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ON THE IRREDUCIBILITY OF POLYNOMIALS TAKING SMALL VALUES

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Dedicated to the memory of W. Ljunggren.

1. Introduction.

In 1919 G. Polya [3] proved that a polynomial $f(x)$ of degree n, with integral coefficients, is irreducible over the rationals if it satisfies the following condition:

There are n integers a_i so that $0<|f(a_i)|<2^{-[(n+1)/2]}[(n+1)/2]!$.

A simple example of such a polynomial is given by

$$f(x) = (x-1)(x-2)\ldots(x-n)+1 .$$

Polya's result has been improved upon later, by (in chronological order) T. Tatuzawa [4], A. Brauer and G. Ehrlich [1], H. Tverberg [5], R. J. Levit [2], and H. Tverberg [6] again. The list (1.1) gives the various results (slightly falsified, for easy comparison). Note that m means, for the rest of the paper, $[(n+1)/2]$.

$$2^{-m}m!,\ m^{-\frac{3}{4}}m!,\ m^{-\frac{1}{2}}m!,\ m!,\ 1.30^m m!,\ 1.68^m m! . \tag{1.1}$$

It is natural to ask for the "exact" version of the theorem, i.e. to find the number $P(n)$ defined by: There exists a *reducible* polynomial $f(x)$ and integers $a_1,a_2,\ldots,a_n$ so that

$$0 < |f(a_i)| \leqq P(n), \quad i = 1,2,\ldots,n . \tag{1.2}$$

There exists *no* reducible polynomial $f'(x)$ for which there are n integers $a_1',a_2',\ldots,a_n'$ so that

$$0 < |f'(a_i')| < P(n), \quad i = 1,2,\ldots,n .$$

The reducible polynomial

$$f_0(x) = \big((x-1)(x-3)\ldots(x-2m+1)+1\big)\big((x-2)(x-4)\ldots (x-2(n-m))+1\big)$$

shows that

$$P(n) \leqq \max\{|f_0(1)|,\ldots,|f_0(n)|\} = f_0(n) < 2^m m! \tag{1.3}$$

which lends perspective to the sequence (1.1).

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In this paper we prove (more or less completely)

THEOREM 1. *$P(n)=(\lambda+o(1))^m m!$, where λ is a fixed number.*

THEOREM 2. *The number λ is effectively computable and satisfies the inequality $1.734<\lambda\leqq(4/3)3^{1/4}=1.754\ldots$.*

THEOREM 3. *$P(n)$ is asymptotically equal to $Q(n)$.*

Here $Q(n)$ is that upper estimate of $P(n)$ which is furnished by consideration of reducible polynomials of the form

$$((x-b_1)(x-b_2)\ldots.(x-b_p)+1)((x-c_1)(x-c_2)\ldots(x-c_{n-p})+1) \tag{1.4}$$

with $\{b_1,\ldots,b_p,c_1,\ldots,c_{n-p}\}=\{1,2,\ldots,n\}$. We shall call such polynomials *special polynomials.* Note that the estimate (1.3) of $P(n)$ was furnished by a special polynomial, so that we have

$$Q(n) < 2^m m!\,. \tag{1.5}$$

2. Polya's lemma.

This is

LEMMA 1. *If $g(x)$ is a polynomial of degree k, with integral coefficients, and $d_0<d_1<\ldots<d_k$ are integers, then, for some i, $|g(d_i)|\geqq k!\,2^{-k}$.*

PROOF (Polya). Consider Lagrange's formula

$$g(x) = (x-d_0)\ldots(x-d_k)\sum_{i=0}^k \frac{g(d_i)}{x-d_i}\,\frac{1}{\prod_{j\neq i}(d_i-d_j)}\,.$$

The leading coefficient of $g(x)$ is given by

$$\sum_{i=0}^k g(a_i)\prod_{j\neq i}(d_i-d_j)^{-1}\,.$$

As the absolute value of this coefficient is 1 or more, we find that $|g(d_0)|,\ldots,|g(d_k)|$ can not all be less than

$$\Big(\sum_{i=0}^k \prod_{j\neq i}|d_i-d_j|^{-1}\Big)^{-1}, \tag{2.1}$$

and hence (because $|d_i-d_j|\geqq|i-j|$) not less than

$$\Big(\sum_{i=0}^k (i!\,(k-i)!)^{-1}\Big)^{-1} = k!\,2^{-k}\,.$$

Polya applies his lemma as follows. If $f(x)$ is reducible, it factorizes over the integers as $g(x)h(x)$. Then $g(x)$, say, has degree k where $m \leqq k < n$. Assume now that $f(x)$ satisfies the condition stated at the beginning of the paper. For every integer a, $g(a)$ divides $f(a)$ and so

$$|g(a_i)| < 2^{-m}m!, \quad i = 1, 2, \ldots, n.$$

On the other hand lemma 1 (with $d_i = a_{i+1}$, $i = 0, 1, \ldots, k$) implies that for some i

$$|g(a_i)| \geqq 2^{-k}k! \geqq 2^{-m}m!.$$

REMARK 1. Note that the argument just given shows that one may obtain irreducibility even with a higher bound on the values of f, provided one has a suitable bound on the degrees of possible factors. For instance, if one knows that f has a factor of degree k with $k > m + 3m/\log m$ one deduces that for some i

$$|f(d_i)| \geqq k!\,2^{-k} \geqq m!\,2^m.$$

This shows that the reducible polynomials which are of interest for the determination of $P(n)$ never have factors of degree exceeding $m + 3m/\log m$.

