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FINITE GROUPS WITH SYLOW 2-INTERSECTIONS OF RANK $\leqq 1$

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In this paper we are going to classify all finite groups satisfying:

(I) Every intersection of two distinct Sylow 2-subgroups is of rank $\leqq 1$.

This classification is, therefore, a generalization of a theorem by M. Suzuki [15], who in 1964 classified all finite groups in which a Sylow 2-intersection is trivial.

Recently W. D. Mazurov [12] classified all finite simple groups satisfying:

(C) Every intersection of two distinct Sylow 2-subgroups is cyclic.

Independently, M. Herzog ([8] and [9]) classified all finite simple groups satisfying:

(C*) Every central Sylow 2-intersection is cyclic.

M. Herzog and E. Shult ([8] and [10]) have just completed the classification by determining all finite simple groups satisfying:

(I*) Every central Sylow 2-intersection is of rank $\leqq 1$.

Here an involution is called central if it belongs to the center of some Sylow 2-subgroup and an intersection of two distinct Sylow 2-subgroups is called a central Sylow 2-intersection if it contains a central involution or is trivial. It turned out that the simple groups satisfying (C) were the same as those satisfying (I*). In particular, all groups with a quaternion Sylow 2-subgroup (by quaternion, we mean generalized or ordinary quaternion) satisfy (I). These groups were already classified partly by Brauer-Suzuki [2] in 1959 and completely in 1965 as a consequence of [6], by using a result of Schur ([13] and [14]).

We are going to prove:

THEOREM. *Let G be a finite group satisfying* (I). *Then G is of one of the following forms*:

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I. *G is solvable.* $G/O(G)$ *is either 2-closed or equals* GL(2,3) *or* T_4, *the representation groups of* S_4.

II. *A Sylow 2-subgroup is quaternion. G contains a normal subgroup* $G_1 \geqq O(G)$ *such that* G/G_1 *is odd, and either* $G_1/O(G) \cong \mathrm{SL}(2,q)$, *q odd,* $q > 3$ *or* $G/O(G)$ *is the perfect extension of a group of order 2 by* A_7, *or an extension of one of these groups by a group of order 2.*

III. *G contains normal subgroups* G_1 *and* G_2 *such that*

$$G \geqq G_1 \geqq G_2 \geqq 1$$

where G/G_1 *is odd, and*

a) $G_2 = O_2(G) \times O(G)$ *and* G_1/G_2 *is one of the following groups:*
 (i) $\mathrm{PSL}(2,q)$, $q = 2^n > 2$,
 (ii) $\mathrm{Sz}(q)$, $q = 2^n \geqq 8$, *n odd,*
 (iii) $\mathrm{PSU}(3,q)$, $q = 2^n > 2$.

Moreover if $O_2(G)$ *is cyclic and* $G_1 = O'(G)$, *the minimal normal subgroup of odd index, then* $G_2 = Z(G_1)$,

b) $G_2 = O(G)$ *and* G_1/G_2 *is one of the following*
 (iv) $\mathrm{PSL}(2,q)$, $q \equiv 3$ *or* 5 (mod 8) $q > 5$,
 (v) J(11), *the Janko group.*

In all 5 cases $R \trianglelefteq RO(G)$, $R \in \mathrm{Syl}_2(G)$.

IV. *G contains a normal subgroup* $G_1 \geqq O(G)$, *such that*

$$G_1/O(G) \cong \mathrm{SL}(2,5) \curlyvee \mathrm{SL}(2,5),$$

the central product with common center (notation of Huppert [11]), *and* G/G_1 *is odd or the direct product of a group of odd order and a group of order 2 permuting the two copies of* SL(2, 5).

In [8], [9] and [10] it is proved that if G satisfying (I*) is simple, then G is of type (i)–(v), and as mentioned above, the same groups appear in [12]. Groups of type (i)–(iii) are called simple (TI)-groups.

The proof will consist of three sections. Section 1 contains general results some of which may be found in papers dealing with related problems. In Section 2 all finite groups satisfying condition (C) are classified, and Section 3 deals with finite groups satisfying condition (I), but with at least one Sylow 2-intersection quaternion, thereby completing the classification.

The problem was proposed by Professor Marcel Herzog to whom the author is very grateful for pointing out weak proofs and for encouraging with useful suggestions.

Shortly before finishing this paper the author learnt that Professor M. Aschbacher has classified the same groups using quite a different method. The classification in the present paper is a little stronger and proofs are independent. The author wants to thank Professor M. Aschbacher for receiving a copy of his manuscript [1].

1. Prerequisites.

NOTATION. Let G be a finite group, $\bar{G}=G/O(G)$. Let $R \in \mathrm{Syl}_2(G)$ and $S(G)$ denote the maximal normal solvable subgroup of G. Concerning other symbols, see [5].

DEFINITION. An involution is called concealed if it is contained in a unique Sylow 2-subgroup.

