. This method is used in Berge [1] with $X \subset \mathsf{R}^m$ and $Y \subset \mathsf{R}^k$.

COROLLARY 2. *Let X be a compact convex subset of a topological vector space, let Y be a convex subset of a vector space, and let $f: X \times Y \to \mathsf{R}$ be a function satisfying:*

(i) *For each $x \in X$ the function $y \mapsto f(x,y)$ is concave on Y.*

(ii) *For each $y \in Y$ the function $x \mapsto f(x,y)$ is lower semi-continuous and convex on X. Then*

$$\sup_{y\in Y} \min_{x\in X} f(x,y) = \min_{x\in X} \sup_{y\in Y} f(x,y) .$$

ALTERNATIVE PROOF. In order to apply the lemma of section 2, let $\alpha \in \mathsf{R}$ and $\{y_1,\ldots,y_n\} \subset Y$ satisfy

$$\alpha < \min_{x\in X} \max_{1\leqq i\leqq n} f(x,y_i) .$$

Let $P \subset \mathsf{R}^n$ be the convex hull of the set

$$\{(f(x,y_1),\ldots,f(x,y_n)) : x \in X\} ,$$

and let Q be the cone of points $(z^1,\ldots,z^n) \in \mathsf{R}^n$ with $z^i \leqq \alpha$, $i=1,\ldots,n$. We show that $P \cap Q = \emptyset$. Any $z \in P$ is of the form

$$z = \sum_{j=1}^m \beta_j (f(x_j,y_1),\ldots,f(x_j,y_n)) ,$$

where $x_j \in X$, $\beta_j \geqq 0$, $j=1,\ldots,m$, and $\sum_{j=1}^m \beta_j = 1$. Setting $x_0 = \sum_{j=1}^m \beta_j x_j$, there exists $i \in \{1,\ldots,n\}$ such that $f(x_0,y_i) > \alpha$. Since $x \mapsto f(x,y_i)$ is convex, the ith coordinate z^i of z satisfies

$$z^i = \sum_{j=1}^m \beta_j f(x_j,y_i) \geqq f(x_0,y_i) > \alpha ,$$

showing that $z \notin Q$. The disjoint convex sets P and Q can be separated by a hyperplane, i.e. there exists a non-zero vector $c = (\gamma_1,\ldots,\gamma_n)$ with $\sup_{z\in Q} c\cdot z \leqq \inf_{z\in P} c\cdot z$. We clearly have $\gamma_i \geqq 0$, $i=1,\ldots,n$, and may assume $\sum_{i=1}^n \gamma_i = 1$. Define $y_0 = \sum_{i=1}^n \gamma_i y_i$. For each $x \in X$ the function $y \mapsto f(x,y)$ is concave, so

$$f(x,y_0) \geqq \sum_{i=1}^n \gamma_i f(x,y_i) \geqq \sup_{z\in Q} c\cdot z = \alpha ,$$

showing that (i) of the lemma holds.

4. Examples.

The examples below serve to illustrate the concepts treated in the note. In particular, the assumptions of Theorem 3 are investigated. We use the notation

$$[\alpha,\beta] = \{x \in \mathsf{R} : \alpha \leqq x \leqq \beta\}, \quad [\alpha,\beta) = \{x \in \mathsf{R} : \alpha \leqq x < \beta\}, \quad \text{etc.}$$

EXAMPLE 1. In each of the cases a.–d. of functions $f: X \times Y \to \mathsf{R}$, all except one of the assumptions of Theorem 3 are satisfied, and $\sup_y \inf_x f < \inf_x \sup_y f$.

a. Let $X=(0,1]$, $Y=[0,+\infty)$, $f(x,y)=xy$. X is not compact, but for example locally compact and σ-compact.

b. Let $X=Y=[0,1]$, $f(x,y)=x$ if $x \in X$ and $y \in [0,1)$, $f(x,1)=1-2x$ if $x \in [0,\frac{1}{2}]$, and $f(x,1)=2x-1$ if $x \in (\frac{1}{2},1]$. There exists no $y_0 \in Y$ with $f(x,y_0) \geqq \frac{1}{2}(f(x,0)+f(x,1))$ for $x \in X$.

c. Let $X=Y=[0,1]$, $f(x,y)=y$ if $x \in [0,1)$, and $f(1,y)=1-y$, for $y \in Y$. The set $\{x \in X : f(x,0) \leqq 0\}$ is not closed.

d. Let $X=Y=[0,1]$, $f(x,y)=x+y$ if $x+y \leqq 1$, and $f(x,y)=2-(x+y)$ if $x+y>1$. The set $\{x \in X : f(x,\frac{1}{2}) \leqq \frac{1}{2}\}$ is not connected.

EXAMPLE 2. Let $X_1=[-2,-1]$, $X_2=[1,2]$, $X=X_1 \cup X_2$ and $Y=[-1,1]$, and define $f(x,y)=xy/|x|$. The restriction of f to $X_i \times Y$ satisfies the assumptions of Theorem 3 for $i=1,2$, but X is not connected, and the conclusion of the theorem fails for f on $X \times Y$.

EXAMPLE 3. Each of the functions $f: X \times Y \to \mathsf{R}$ of a.–c. below satisfies the assumptions of precisely one of the following three theorems: Fan's theorem, Sion's theorem and Theorem 3 (cf. section 3). Hence,

$$\sup_y \min_x f = \min_x \sup_y f$$

holds in each case.

a. Let $X=Y=[0,1]$, $f(0,y)=0$ and $f(1,y)=0$ for $y \in Y$, $f(x,y)=1$ elsewhere. Fan's theorem applies (take $x_0=0$ to verify that f is convex-like on X). The sets $\{x \in X : f(x,y) \leqq 0\}$ are not connected, violating (ii) of Theorem 3, hence (ii) of Sion's theorem.

b. Let $X=Y=[0,1]$, $f(0,y)=y-1$ for $y \in Y$, $f(x,0)=x$ for $x \in (0,1]$, $f(x,y)=0$ elsewhere. Sion's theorem applies. There exists no $y_0 \in Y$ with $f(x,y_0) \geqq \frac{1}{2}(f(x,0)+f(x,1))$ for $x \in X$, so neither (i) of Theorem 3 nor (i) of Fan's theorem holds.

c. Let X be the half-circle in the Euclidean plane consisting of points $x=(x^1,x^2)$ with $(x^1)^2+(x^2)^2=1$ and $x^2 \geqq 0$, and let $Y=[0,1]$. Define

$f(1,0,y)=0$ if $y \in [0,1)$, $f(x,1)=0$ if $x \in X \setminus \{(-1,0)\}$, $f(-1,0,1)=-1$, and $f(x,y)=1$ elsewhere. Theorem 3 can be applied. There exists no $x_0 \in X$ with $f(x_0,y) \leqq \frac{1}{2}(f(1,0,y)+f(-1,0,y))$ for $y \in Y$, implying that f is not convexlike on X, i.e. (ii) of Fan's theorem is violated. The functions $x \mapsto f(x,y)$ are not quasi-convex (X is not convex), i.e. the assumptions of Sion's theorem are violated.

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