REMARK 2. The results listed in (1.1), except the last one, are all based on Polya's idea of considering the factor $g(x)$ of degree $\geqq m$. The improvements have consisted in raising the lower estimate of how well $g(x)$ can approximate 0 on $\{a_1, a_2, \ldots, a_n\}$. This method reached its limit in [2], where it was proved (Theorem 1) that the error must be at least

$$(2.2) \qquad 2^{1-k}\tfrac{1}{2}(n-k)\left(\tfrac{1}{2}(n-k)+1\right)\left(\tfrac{1}{2}(n-k)+2\right)\ldots\left(\tfrac{1}{2}(n-k)+k-1\right).$$

In the domain which interests us most, i.e. for $|k-m| < 3m/\log m$ this expression equals

$$(2.3) \qquad \left(\tfrac{3}{4}\sqrt{3}+o(1)\right)^k k!.$$

This result, which is useful for the estimation of λ from below, may also be deduced by putting $d_i = a_{p_i}$, $i = 0, 1, \ldots, k$ in the expression (2.1). Here the p's are constructed as follows: In section 4, put

$$\varphi(t) = 1 - 2\pi^{-1}\operatorname{Arctan}\left(\operatorname{Re}\left(3-4(2tn^{-1}-1)^2\right)^{\frac{1}{2}}\right), \quad \varrho = m^{-1}(k+1-m)\log m.$$

Then $p_{i-1} = b_i$, as given by (4.5).

We shall not effect this deduction, which is, however, quite easy for one who has mastered the contents of sections 4 and 7. It is probably easy to obtain Levit's more general result (2.2) in a similar way.

It is a well known fact from approximation theory that (2.1) is capable of yielding the best possible result (concerning approximation by monic real (or complex) polynomials), but the problem is to find the optimal choice of the d_i's.

3. Proof of Theorem 3.

By definition $P(n)$ is a minimum, taken over the set of all reducible polynomials of degree n. $Q(n)$ is the minimum of the same quantity, taken over the subset of special polynomials. Thus

$$P(n) \leqq Q(n) . \tag{3.1}$$

We now want to prove the inequality

$$Q(n) \leqq P(n)(1+o(1)) \tag{3.2}$$

which, together with (3.1) yields Theorem 3. For this purpose it is necessary to locate the zeros of $f(x)$.

Consider an *extremal* polynomial, i.e. a reducible polynomial $f(x)$ satisfying (1.2). Let ζ be any zero of f and put $g(x)=f(x)/(x-\zeta)$. Then $g(x)$ has a leading coefficient whose absolute value is at least 1, and the proof of Lemma 1 shows that, for some i,

$$|g(a_i)| \geqq (n-1)!\, 2^{1-n} .$$

Thus

$$|\zeta - a_i| \leqq (2^{n-1}/(n-1)!)\,|f(a_i)| \leqq (2^{n-1}/(n-1)!)P(n)$$

so that, by (1.3),

$$|\zeta-a_i| \leqq 2^{m+n-1}m!/(n-1)! = m^{-m(1+o(1))}. \tag{3.3}$$

Thus for large n, every zero of $f(x)$ is real and very near one of the integers a_i.

Moreover, different zeros are near different integers. Because, if ζ_1 and ζ_2, say, satisfy (3.3) for the same i, we may consider the polynomial $g(x)/(x-\zeta_1)(x-\zeta_2)$ on the set $\{a_1,\ldots,a_n\}\setminus\{a_i\}$ and deduce that for some $j, j \neq i$,

$$|\zeta_1-a_j|\,|\zeta_2-a_j| < m^{-m(1+o(1))}$$

which contradicts $|a_i-a_j| \geqq 1$.

Having located the zeros of $f(x)$ we are ready for the proof of (3.2). Let the zeros be $\zeta_1<\zeta_2<\ldots<\zeta_n$. Thus

$$|\zeta_i-a_i| < m^{-m(1+o(1))}. \tag{3.4}$$

Let $f(x)$ factorize over the integers as $g(x)h(x)$. With $f(x)$ we associate the special polynomial

$$s(x) = \left(\prod_{b\in B}(x-b)+1\right)\left(\prod_{c\in C}(x-c)+1\right)$$

where

$$B = \{i; g(\zeta_i) = 0\}, \quad C = \{i; h(\zeta_i) = 0\}.$$

For any $b \in B$ we have

$$\begin{aligned}|s(b)-1| &= \prod_{c\in C}|b-c| \leqq \prod_{c\in C}|a_b-a_c| \\ &= \left|\prod_{c\in C}(a_b-\zeta_c)\right| \prod_{c\in C}|1+(\zeta_c-a_c)(a_b-\zeta_c)^{-1}| \\ &= |h_0|^{-1}|h(a_b)| \prod_{c\in C}|1+(\zeta_c-a_c)(a_b-\zeta_c)^{-1}| ,\end{aligned}$$

where h_0 is the leading coefficient of $h(x)$. If we note that $|b-c| \geqq 1$ for all $c \in C$, and use (3.4), we get

$$(3.5) \qquad |s(b)-1| \leqq |h_0|^{-1}|h(a_b)|\left(1+o(1)\right) \leqq |h_0^{-1}|\,|f(a_b)|\left(1+o(1)\right).$$

Thus

$$(3.6) \qquad |s(b)-1| \leqq |h_0^{-1}|P(n)\left(1+o(1)\right) \leqq P(n)\left(1+o(1)\right) ,$$

and similarly for any $c \in C$,

$$(3.7) \qquad |s(c)-1| \leqq P(n)\left(1+o(1)\right) .$$

Now, since $s(x)$ is a special polynomial, the definition of $Q(n)$ shows that

$$Q(n) \leqq \max\{|s(1)|, |s(2)|, \ldots, |s(n)|\} ,$$

and so (3.6) and (3.7) yield (3.2), and hence theorem 3.