The following theorem is a corollary of Shult's theorem on fusion, proved in [7]:

THEOREM 1.1. *Let G be a finite group and suppose that G contains a central concealed involution z. Let $N=\langle z^G\rangle$. Then*

$$N/Z(N) \cong N_1 \times \ldots \times N_r \times M$$

where M has an elementary abelian Sylow 2-subgroup and a normal 2-complement and each N_i is a simple (TI)*-group.*

LEMMA 1.2. *Let G satisfy* (I), *$H \leq G$. Then*
(i) *H satisfies* (I).
(ii) *If $H \trianglelefteq G$ and $|G/H|_2=|G|_2$, then G/H satisfies* (I).

PROOF. See Lemma 1 of [8]. In the following we will use this lemma without mentioning it explicitly. In Section 2 we will see how it may be strengthened if G satisfies (C).

LEMMA 1.3. *Let G satisfy* (I) *and $R_1 \in \mathrm{Syl}_2(G)$. Let $z \in Z(R_1)$ be an involution and $z^g \in R_2$ for some $R_2 \in \mathrm{Syl}_2(G)$ and $g \in G$. Then $z^g \in Z(R_2)$.*

PROOF. See Lemma 3 of [8].

LEMMA 1.4. *Let S satisfy* (I), *$T \in \mathrm{Syl}_2(S)$. Suppose $S=O(S)T$ and $|\Omega_1(Z(T))|>2$. Then either T is normal in S or* 2-rank$(S)=2$.

PROOF. See Lemma 5 of [8].

LEMMA 1.5. *Suppose H satisfies* (I), $Z(H/S(H))=1$ *and* $H>S(H)=VO(H)$, *where* $1<V\leqq T\in \mathrm{Syl}_2(H)$. *Then* $|\Omega_1(Z(T))|=2$ *or T is normal in* $TO(H)$.

PROOF. By the Frattini argument, $H=N_H(V)S(H)$. Since H is non-solvable, $N_H(V)$ is not 2-closed. Hence by assumption $\mathrm{rank}(V)=1$. If T is not normal in $TO(H)$ then by Lemma 1.4 either $|\Omega_1(Z(T))|=2$ or $\mathrm{rank}(T)=2$ and $|\Omega_1(Z(T))|=4$, as every involution belongs to the center. Consider the last case and let j be the involution of V and $i\in T\setminus V$ another involution in the center. Then $iS(H)$ is an involution in $TO(H)/S(H)$ which is a member of $\mathrm{Syl}_2(H/S(H))$. Let $h^{-1}ihS(H)$ be a conjugate that also belongs to $TO(H)/S(H)$, $h\in H$. Then $h^{-1}ih=tvs$ for some $t\in T$, $v\in V$ and $s\in O(H)$. But $(tvs)^2=1=(tv)^2s's$ for some $s'\in O(H)$. Hence $(tv)^2=(s's)^{-1}\in T\cap O(H)=1$ and $tv\in T\setminus V$ is an involution. Since i and ij are the only involutions in $T\setminus V$, we have $h^{-1}ihS(H)=iS(H)$ and $iS(H)$ is isolated. By Glauberman [4], $iS(H)\in Z(H/S(H))$ in contradiction to the assumption.

The next two theorems deal with extensions of certain groups that satisfy (I) and are just weak generalizations of results proved in [9] and [10].

THEOREM 1.6. *Suppose G satisfying* (I) *contains a unique minimal normal subgroup H which is simple. Then G/H is odd and G satisfies* (C).

PROOF. By [8], [9] and [10] H is of type (i)–(v). The proof of Lemma 4 in [9] may then be applied.

THEOREM 1.7. *Suppose G satisfying* (I) *is of* 2-rank >1 *and contains a normal subgroup* $G_2=VO(G)$, *V a cyclic 2-group and that G/G_2 contains a unique minimal normal subgroup H/G_2 which is a simple* (TI)*-group. Then G/H is odd, and G satisfies* (C).

PROOF. By Theorem 1.6 we may assume $V\neq 1$. Suppose $|G/H|$ is even. In order to reach a contradiction, we may assume that $|G:H|=2$ by Lemma 1.2. By Theorem 1.6, it will be enough to prove that G/G_2 satisfies (C). Let

$$R_1/G_2\cap R_2/G_2 = D/G_2$$

and

$$Q_i/G_2 = R_i/G_2\cap H/G_2\,,$$

where $R_i/G_2 \in \mathrm{Syl}_2(G/G_2)$ $i=1,2$. If $|D/G_2 \cap H/G_2|=1$, then $|D/G_2| \leqq 2$ and D/G_2 is cyclic. If not, then since H/G_2 is a (TI)-group,

$$Q_1/G_2 = Q_2/G_2 = D/G_2 \cap H/G_2$$

and hence a Sylow 2-subgroup of H is quaternion. Thus $H/O(G)=\mathrm{SL}(2,5)$, since H/G_2 is a simple (TI)-group. Now we may assume that 2-rank$(G)>1$. Hence $G/H=\langle zH\rangle$, z an involution. We will be done by proving the following (for $L=G/O(G)$):

LEMMA 1.8. *Suppose L contains* SL(2,5) *as a normal subgroup of index* 2. *Then L does not satisfy condition* (I) (*or* (C)).