The proof just carried through gives some information on $g(x)$ and $h(x)$ for large n, namely

Corollary 1. *At least one of the polynomials* $\pm g(x)$ *and* $\pm h(x)$ *is monic.*

Corollary 2. *At least one of the values taken by* $g(x)$ *and* $h(x)$ *on* $\{a_1, a_2, \ldots, a_n\}$ *equals* ± 1.

Proof. If neither $\pm g(x)$ nor $\pm h(x)$ were monic, the inequality (3.6) would become

$$|s(b)-1| \leqq \tfrac{1}{2}P(n)\left(1+o(1)\right) ,$$

and similarly for (3.7), so that we would get

$$Q(n) \leqq \tfrac{1}{2}P(n)\left(1+o(1)\right) ,$$

in contradiction to (3.1).

If Corollary 2 were false, (3.5) would become

$$|s(b)-1| \leqq \tfrac{1}{2}|h_0^{-1}|\,|f(a_b)|\,(1+o(1)),$$

which together with a similar inequality for $s(c)$ would give the same contradiction as above.

4. Smoothing of the problem of determining $Q(n)$.

It is easy to see that $Q(n)/Q(n\pm 1)=O(n)$, so that the restriction to the case $n=2m$, which we now make, is relatively harmless, considering our main project, which is to prove Theorem 1 and 2, using the already established Theorem 3.

Let φ be a continuous function on $[0,n]$ and ϱ a real number, satisfying the conditions

$$0 \leqq \varphi \leqq 1, \quad \int_0^n \varphi(t)dt = m, \quad |\varrho| < 3. \tag{4.1}$$

Let G and H be defined by

$$G(x) = \varrho m+\int_0^n \varphi(t)\log|x-t|dt \tag{4.2}$$

$$H(x) = -\varrho m+\int_0^n (1-\varphi(t))\log|x-t|dt. \tag{4.3}$$

We shall construct a special polynomial $s(x)$ such that the values taken by it on $\{1,2,\ldots,n\}$ are given, up to error terms, by the values of $\exp G$ and $\exp H$. The factors of this special polynomial will have degrees $m\pm[\varrho m/\log m]$. Our construction may easily be reversed so that, starting with a special polynomial ($s(x)$ whose factors have degrees $m\pm \varrho m/\log m$, $|\varrho|<3$, one arrives at a pair φ,ϱ for which the corresponding special polynomial is exactly $s(x)$. The problem of estimating $Q(n)$ is thus carried over to a smoother problem about pairs (φ,ϱ).

Now the construction of $s(x)$. We first construct a continuous function ψ, defined on $[0,n]$, which satisfies

$$\int_0^n \psi(t)dt = m+[\varrho m/\log m], \quad 0 \leqq \psi \leqq 1, \quad (\psi-\varphi)\varrho \geqq 0.$$

$$x\in \operatorname{supp}(\cdot) \Rightarrow \operatorname{dist}(x,\operatorname{supp}(\cdot\cdot)) < 3m/\log m.$$

$$(\cdot,\cdot\cdot) = (\varphi,\psi),\ (\psi,\varphi),\ (1-\varphi,1-\psi),\ (1-\psi,1-\varphi) \tag{4.4}$$

Here $\operatorname{supp}(\varphi)=$the support of $\varphi=$the closure of the set $\{x;\varphi(x)\neq 0\}$. The relevance to us of this concept will become clear later on.

The existence of such a ψ is trivial if $\varrho>0$ and the set $\varphi^{-1}(\{0\})$ contains an interval of length $\geqq 3m/\log m$. If $\varrho>0$, but there is no such interval, we put

$$\psi(t) = \max(\varphi(t),a)$$

where $a\in(0,1)$ is chosen so that $\int_0^n\psi(t)dt=m+[\varrho m/\log m]$.

If $\varrho < 0$ we apply the preceding construction to $1-\varphi$, $-\varrho$, obtain a function ψ_1 and put $\psi = 1-\psi_1$.

We now define $B = \{b_1, b_2, \ldots\} \subset \{1, 2, \ldots, n\}$ by

$$(4.5) \quad \int_0^{b_i} \psi(t)dt \geqq i, \ \int_0^{b_i-1} \psi(t)dt < i, \quad i = 1, 2, \ldots, m + [\varrho m/\log m],$$

put $C = \{1, 2, \ldots, n\} \setminus B$, and then define $s(x)$ by

$$(4.6) \qquad s(x) = \left(\prod_{b\in B}(x-b)+1\right)\left(\prod_{c\in C}(x-c)+1\right).$$

The values of $s(x)$ on $\{1, 2, \ldots, n\}$ may easily be compared with those of G and H on $[0, n]$. Consider, say, $\log|s(c)|$, $c \in C$. As $|B| \sim |C| \sim m$ we have

$$(4.7) \qquad \log|s(c)| + o(1) = \sum_i \log|c-b_i| .$$

By (4.5) there is a $y \in \operatorname{supp}(1-\psi)$, with $c-1 < y < c$ and by (4.4) there is an $x \in \operatorname{supp}(1-\varphi)$, with $|x-y| < 3m/\log m$.