PROOF. Let $R \in \mathrm{Syl}_2(L)$. From the known structure of L, rank$(R)>1$. Let $R_1 \in \mathrm{Syl}_2(\mathrm{SL}(2,5))$, say $R_1 \leqq R$. Then $R/R_1=\langle xR_1\rangle$ for some involution $x \in R$, and x acts on SL(2,5). If x acts trivial we are done, since then x and z, the involution of SL(2,5), belong to every Sylow 2-intersection. Hence we may assume that x acts non-trivial on SL(2,5) and $L=\mathrm{SL}^*(2,5)$ then with the notation of [16]. Also by [16], $|C_L(x)|=2(q+1)=12$, and R is semi-dihedral. But then R contains a characteristic subgroup of index 2, which is cyclic. Hence $N_L(R)=R$, since no element of odd order centralizes the cyclic subgroup by the fact that a Sylow 2-subgroup of PSL(2,5) is self-centralizing. Thus there exists an element that centralizes x and z but does not normalize R and again $\langle x,z\rangle$ is contained in a Sylow 2-intersection.

2. Finite groups satisfying (C)

THEOREM A. *If G satisfies* (C), *then G is of one of the following forms*:

I. *G solvable. Then $G/O(G)$ is 2-closed.*

II. *A Sylow 2-subgroup is ordinary quaternion, and G contains a normal subgroup $G_1 \geqq S(G)$ such that G/G_1 is odd and*

$$G_1/O(G) = \mathrm{SL}(2,q), \quad q \equiv 3 \text{ or } 5 \pmod 8,\ q>3\,.$$

III. *G contains a normal subgroup $G_1 \geqq S(G)$ such that G/G_1 is odd, $R \trianglelefteq RO(G)$, $R \in \mathrm{Syl}_2(G)$, and*

a) *$S(G)=O_2(G)\times O(G)$ and $G_1/S(G)$ is isomorphic to*
  (i) PSL$(2,q)$, $q=2^n>2$,
  (ii) Sz(q), $q=2^n \geqq 8$, *n odd*,
  (iii) PSU$(3,q)$, $q=2^n>2$,

b) *$S(G)=O(G)$ and $G_1/S(G)$ is isomorphic to*
(iv) PSL$(2,q)$, $q\equiv 3$ *or* 5 (mod 8) $q>5$
(v) J(11), *the Janko group.*

To prove the above theorem, assume in this section that G satisfies (C). Let $\bar{G}=G/O(G)$ and $R\in \mathrm{Syl}_2(G)$.

LEMMA 2.1. *If G is solvable, then $\bar{G}$ is 2-closed.*

PROOF. Since G is solvable, $\bar{C}=C_{\bar{G}}(O_2(\bar{G}))\leqq O_2(\bar{G})\neq 1$ (see [5, Theorem 6.3.2, p. 228]). If G is not 2-closed, then $O_2(\bar{G})$ is cyclic by assumption. But then $\bar{G}/\bar{C}$ and hence also $\bar{G}/O_2(\bar{G})$ is a 2-group, a contradiction. (See also Lemma 1.4.)

We will in the following assume that G is non-solvable.

THEOREM 2.2. *$S(G)=VO(G)$, V a cyclic 2-group and G contains a normal subgroup G_1, such that G/G_1 is of odd order. $G_1/S(G)$ is simple of type* (i)–(v).

To prove this, we need the following two lemmas.

LEMMA 2.3. *If $H\leqq G$ then:*
(i) *H satisfies* (C).
(ii) *If $H\trianglelefteq G$, G/H satisfies* (C).

PROOF. See also Lemma 3 of [12]. (i) is trivial.

(ii) Let $S_1/H, S_2/H\in \mathrm{Syl}_2(G/H)$ such that $D/H=S_1/H\cap S_2/H$ is non-cyclic and maximal. Clearly $N/H=N_{G/H}(D/H)$ is not 2-closed. Now let $T\in\mathrm{Syl}_2(D)$. Then $D=TH$ and $D/H\simeq T/H\cap T$ showing that T is non-cyclic. $T\leqq N$ and by (i), $N_N(T)$ is 2-closed. But $N=DN_N(T)$, and therefore $N/H=DN_N(T)/H$ which clearly is 2-closed, a contradiction.

LEMMA 2.4. *$S(G)=VO(G)$, where V is a cyclic 2-group.*

PROOF. Let $V\in\mathrm{Syl}_2(S(G))$. By the Frattini argument $G=N_G(V)S(G)$. Since G is non-solvable, $N_G(V)$ is not 2-closed. So V is cyclic and by Burnside, V has a normal complement in $S(G)$. By maximality this has to be $O(G)$.

Now to complete the proof of Theorem 2.2, define $G_1/S(G)\trianglelefteq G/S(G)$ to be the product of all minimal normal subgroups of $G/S(G)$.

LEMMA 2.5. *$G_1/S(G)$ is simple of type* (i)–(v).