Now, *firstly*,

$$(4.8) \qquad \sum_i \log|c-b_i| - G_\psi(c) = \sum_i \left(\log|c-b_i| - \int_{\beta_{i-1}}^{\beta_i} \psi(t) \log|c-t|dt\right)$$

where we have put

$$\beta_i = \inf\{x;\ \int_0^x \psi(t)dt = i\}, \quad i = 0, 1, \ldots,$$

and

$$(4.9) \qquad G_\psi(x) = \int_0^n \psi(t) \log|x-t|\,dt .$$

Hence, by (4.5), $b_{i-1} < \beta_i \leqq b_i$. If $b_i < c$, then, with $b_{-1} = b_0 = 0$,

$$0 < \int_{\beta_{i-1}}^{\beta_i} \psi(t)\log(c-t)dt - \log(c-b_i) < \log(c-b_{i-2}) - \log(c-b_i)$$

and if $b_{i-2} > c$, then

$$0 < \log(b_i - c) - \int_{\beta_{i-1}}^{\beta_i} \psi(t)\log(t-c)dt < \log(b_i-c) - \log(b_{i-2}-c) .$$

From this it follows that the right hand side, and hence the left hand side of (4.8) is $O(\log m)$.

Secondly, denoting $x-c$ by $2a$, we have

$$\begin{aligned} |G_\psi(c) - G_\psi(x)| &= \left|\int_0^n \psi(t)(\log|x-t| - \log|c-t|)dt\right| \\ &= \left|\int_{[c,x]} + \int_{[0,n]\setminus[c,x]}\right| \\ &< \int_0^a (\log(a+t) - \log(a-t))dt + \int_0^{n-2a} (\log(t+2a) - \log t)dt \\ &= 2a\log m - 2(m-a)\log(1-a/m) - 2a\log(\tfrac{1}{2}a) \\ &= O(m(\log\log m)/\log m) . \end{aligned}$$

The last equality holds because $2a = |x-c| = |x-y+y-c| < 4m/\log m$.

Thirdly,

$$\begin{aligned} G_\psi(x) - G(x) &= \int_0^n (\psi(t) - \varphi(t))\log|x-t|dt - \varrho m \\ &= \int_0^n (\psi(t) - \varphi(t))\log|m^{-1}(x-t)|dt + [\varrho m/\log m]\log m - \varrho m \,. \end{aligned}$$

As $0 \leqq (\psi - \varphi)\operatorname{sgn}\varrho \leqq 1$ it is clear, because of (4.1) and (4.4), that the absolute value of the last integral does not exceed

$$\max\left(2\int_0^a |\log tm^{-1}|dt, \int_{n-2a}^n \log tm^{-1}dt\right) = O(m(\log\log m)/\log m) \,.$$

(for $2a = [\varrho m/\log m]$).

Combining the estimates we find that for any $c \in C$ there is an $x \in \operatorname{supp}(1-\varphi)$ so that $\log|s(c)| = G(x) + o(m)$. The same procedure may now be followed to show that for any $b \in B$ there is an $x \in \operatorname{supp}\varphi$ so that $\log|s(b)| = H(x) + o(m)$. The definition of B shows namely that there is a $y \in \operatorname{supp}\psi$ so that $b-1 < y < b$. By (4.4) there is an $x \in \operatorname{supp}\varphi$ so that $|x-y| < 3m/\log m$ and so $|x-b| < 4m/\log m$. The difference $H(x) - \log|s(b)|$ can be estimated in steps, as above, with $G_{1-\psi}$ replacing G_ψ. Only the estimate of $\sum_i \log|c-b| - G_{1-\psi}(b)$ causes some trouble because the relationship of C to $1-\psi$ is not exactly the same as that of B to ψ.

Here the identity

$$\begin{aligned} \sum_i \log|c_i - b| - G_{1-\psi}(b) &= \left(\log((b-1)!\,(n-b)!) - \int_0^n \log|b-t|dt\right) \\ &\quad - \left(\sum_{d\in B-\{b\}} \log|b-d| - G_\psi(b)\right) \end{aligned}$$

carries us back to a well known situation and allows us to conclude that

$$\sum_{c\in C} \log|c-b| - G_{1-\psi}(b) = O(\log m) \,.$$

As said before, it is possible, given a special polynomial $s(x)$, with $|\deg(s(x)) - m| < 3m/\log m$ to reverse the procedures above and construct a pair (φ, ϱ) satisfying (4.1), so that the following holds. For any $x \in \operatorname{supp}\varphi$ there is a $b \in B$ so that $H(x) = \log|s(b)| + o(m)$, while for any $x \in \operatorname{supp}(1-\varphi)$ there is a $c \in C$ so that $G(x) = \log|s(c)| + o(m)$.

5. Proof of theorem 1.

In the preceding section the problem of estimating $Q(n)$ was smoothed. We now normalize the smooth version so that we obtain a problem which is independent of n. This is easy. A linear transformation changes $[0,n]$ to $[-1,1]$ and then the outcome of section 4 may be expressed as follows.

LEMMA 2. *Let φ be a continuous function on $[-1,1]$ and let ϱ be a real number, satisfying*

$$(5.1) \qquad 0 \leqq \varphi \leqq 1, \quad \int_{-1}^1 \varphi(t)dt = 1, \quad |\varrho| \leqq 3 \,.$$

Let G and H be defined by

(5.2) $G(x) = \int_{-1}^{1}\varphi(t)\log|x-t|dt, \quad H(x) = \int_{-1}^{1}(1-\varphi(t))\log|x-t|dt\,,$

and put

(5.3) $M_G = \max\{G(x)\,;\,x \in \operatorname{supp}(1-\varphi)\}, \quad M_H = \max\{H(x)\,;\,x \in \operatorname{supp}\varphi\},$