PROOF. $G_1/S(G)$ is the direct product of simple groups since a minimal normal subgroup is characteristically simple. The Sylow 2-subgroups of the simple groups are non-cyclic. Since $G_1/S(G)$ satisfies condition (C) by Lemma 2.2, $G_1/S(G)$ itself has to be simple. By [9] or [12], $G_1/S(G)$ is of type (i)–(v).

Now it follows immediately by Theorem 1.6, that G/G_1 is odd, and Theorem 2.2 is proved.

Assume in the following that $V \leqq R \in \mathrm{Syl}_2(G)$. Clearly $V \trianglelefteq R$. Also we let G still be non-solvable and $\bar{G} = G/O(G)$.

LEMMA 2.6. *$VO(G)/O(G) \leqq Z(\bar{G})$. In particular $V \leqq Z(R)$.*

PROOF. $VO(G)/O(G) \trianglelefteq \bar{G}$. Let $\bar{C} = C_{\bar{G}}(VO(G)/O(G))$. If $V \neq 1$, then since V is a cyclic 2-group, $\bar{G}/\bar{C}$ is an abelian 2-group. By the known structure of $\bar{G}$ (Theorem 2.2), $\bar{G} = \bar{C}$ then. As $V \trianglelefteq R$ and R centralizes $V \bmod O(G)$, $V \leqq Z(R)$.

THEOREM 2.7. *If R is quaternion then R is ordinary quaternion, V is of order 2 and*

$$G_1/O(G) = \mathrm{SL}(2,q), \quad q \equiv 3 \text{ or } 5 \pmod 8, \ q > 3 .$$

PROOF. By Brauer–Suzuki [2], $Z(\bar{G})$ is of order 2 and thus by Lemma 2.6, $Z(\bar{G}) = VO(G)/O(G)$. By Theorem 2.2, $G_1/S(G)$ is simple of type (i)–(v). But since R/V is dihedral, the only possibility among these groups is a group satisfying: R/V is elementary abelian of order 4 and $G_1/S(G) = \mathrm{PSL}(2,q)$, $q \equiv 3$ or $5 \pmod 8$, $q > 3$, and hence R is ordinary quaternion. By Schur [13, p. 120], $G_1/O(G) = \mathrm{SL}(2,q)$.

So assume in the following that G is non-solvable satisfying (C) and that G has 2-rank > 1.

THEOREM 2.8. *V is normal in G. Moreover, if $G_1/S(G)$ is of type* (iv) *or* (v), *then $V = 1$.*

The proof will consist of a series of lemmas.

LEMMA 2.9. *Suppose $1 < V \leqq R$. Then either $|\Omega_1(Z(R))| = 2$ or $V = O_2(G)$.*

Proof. By Lemma 1.5, either $R \trianglelefteq RO(G)$ or R has only one involution in the center. In the first case, we get that $V \trianglelefteq VO(G)$, since $V \trianglelefteq R$. Hence $V = O_2(G)$.

Lemma 2.10. *If* $V \neq 1$, *then* $G_1/S(G)$ *is not of type* (iv) *nor* (v).

Proof. In both cases a non-trivial Sylow 2-intersection consists of the group generated by one involution. Moreover, it is known that every involution generates a Sylow 2-intersection, since they are conjugate. Let $i \notin V$ be an involution of R. Hence $iS(G)$ is an involution of $RO(G)/S(G)$ and we may assume

$$\{S(G), iS(G)\} = RO(G)/S(G) \cap g^{-1}RgO(G)/S(G)$$

for some $g \in G$. Now $\langle i \rangle V \leqq R$ and $\langle i \rangle V \leqq T$ for some $T \in \mathrm{Syl}_2(g^{-1}RgO(G))$. But $T \neq R$ since $RO(G) \neq g^{-1}RgO(G)$. Therefore $\langle i \rangle V$ is contained in a proper Sylow 2-intersection.

So to complete the proof of Theorem 2.8 we only have to consider the cases where $G_1/S(G)$ is a simple (TI)-group. By Lemma 2.9 we are done, if R has more than one involution in the center or $R \trianglelefteq RO(G)$. So assume $j \in V \neq 1$ is the central involution of R, R not normal in $RO(G)$ and $\mathrm{rank}(R) > 1$.

We need the following general fact on p-groups (Huppert [11, Satz 13.7, p. 353]).

Lemma 2.11. *Let P be a non-abelian p-group such that $P/Z(P)$ is elementary abelian and $Z(P)$ is cyclic. Then the following hold*:

(i) $|P'| = p$.

(ii) $|P/Z(P)|$ *is a square.*

(iii) *If* $|P/Z(P)| = p^{2m}$, *then every maximal abelian normal subgroup* A *of* P *is of order* $p^m|Z(P)|$.

Now define

$$R_1/V = \Omega_1(Z(R/V)) = Z(R/V) = \Omega_1(R/V)$$

by the known structure of the Sylow 2-subgroup of a simple (TI)-group. Consider $R_1O(G)/VO(G) \cong R_1/V$. Let $i \notin V$ be an involution of R. Then $i \in R_1$. In a simple (TI)-group all involutions are conjugate. Also if two involutions belong to the same Sylow 2-subgroup, they are conjugate in the normalizer of the Sylow 2-subgroup.

LEMMA 2.12. $Z(R_1) = V$. *In particular* R_1 *satisfies the conditions of Lemma* 2.11.

PROOF. By the remarks above,

$$R_1O(G)/S(G) = \{S(G), iS(G), i^{g_3}S(G), \ldots, i^{g_q}S(G)\}, \quad |R_1/V| = q,$$

where $g_i \in N_{G_1}(R)$. So $i^{g_k} \in R$, and therefore

$$R_1/V = \{V, iV, i^{g_3}V, \ldots, i^{g_q}V\}.$$

By assumption none of the i^{g_k}'s is central in R. From Lemma 2.6 we know that $V \leqq Z(R_1)$. If $G_1/S(G) = \mathrm{PSL}(2,q)$, $q = 2^n > 2$, then $R_1 = R$. Let $r \in R \setminus V$. Since $r \in i^{g_k}V$ for some k we see that r is not central, and therefore $Z(R_1) \leqq V$. Consider

$$G_1/S(G) \simeq \mathrm{Sz}(q), \quad q = 2^n \geqq 8$$

or

$$G_1/S(G) \simeq \mathrm{PSU}(3,q), \quad q = 2^n > 2.$$

Suppose $i \in Z(R_1)$. Then by Lemma 1.3, R_1 is abelian, so $|\Omega_1(Z(R))| > 2$, in fact $|\Omega_1(Z(R))| \geqq 8$ from the known structure of $G_1/S(G)$. Hence by applying Lemma 1.4 on $R_1O(G)$, we get that $R_1 \trianglelefteq R_1O(G)$. Since $\mathrm{rank}(R_1) > 1$ and $RO(G)$ satisfies condition (C), $R \trianglelefteq RO(G)$—a contradiction to the assumption—and the lemma follows.

LEMMA 2.13. $|R_1/V| \leqq 4$.

PROOF. Suppose $|R_1/V| = q > 4$. By Lemma 2.11 (ii), $|R/V|$ is a square, say $q = 2^{2m}$. Let A be a maximal abelian normal subgroup of R. By 2.11 (iii), $|A| = 2^m|V|$. Since

$$R/V = \{V, iV, \ldots, i^{g_q}V\},$$
$$A/V = \{V, iV, \ldots, i^{g_r}V\}, \quad r = q^{\frac{1}{2}}, \text{ say}.$$

If $q > 4$, then since A is abelian, $|\Omega_1(Z(A))| > 4$ and again by Lemma 1.4 applied to $A\,O(G)$ we get $A \trianglelefteq A\,O(G)$, a contradiction since then $R \trianglelefteq RO(G)$.

COROLLARY 2.14. $G_1/S(G)$ *is not isomorphic to*

(i) $\mathrm{PSL}(2,q)$ $\quad q = 2^n > 4$,
(ii) $\mathrm{Sz}(q)$ $\quad q = 2^n \geqq 8$, *n odd*,
(iii) $\mathrm{PSU}(3,q)$ $\quad q = 2^n > 4$.

Now consider the case $G_1/S(G) \simeq \mathrm{PSL}(2,4) \simeq \mathrm{PSL}(2,5)$. Let $H = G_1/O(G)$. We need the following result of Schur [13].

LEMMA 2.15. *Let H be a finite group and U a subgroup satisfying*
(i) $1 \neq U \leqq Z(H)$,
(ii) $H/U \cong \mathrm{PSL}(2,p)$, *$p$ an odd prime,*
(iii) $U \leqq H'$.
Then $|U| = 2$ *and* $H \cong \mathrm{SL}(2,p)$.

By this lemma we are done if H is perfect, that is $H' = H$. So assume $H' < H$, and define $H^{(n)} = (H^{(n-1)})'$. Hence $H^{(n)'} = H^{(n)}$ for some n. Therefore either $H^{(n)} = \mathrm{PSL}(2,5)$ or if $O_2(H^{(n)}) \neq 1$, we have $H^{(n)} = \mathrm{SL}(2,5)$ by Lemma 2.15 since then $H^{(n)}$ satisfies the conditions of Lemma 2.15 with $U = O_2(H^{(n)})$. $H^{(n)} = \mathrm{SL}(2,5)$ is impossible by Lemma 1.8. $H^{(n)} = \mathrm{PSL}(2,5)$ is impossible since then $H = \mathrm{PSL}(2,5) \times O_2(H)$, which has an abelian Sylow 2-subgroup. So we have proved

LEMMA 2.16. *$G_1/S(G)$ is not isomorphic to* $\mathrm{PSL}(2,4)$.

LEMMA 2.17. *$G_1/S(G)$ is not isomorphic to* $\mathrm{PSU}(3,4)$.