(5.4) $$M_{\varphi,\varrho} = \max(M_G+\varrho,\, M_H-\varrho)\,.$$

Then

(5.5) $$\log Q(n) = m\log m + m(\inf_{\varphi,\varrho} M_{\varphi,\varrho} + o(1))$$

This lemma has a simpler version, as the parameter ϱ may be eliminated. We have, using the notation of lemma 2,

Lemma 3. *Put*

(5.6) $$M_\varphi = \tfrac{1}{2}(M_G+M_H)\,.$$

Then

$$\log Q(n) = m\log m + m(\inf_\varphi M_\varphi + o(1))\,.$$

Proof. For any real numbers a,b we have

$$\inf_{|\varrho|\leqq 3}\max(a+\varrho,\, b-\varrho) = \max\left(\tfrac{1}{2}(a+b),\, \max(a,\, b)-3\right)$$

Hence it suffices to prove that, say, $M_G - 3 \leqq \frac{1}{2}(M_G+M_H)$ that is $M_G - M_H \leqq 6$. But it is obvious that

$$M_G \leqq \int_0^1 \log(1+t)dt, \quad M_H \geqq \int_{-\frac{1}{2}}^{\frac{1}{2}} \log|t|dt$$

so that $M_G - H_H \leqq \log 8 < 6$.

The proof of theorem 1 is now trivial. Theorem 3 says that $P(n) \sim Q(n)$ and lemma 3 then gives

$$m^{-1}(\log P(n) - \log m!) = 1 + \inf_\varphi M_\varphi + o(1)\,.$$

Thus

(5.7) $$P(n) = (\exp(1+\inf_\varphi M_\varphi) + o(1))^m m!\,,$$

which is theorem 1. The number λ is expressed by

(5.8) $$\lambda = \exp(1+\inf_\varphi M_\varphi)\,.$$

6. Computability of λ. Existence of extremal functions.

Let φ be a function satisfying (5.1) and let N be a natural number. For $k = 1, 2, \ldots, 2N$ there is a minimal integer q_k for which

$$\int_{-1+(k-1)/N}^{-1+k/N} \varphi(t)dt \leqq q_k/N^2 .$$

Let q be a $4N$-vector whose first $2N$ components are the q_k, while the component q_{2N+k} is 0, 1 or 2 according to whether the interval $[-1+(k-1)/N, -1+k/N]$ meets only suppφ, only supp$(1-\varphi)$ or both.

Using only the information about φ given by q it is easy to give twosided estimates of G and of M_G. Similarly one finds an estimate of M_H, and hence of M_φ. Consider now all the possible vectors for a given N. One of them at least, yields a minimal upper estimate of M_φ. Denote this minimum by M_N, and let φ_N be a function which yields the vector in question.

It is now easy to see that $\lim_{N\to\infty} M_N = \lim_{N\to\infty} M_{\varphi N} = \inf_\varphi M_\varphi$. Looking into the details of the estimation above one finds immediately that $\inf_\varphi M_\varphi$, and hence λ, is an effectively computable number.

REMARK. The continuity of φ was not used in the considerations above. This means that, for instance the characteristic function of a finite union of intervals of total length 1 may be considered. If one defines M_φ as for a continuous φ, it is clear from the above that $\inf_\varphi M_\varphi$ is the same whether taken over the continuous φ's or over the discontinuous φ's just mentioned. This remark will be useful later on, but ought to be forgotten at the moment.

It would now be nice if $\{\varphi_N\}$, or at least some subsequence of $\{\varphi_N\}$ converged towards a continuous function φ_0. If that happens (and it probably does) for a suitable choice of the φ_N's consideration of the support of φ_0 and $1-\varphi_0$ shows that $M_{\varphi_0} \leqq \lim_{N\to\infty} M_{\varphi N} = \inf_\varphi M_\varphi$. Hence $\inf_\varphi M_\varphi = M_{\varphi_0}$, so that an extremal function exists.

Not being able to prove the existence of an extremal function, however, we shall mention a different (but equivalent) formulation of our problem, which allows one to speak with certainty of extremal functions.

Let Φ be any function defined on $[-1,1]$, having the properties

$$\Phi(-1) = 0, \quad \Phi(1) = 1, \quad y \leqq x \Rightarrow 0 \leqq \Phi(x)-\Phi(y) \leqq x-y .$$

Let G and H be defined by

$$G(x) = \int_{-1}^{1} \log|x-t| d\Phi(t), \quad H(x) = \int_{-1}^{1} \log|x-t| d(t-\Phi(t)) ;$$

let M_G be the maximum of G on the set $[-1,1] \setminus \operatorname{int}\{x; \Phi'(x)=1\}$ and M_H the maximum of H on the set $[-1,1] \setminus \operatorname{int}\{x; \Phi'(x)=0\}$. Finally put $M_\Phi = \frac{1}{2}(M_G+M_H)$. Then one can prove rather easily, using Ascoli's theorem, that there exists a function Φ_0 for which

$$M_{\Phi_0} = \inf_\Phi M_\Phi = \inf_\varphi M_\varphi .$$

We shall not go further into this fact, which will not be used in this paper.

7. Upper bound on λ.

From the formula (5.8) it follows that any allowable φ yields an upper estimate of λ. We choose φ as follows

$$(7.1)\qquad \begin{aligned} \varphi(t) &= 2\pi^{-1}\operatorname{Arc tan}(3-4t^2)^{\frac12}\,; && |t| \leqq \tfrac12\sqrt{3} \\ \varphi(t) &= 0\,; && \tfrac12\sqrt{3} \leqq |t| \leqq 1\,. \end{aligned}$$

This choice, which may seem unmotivated at the moment, has been made in order to make φ satisfy the integral equation

$$(7.2)\qquad \int_{-1}^{1}(1-\varphi(t))\log|x-t|dt = -1+\tfrac34\log 3-\log 4; \qquad |x| \leqq \tfrac12\sqrt{3}\,.$$

The idea of considering the solutions of this equation originated in the study of 1065 stepfunctions by means of a computer. For that stepfunction φ_0 which gave the smallest estimate of λ, the left hand side of (7.2) varies very little as x varies over the support of φ_0. It turns out that if one requires the left hand side of (7.2) to be constant on the support of φ for a continuous function φ satisfying (5.1) and having support $[-c,c]$, then c has to be $\frac12\sqrt{3}$ and φ has to be the function defined by (7.1).

We now check that (7.2) holds. Let F be the function defined on R^1 by

$$(7.3)\qquad F(x) = \int_{-c}^{c}\varphi(t)\log|x-t|dt, \qquad c = \tfrac12\sqrt{3}\,.$$

Then, as is well known

$$(7.4)\qquad F'(x) = \int_{-c}^{c}\varphi(t)/(x-t)dt$$

where the integral means the Cauchy principal value, (that is

$$\lim_{\varepsilon\to 0}\Big(\int_{-c}^{x-\varepsilon}+\int_{x+\varepsilon}^{c}\Big)\Big) \quad \text{if} \quad |x| < c\,.$$

Partial integration gives

$$\begin{aligned} F'(x) &= -\lim_{\varepsilon\to 0+}\big(\big|_{-c}^{x-\varepsilon}+\big|_{x+\varepsilon}^{c}\big)\varphi(t)\log|x-t|+\int_{-c}^{c}\varphi'(t)\log|x-t|dt \\ &= \int_{-c}^{c}\varphi'(t)\log|x-t|dt\,, \end{aligned}$$

and so

$$F''(x) = \int_{-c}^{c}\varphi'(t)/(x-t)dt = -\int_{-c}^{c}\frac{tdt}{(1-t^2)(3-4t^2)^{\frac12}(x-t)},$$

whence

$$F''(x) = \tfrac12(1+x)^{-1}(x^2-\tfrac34)^{-\frac12}-\tfrac12(1-x)^{-1}(x^2-\tfrac34)^{-\frac12}+(1+x)^{-1}+(1-x)^{-1}; \qquad x > c$$

$$F''(x) = 2/(1-x^2); \qquad |x| < c\,.$$

From this we obtain

$$(7.5)\quad F(x) = (1-x)\log\left(2x-\tfrac{3}{2}+(x^2-\tfrac{3}{4})^{\frac{1}{2}}\right)-(1+x)\log\left(2x+\tfrac{3}{2}-(x^2-\tfrac{3}{4})^{\frac{1}{2}}\right)$$
$$-\log\left(x+(x^2-\tfrac{3}{4})^{\frac{1}{2}}\right)+2(1+x)\log(1+x)+x\log 3$$
$$-1+\log 2-\log 3;\qquad x > c$$

$$(7.6)\quad F(x) = (1+x)\log(1+x)+(1-x)\log(1-x)-1-\tfrac{3}{2}\log 3+\log 4\;;$$
$$|x| \leqq c\,.$$

In order to determine the constants of integration we used the following facts:

$$F(x) = F(-x),\quad F'(0) = 0,\quad \lim_{x\to\infty}F'(x) = 0\,,$$
$$\lim_{x\to\infty}(F(x)-\log x) = 0,\quad \lim_{x\to c}F(x) = F(c)\,,$$

which all are easy consequences of our definitions (note that

$$F(x)-\log x = \int_{-c}^{c}\varphi(t)\log(1-t/x)dt \quad \text{when } x > c)\,.$$

It now follows from (7.3) and (7.6) that (7.2) is satisfied. Thus

$$(7.7)\qquad \max_{\operatorname{supp}\varphi}\int_{-1}^{1}(1-\varphi(t))\log|x-t|dt = -1+\tfrac{1}{2}\log 27-\log 4\,.$$

The other maximum to be computed is that of F on $[-1,1]$. It turns out that $F' \geqq 0$ on $[0,1]$, and, as $F(x)=F(-x)$ it follows that the maximum equals $F(1)$,

$$(7.8)\qquad F(1) = \log 64-\log 27-1$$

Thus $M_\varphi = -1+\log 4-\frac{1}{4}\log 27$, which yields the estimate

$$(7.9)\qquad \lambda \leqq \tfrac{4}{3}\,3^{\frac{1}{4}} = 1.754\ldots\,.$$

8. Lower bound on λ.

The estimation of $P(\lambda)$ from below will only be sketched as the result arrived at is not the definitive one. A complete discussion would fill at least 5 pages.

Let $f(x)=g(x)h(x)$ be a special polynomial. We want to prove that for some $i \in \{1,\ldots,n\}$, $|f(i)| > (1.7341)^m m!$, provided n is large.

Case 1. Degree$(g(x)) \geqq m+\varrho_0 m/\log m$.

Here $\varrho_0=\log(1.73411/\frac{3}{4}\sqrt{3})$. We know (cf. (2.3)) that for some i

$$|f(i)| \geqq |g(i)| \geqq \left(\tfrac{3}{4}\sqrt{3}+o(1)\right)^{m+\varrho_0 m/\log m}(m+\varrho_0 m/\log m)!$$
$$(8.1)\qquad = \left(1.73411+o(1)\right)^m m!\,.$$

CASE 2. Degree$(g(x)) \leqq m + \varrho_0 m/\log m$.