PROOF. $|R_1/V| = \{V, iV, i^{g_3}V, i^{g_4}V\}$. Since i is not central and $R' = \langle j \rangle$, we have $r^{-1}ir = ij$ for some $r \in R$, where j is the involution of V. Hence R contains exactly 2 conjugacy classes of involutions in G_1, namely

$$\{j\} \quad \text{and} \quad \{i, ij, i^{g_3}, i^{g_3}j, i^{g_4}, i^{g_4}j\}\ .$$

Consider the order of $C_{G_1}(i)$. Each Sylow 2-subgroup contains 6 conjugates of i. Suppose i is contained in r Sylow 2-subgroups. Then

$$|G_1/C_{G_1}(i)| \;=\; 6r^{-1}|G_1 : N_{G_1}(R)|\ ,$$

that is $|R/C_R(i)| \leqq 2$, hence $|R/C_R(i)| = 2$, since i is non-central. Now i^{g_3} does not centralize i since R_1 is non-abelian as we saw. By the known structure of a Sylow 2-subgroup of $\mathrm{PSU}(3,4)$, every involution is a square. Hence $y^2V = i^{g_3}V$ for some $y \in R$. It follows that $yC_R(i)$ is of order 4 in $R/C_R(i)$, a contradiction.

By this Theorem 2.8 is proved. As a matter of fact, we have proved

LEMMA 2.18. *If $R \in \mathrm{Syl}_2(G)$, then $R \trianglelefteq RO(G)$.*

REMARK. From Lemma 2.6 we know that $VO(G)/O(G) \leqq Z(G/O(G))$. Hence, since V is normal in G, also $V \leqq Z(G)$. If we define G_1 to be the subgroup generated by all 2-elements of G, that is the minimal normal subgroup of odd index, then it follows from Lemma 2.18 and the definition of G_1 that $S(G_1) = Z(G_1)$.

3. Finite groups satisfying (I) but not (C).

THEOREM B. *If G satisfies condition* (I) *but not* (C), *then G is of one of the following forms*:

I. *G is solvable. $G/O(G)$ is either 2-closed or equals* GL(2,3) *or* T_4, *the representation groups of* S_4.

II. *A Sylow 2-subgroup is generalized quaternion. Then either $G/O(G)$ is the perfect extension of a group of order 2 by* A_7 *or G contains a normal subgroup G_1, such that G/G_1 is odd and $G_1/O(G) = \mathrm{SL}(2,q)$, q odd* ($q \not\equiv 3$ *or* 5 (mod 8)), *or an extension of one of these groups by a group of order* 2.

III. *G contains a normal subgroup $G_1 \geqq O_2(G) \times O(G)$, where $O_2(G)$ is quaternion and $G_1/O_2(G) \times O(G)$ is one of the following groups*:

(i) $\mathrm{PSL}(2,q)$, $q = 2^n > 2$,
(ii) $\mathrm{Sz}(q)$, $q = 2^n \geqq 8$, *n odd*,
(iii) $\mathrm{PSU}(3,q)$, $q = 2^n > 2$.

IV. *G contains a normal subgroup $G_1 \geqq O(G)$ such that*

$$G_1/O(G) = \mathrm{SL}(2,5) \curlyvee \mathrm{SL}(2,5)$$

the central product with common center (*notation of Huppert*) *and G/G_1 is odd or the direct product of a group of odd order and a group of order* 2 *permuting the two copies of* SL(2,5).

So assume in the following that G satisfies (I) but not (C). Let $\bar{G} = G/O(G)$, $R \in \mathrm{Syl}_2(G)$.

THEOREM 3.1. *Let G be solvable. Then $G/O(G)$ is 2-closed or equals* GL(2,3) *or* T_4 (*in the notation of Schur, see* [14]).

PROOF. Suppose $\bar{G}$ is not 2-closed. Since G is solvable $O_2(\bar{G}) \neq 1$ and cyclic or quaternion by assumption. Since the automorphism group of a cyclic 2-group or a generalized quaternion group is a 2-group and since $\bar{C} = C_{\bar{G}}(O_2(\bar{G})) \leqq O_2(\bar{G})$ (by Theorem 6.3.2 in [5]), we may assume that $O_2(\bar{G})$ is ordinary quaternion. The automorphism group of Q_8 is $S_4 = \mathrm{PGL}(2,3)$. Hence $\bar{G}/\bar{C} \leqq \mathrm{PGL}(2,3)$, and since it is not 2-closed, the only possibility is $\bar{G}/\bar{C} = \mathrm{PGL}(2,3)$. Hence $\bar{G}$ is a representation group of S_4. By Schur [14, p. 164], $\bar{G}$ equals GL(2,3) or T_4, where T_4 is defined by

$$j^2 = 1, \quad t_1^2 = t_2^2 = t_3^2 = j, \quad t_1t_3 = jt_3t_1 .$$

Its Sylow 2-subgroup is quaternion of order 16 and $t_1t_2t_3$ is an element of order 8. T_4 may also be considered as an extension of SL(2,3) by a group of order 2.

So assume in the following that G is non-solvable. Let $\bar{G}_1 = G_1/O(G)$ be the subgroup of $\bar{G}$ generated by all 2-elements and $G_2 = S(G_1)$.