The discussion in sections 4 and 5 shows that for some $i, |f(i)| \geqq$ the right hand side of (5.5), where one of the original conditions in connection with (5.5), namely $|\varrho| \leqq 3$, is replaced by $|\varrho| \leqq \varrho_0$. We thus have to give a lower estimate of

$$\inf_{\varphi,\varrho} M_{\varphi,\varrho} \tag{8.2}$$

under the new condition $|\varrho| \leqq \varrho_0$, the conditions on φ being the same as before. We now replace the quantity $M_{\varphi,\varrho}$ by a smaller (or equal) one and estimate the latter from below. Let $[\alpha,\beta]$ (respectively $[\gamma,\delta]$ be the interval spanned by suppφ (respectively supp$(1-\varphi)$) and put

$$\begin{aligned} M'_G &= \max(G(\gamma), G(\delta)), \quad M'_H = \max(H(\alpha), H(\beta)) \\ M'_{\varphi,\varrho} &= \max(M'_G + \varrho, M'_H - \varrho) \\ M' &= \inf_{\varphi,\varrho} M'_{\varphi,\varrho} \,. \end{aligned} \tag{8.3}$$

Then M' is clearly a lower estimate of (8.2). It turns out that the infimum in (8.3) is not attained. According to the remark in section 6 we are, however, entitled to consider, instead of our original φ's, φ's which are characteristic functions of finite unions of intervals. Then M' is attained for the characteristic function of the set

$$\{-c\} \cup [-b, -a] \cup [a, b] \cup \{c\} \tag{8.4}$$

where $a = 0.239\ldots, b = a + 0.5, c = (0.5 + a^2 + a/2)^{\frac{1}{2}}$.

In case the reader should feel cheated by just having been given the hints above, we shall *prove* a weaker result, so as at least to convince him that the approach in this paper leads somewhere. The weaker result is:

$$\lambda \geqq \sqrt{2}\,. \tag{8.5}$$

In order to prove (8.5) we consider M_φ, defined by (5.6). Assume that $\varphi(-1) < 1$, which is no essential restriction, as $M_\varphi = M_{1-\varphi}$. Let $[\alpha,\beta]$ be the interval spanned by suppφ. We have

$$\begin{aligned} 2M_\varphi &\geqq G(-1) + H(\alpha) \qquad (8.6) \\ &= \textstyle\int_\alpha^1 \varphi(t)(\log(t+1) - \log(t-\alpha))dt + \int_{-1}^1 \log|t-\alpha|dt \\ &\geqq \textstyle\int_0^1 (\log(t+1) - \log(t-\alpha))dt + \int_{-1}^1 \log|t-\alpha|dt \\ &= 2\log 2 - \alpha\log(-\alpha) + (1+\alpha)\log(1+\alpha) - 2 \qquad (8.7) \end{aligned}$$

where the second inequality holds because $\log(t+1) - \log(t-\alpha)$ is a decreasing function on $[\alpha, 1]$.

The expression (8.7) is seen to take its minimum, $\log 2-2$, at $\alpha=-\frac{1}{2}$. Hence

$$\lambda = \exp(1+\inf_\varphi M_\varphi) \geqq \exp(1+\tfrac{1}{2}(\log 2-2)) = \sqrt{2}\,.$$

9. Some simplifications of the unsolved problem.

The unsolved problem is, of course, that of determining λ, using lemma 3. The simplifications are, in the terminology of that lemma.

(A) Only functions φ which are identically zero near -1 and $+1$ need to be considered.

(B) In (5.6) the quantity M_G may be replaced by

$$\max(G(-1), G(1))\,. \tag{9.1}$$

(C) In (5.6) the quantity M_H may be replaced by

$$\max_{x\in[\alpha,\beta]} H(x) \tag{9.2}$$

where $[\alpha,\beta]$ is the convex hull of the support of φ.

As to (A), it is clear that the normalization $\varphi(-1)<1$ may be made. If $\alpha=-1$, (8.6) gives $M_\varphi \geqq \log 2-1=-0.306\ldots$. But we have the estimate

$$\inf M_\varphi \leqq -1+\log 4-\tfrac{1}{4}\log 27 = -0.437\ldots, \tag{9.3}$$

(cf. (7.9)) and so the case $\alpha=-1$ can be excluded. Using (8.6) we find that we may even assume

$$\alpha > -0.93\,.$$

There are now two distinct cases to consider. If $\varphi(1)<1$ the argument just given shows that

$$\beta < 0.93, \quad \varphi(1) = 0\,,$$

but the other case, $\varphi(1)=1$, may also occur. If so, we take also ϱ into consideration. After interchanging $(\varphi(t),\varrho)$ with $(1-\varphi(-t),-\varrho)$ if necessary, we may assume that $\varrho\geqq 0$. We then estimate the quantity

$$\inf_{\varphi,\varrho}\max(G(-1)+\varrho\,, H(\alpha)-\varrho) \tag{9.4}$$

where φ and ϱ are restricted only by

$$\varrho \geqq 0, \quad \varphi(-1) = 0, \quad 0 \leqq \varphi \leqq 1 = \textstyle\int_{-1}^{1}\varphi(t)dt\,.$$

This problem is similar to, but easier than, the one which was discussed under case 2, section 8. (That discussion would, if written out in full, include the present one, as the possibility $\varphi(-1)=0, \varphi(1)=1$, would have to be dealt with.) The function which corresponds to the one described by (8.4) is now the characteristic function of a certain set

$$\{p\} \cup [q, q+1], \quad p = -0.84\ldots, q = -(1-p)^2/2p .$$

It turns out that the value of (9.4) is > -0.42. Compare this to (9.3); the case $\varphi(1)=1$ is excluded.