Theorem 3.2. *If $V \in \mathrm{Syl}_2(G_2)$ is quaternion, then G/G_2 satisfies condition* (C) *and G_1/G_2 is a simple* (TI)*-group.*

Proof. V is quaternion, so $\mathrm{rank}(R) > 1$ since G is non-solvable. By the Frattini argument, $G = N_G(V)G_2$. Let

$$RG_2/G_2 \cap R^g G_2/G_2 \;=\; D/G_2$$

be a Sylow 2-intersection in G/G_2. We may assume $V \leqq R$ and $g \in N_G(V)$, that is $V \leqq R \cap R^g \leqq S \in \mathrm{Syl}_2(D)$. Since G satisfies (I), S is therefore quaternion, and $|S : V| \leqq 2$ since $V \trianglelefteq S$. Hence G/G_2 satisfies condition (C). So by Theorem A, G_1/G_2 is a simple group of type (i)–(v). If $i \in R \setminus V$ is an involution, then iG_2 is contained in a unique Sylow 2-subgroup (see Proof of Lemma 2.9). Therefore G_1/G_2 is not of type (iv) nor (v).

Lemma 3.3. $G_2 = VO(G)$, *where V is a 2-group of rank* $\leqq 1$.

Proof. $V \in \mathrm{Syl}_2(G_2)$. If V is cyclic we are done. If not then again by using the Frattini argument V is quaternion. By Theorem 3.1, $G_2/O(G)$ is 2-closed or equals T_4. Let $\bar{G}_2 = G_2/O(G)$. If $O_2(\bar{G}_2) < \bar{G}_2$, then since

$$C_{\bar{G}_2}(O_2(\bar{G}_2)) \;\leqq\; O_2(\bar{G}_2)\;,$$

$O_2(\bar{G}_2)$ is ordinary quaternion. If $V = O_2(\bar{G}_2)$, then $\bar{G}_1 = G_1/O(G)$ contains a normal subgroup $\bar{C}$ such that $\bar{G}_1/\bar{C} = \mathrm{PSL}(2,3)$. So G_1 contains a normal subgroup of odd index containing $O(G)$, contrary to the definition of G_1. If $\bar{G}_2 = \mathrm{T}_4$, then consider $\bar{G}_1/O_2(\bar{G}_2)$, which contains a copy of $\mathrm{T}_4/\mathrm{Q}_8$ as a normal subgroup, the factorgroup being G_1/G_2. Since $\mathrm{T}_4/\mathrm{Q}_8 \cong \mathrm{PSL}(2,2)$ is of order 6, it easily follows that $H = \bar{G}_1/O_2(\bar{G}_2)$ satisfies condition (C) by use of Theorem 3.2. By Theorem A, $S(H)$ is 2-closed, contrary to the fact that $S(H) \cong \mathrm{PSL}(2,2)$.

Remark. G_2 does not have to be $S(G)$. As an example consider $N \times \mathrm{SL}(2,3)$, N a simple (TI)-group.

Theorem 3.4. *If V is quaternion, then $V = O_2(G)$.*

Proof. Let $R_1 \in \mathrm{Syl}_2(C)$ where $C/O(G) = \bar{C} = C_{\bar{G}}(VO(G)/O(G))$, say $V \leqq R_1$. If j is the central involution of V_1 then $\langle jO(G)\rangle/O(G) \trianglelefteq \bar{C}$, and

C satisfies Theorem 1.7 by Theorem 3.2. So C satisfies (C) and by Theorem A, $R_1 \trianglelefteq R_1 O(G)$. If $\operatorname{rank}(R_1) > 1$, then $R \trianglelefteq RO(G)$ and we are done since $V \trianglelefteq R$. If $\operatorname{rank}(R_1) = 1$, then $\bar{C} = \mathrm{SL}(2,5)$. But by definition of C G/C is not odd and we are done by Lemma 1.8, since then G does not satisfy (I).

So in the following, we may assume that V is cyclic.

LEMMA 3.5. *The product of all minimal normal subgroups of G/G_2 of even order is either the product of two simple groups or simple itself.*

PROOF. Define $H/G_2 = \hat{H}$ to be the direct product of all minimal normal subgroups. Then $\hat{H}$ is the product of simple groups, say $\hat{S}_1 \times \ldots \times \hat{S}_n$. Suppose $n > 1$. Let $S \in \mathrm{Syl}_2(I)$,

$$I/G_2 = \hat{R}_1 \times \ldots \times \hat{R}_{n-1}, \quad \hat{R}_i \in \mathrm{Syl}_2(\hat{S}_i) .$$

Then S is contained in a Sylow 2-intersection. It follows that $n \leqq 2$.

The following theorem is the main step in this section. As mentioned we may assume that V is cyclic, and $V \trianglelefteq R \in \mathrm{Syl}_2(G)$.

THEOREM 3.6. *Suppose $\hat{H}$ is simple. Then R is generalized quaternion.*

The proof will consist of a series of lemmas.

LEMMA 3.7. *R is quaternion, dihedral or semi-dihedral, and $V = Z(R) = \langle j \rangle$, where j is an involution.*

PROOF. If $V = 1$ then, by Theorem 1.6, G satisfies (C) contrary to the assumption on G. Let j be the involution of V. If R contains another involution, z in the center, then zG_2 would be central concealed in G/G_2, and by Theorem 1.1 and Theorem 1.7, G satisfies (C). So j is the only involution of $Z(R)$. If R contains an elementary abelian normal subgroup $A = \langle z,j \rangle$ of order 4, zG_2 again is central concealed. This proves the first part of the lemma.

By assumption V is cyclic, hence $VO(G)/O(G) \leqq Z(\bar{H})$, where $\bar{H} = H/O(G)$, since $H/VO(G)$ is simple. But $V \trianglelefteq R_1$, thus $V \leqq Z(R_1)$ and we are done.

LEMMA 3.8. *Suppose* $\operatorname{rank}(R) > 1$. *Then R_1 is quaternion and R is semi-dihedral.*

Proof. If R_1 is not quaternion, then $H/O(G)$ contains two conjugacy classes of involutions and since it is non-solvable, it contains by transfer a normal subgroup of index 2, contrary to the known structure of R_1 and $H/O(G)$. It follows immediately from the classification of Gorenstein–Walter [6], that R is not dihedral, again because of the known structure of $G_1/O(G)$.

Let $i \in R \setminus R_1$ be an involution. Then $iO(G)$ acts on $H/O(G) = \mathrm{SL}(2,q)$, and the semi-direct product of $H/O(G)$ and $\langle iO(G)\rangle/O(G)$ is $\mathrm{SL}^*(2,q)$ with notation of [16].

To finish the proof of Theorem 3.6 we only have to show:

Lemma 3.9. $\mathrm{SL}^*(2,q)$, $q>3$ *does not satisfy condition* (I) *nor condition* (C).

Proof. Let $L = \mathrm{SL}^*(2,q)$, $R \in \mathrm{Syl}_2(L)$.

1. Let $q \equiv 3 \pmod 4$. Then $|C_L(i)| = 2(q-1)$ by [16], and

$$\mathrm{SL}^*(2,q) \leqq \mathrm{GL}(2,q)$$

of odd index. By [3], $N(R) = R$ in $\mathrm{GL}(2,q)$. So $C_L(i)$ does not normalize R implying that $\langle i,j\rangle$ is contained in a Sylow 2-intersection.

2. $q \equiv 1 \pmod 4$. Then $|C_L(i)| = 2(q+1)$ (by [16]). Let

$$R_1 \in \mathrm{Syl}_2(\mathrm{SL}(2,q)), \quad R_1 \leqq R .$$

It is well-known that R_1 is its own normalizer in $\mathrm{SL}(2,q)$ unless $q \equiv 3$ or $5 \pmod 8$. Moreover, in the last case, 3 is the only divisor of odd order occuring in the order of the normalizer. Hence we have reached a contradiction unless $2(q+1) = 3 \cdot 2^n$. But then, since $q \equiv 1 \pmod 4$, $n=2$ and $q=5$, a case considered in Lemma 1.8.

This completes the proof, since we have shown that $G/O(G)$ contains $\mathrm{SL}^*(2,q)$ if $\mathrm{rank}(R) > 1$, and a subgroup of $G/O(G)$ satisfies (I) by Lemma 1.2. By Theorem A, R is generalized quaternion.

Theorem 3.10. *Suppose H/G_2 is not simple. Then*

$$H/O(G) = \mathrm{SL}(2,5) \curlyvee \mathrm{SL}(2,5) .$$

Moreover, $G_2 = O_2(G) \times O(G)$ and G/H is odd or the direct product of a group of odd order and a group of order 2 permuting the two copies of $\mathrm{SL}(2,5)$.

Proof. We have seen that $H/VO(G) = \hat{S}_1 \times \hat{S}_2$, $\hat{S}_i$ simple. Since a Sylow 2-subgroup of S_i is contained in a Sylow 2-intersection, it has to be quaternion. Again we know that $V=\langle j\rangle$, j an involution in the center. Let $R_i \leqq T_i$, T_i a maximal Sylow 2-intersection in H and $R_i \in \mathrm{Syl}_2(S_i)$. Then $|T_i/R_i| \leqq 2$, so the only possibility for $S_i/VO(G)$ is $\mathrm{PSL}(2,q)$, $q \equiv 3$ or $5 \pmod 8$, and $R_i \cong Q_8$. If $q>5$ then since S_i satisfies (C) we may apply Theorem A and get that $V=1$, a contradiction. So $q=5$.

Suppose $|G/H|$ is even. Clearly it is 2-closed, since G satisfies (I). Let tH be a 2-element of G/H. Then $tO(G)/O(G)$ acts on $H/O(G)$. If it normalizes the copies of $\mathrm{SL}(2,5)$, then $H/O(G)$ contains a copy of $\mathrm{SL}^*(2,5)$ and again we have a contradiction by Lemma 1.7. Therefore, $tO(G)/O(G)$ permutes the copies. Since t was arbitrary, tH has to be the only 2-element of G/H. Now the structure of G/H follows from Burnsides Theorem. The structure of $G_2 = VO(G)$ follows from Theorem A.

By this the classification is completed.

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