We are now ready for the simplification (B). By (A) we may assume

$$\operatorname{supp}\varphi \subset [-0.93, 0.93] . \tag{9.5}$$

It turns out that this condition implies

$$\max_{x\in[-1,1]} G(x) = \max(G(-1), G(1)) \tag{9.6}$$

which clearly justifies (B). In order to prove (9.6) we first note that

$$\max_{x\in[-1,\alpha]} G(x) = G(-1), \quad \max_{x\in[\beta,1]} G(x) = G(1)$$

so that the falsity of (9.6) would imply the existence of a $y \in (\alpha,\beta)$ for which

$$G(y) \geqq G(-1), \quad G(y) \geqq G(1) . \tag{9.7}$$

If a point in, say, (y,β) is moved to the right, the logarithm of its distance from y increases more than the logarithm of its distance from -1, while its distance to $+1$ goes down. This means that the inequalities (9.7) imply

$$\int_{-1}^{1} \psi(t)\log|y-t|dt \geqq \max\left\{\int_{-1}^{1} \psi(t)\log|x-t|dt;\, x = \pm 1\right\} \tag{9.8}$$

where $\psi(t)$ is the characteristic function of the set

$$\left[-0.93, -0.93+\int_{\alpha}^{y}\varphi(t)dt\right], \quad \left[0.93-\int_{y}^{\beta}\varphi(t)dt,\, 0.93\right] .$$

It is now an exercise in calculus to see that (9.8) is impossible.

The final simplification (C) is justified by the fact that

$$\max_{x\in[\alpha,\beta]} H(x) = \max_{x\in\operatorname{supp}\varphi} H(x) . \tag{9.9}$$

If (9.9) were false, H would take its maximum on $[\alpha,\beta]$ at a point c for which, for some $\varepsilon > 0$,

$$|x-c| < \varepsilon \Rightarrow \varphi(x) = 0 .$$

Differentiating (5.2), we get, for $|x-c| < \varepsilon$

$$H'(x) = \left(\int_{-1}^{c-\varepsilon} + \int_{c+\varepsilon}^{1}\right)(1-\varphi(t))(x-t)^{-1}dt + \log(x-c+\varepsilon)(c-x+\varepsilon)^{-1}$$

$$H''(c) = -\left(\int_{-1}^{c-\varepsilon} + \int_{c+\varepsilon}^{1}\right)(1-\varphi(t))(c-t)^{-2}dt + 2\varepsilon^{-1} > -2\int_{\varepsilon}^{\infty} t^{-2}dt + 2\varepsilon^{-1} = 0 ,$$

a contradiction.

10. Numerical results on $P(n)$ and $Q(n)$.

Computation by hand gave $P(2),\ldots,P(7) = 1, 2, 1, 3, 5, 16$. The extremal polynomials are (after a linear substitution) x^2 and $(2x+1)^2$ for $n=2$, $x^3-(1\pm 1)x^2-(1\mp 1)x$, x^3-x^2-x and x^3-3x for $n=3$, $(x^2+x-1)^2$ for $n=4$. For $n=5$ one has with $f(x)=g(x)h(x)$,

$$g(1),\ldots,g(5) = 3,\ -1,\ -3,\ -3,\ -1$$
$$h(1),\ldots,h(5) = -1,\ 3,\ 1,\ -1,\ 3\ .$$

For $n=6$, the corresponding sequences are $5, 1, -1, -1, 1, 5$ and $-1, -5, 5, 5, -5, -1$. For $n=7$ we have

$$g(x) = (x-2)(x-4)(x-6)+1\,,$$
$$h(x) = (x-1)(x-3)(x-5)(x-7)\pm 1\ .$$

The extremal special polynomials were found, by computer, for $n=2,\ldots,26$. We describe them by $0-1-$sequences$s_2,\ldots,s_{26}$, where

$$s_n(i) = 1 \Leftrightarrow i \in \{b_1,\ldots,b_p\}$$

(cf. (1.4)). The sequences are

01, 011, 0110, 01010, 010010, $s_5$10, 01101001, $s_7$10, 0110101001, $s_9$10, 001010110100, s_{11}10, 00101011010100, s_{13}10, 0010101011010100, s_{15}10, 001010110100110100, s_{17}10, 00101011100010110100, 00010101101011010100 0,

The corresponding values of $Q(n)$, when written in the form $\lambda_n{}^m m!$ give the following sequence of λ's (rounded values): 2.00, 1.23, 1.22, 0.88, 1.05, 0.91, 1.24, 0.98, 1.29, 1.05, 1.33, 1.11, 1.31, 1.16, 1.37, 1.21, 1.39, 1.25, 1.43, 1.27, 1.45, 1.30, 1.48, 1.30, 1.52.

11. Concluding remarks.

It is clear that much remains to be done on the problem of estimating $P(n)$ and $Q(n)$. The outstanding problem seems to be the determination of λ and the (probably existing and unique) corresponding continuous function φ. But it would also be interesting to find out something more on the extremal polynomials.

It is tempting to conjecture that they are either special polynomials or obtainable from such by linear transformations, or by replacing one or both of the "1's" in (1.4) by -1. It is rather easy to see, using lemma

1, that if $g(x) = ax^p + \ldots$ is one of the two factors in an extremal polynomial, then

$$g(x) = a(x-b_1)(x-b_2)\ldots(x-b_p)+g_1(x)$$

where $b_1,\ldots,b_p$ are the integers near the zeros of g and $g_1(x)$ is of degree k satisfying

$$k \log k = O(m) .$$

It seems probable that here $O(m)$ can be replaced by $O(1)$.

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