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MARTIN'S AXIOM APPLIED TO EXISTENTIALLY CLOSED GROUPS

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0. Introduction.

In [7], we proved that every countable existentially closed group has an $L_{\infty,\omega}$-elementary extension of cardinality $\aleph_1$. In particular, there are generic groups of cardinality $\aleph_1$. (In this paper, generic structures will be structures generic for finite forcing. We shall also consider filters generic for a certain partially ordered set $\mathscr{C}$ different from the customary sets of finite conditions [1]. We do not think any confusion can arise.) We raised the question of whether there are generic groups of all infinite cardinalities. We are optimistic that this question has an affirmative answer in ZFC, but so far a suitable combinatorial method has eluded us.

At the start of our work on this question we had the idea of bringing Martin's Axiom [10] into use, but we didn't get far. Recently John Derrick showed us some very interesting notes [17] on forcing by J. Stern, and by using an idea from these notes we have proved the theorem below, which is the main result of this paper.

THEOREM. *Assume Martin's Axiom. Let G be a countable existentially closed group, and $\varkappa$ an infinite cardinal $< 2^{\aleph_0}$. Then G has an $L_{\infty,\omega}$-elementary extension H of cardinality $\varkappa$.*

By [15], H is existentially closed. From [8, Lemma 3] and [10, Theorem 2] it follows that it is consistent with ZFC that there are generic groups of cardinalities $\aleph_2$, $\aleph_\omega$, $\aleph_{\omega_1}$, etc.

We are indebted to Derrick for putting Stern's notes our way, and to Stern for the many valuable ideas in the notes.

1. Preliminaries.

Since we sincerely hope that the present paper is not the last word on the subject, we have suppressed routine details. We refer to [7] for all required information about existentially closed groups and finite forcing in group theory.

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1.1. Let L be the usual logic for group theory, with the extralogical symbols $\cdot$, $^{-1}$, e. $L_{\infty,\omega}$ is the standard infinitary extension of L. L_Q is the extension of L with the quantifier Q whose interpretation is "there are uncountably many". See [4].

1.2. For each infinite cardinal $\varkappa$, $MA_\varkappa$ is the following statement:

If $\mathscr{P}$ is a partial ordering satisfying the countable chain condition, and $\mathscr{I}$ is a set of dense open subsets of $\mathscr{P}$ of cardinality $\leqq \varkappa$, then there is an $\mathscr{I}$-generic filter on $\mathscr{P}$.

For definitions and background, see [10].

Martin's Axiom, is the statement that $MA_\varkappa$ holds for each $\varkappa < 2^{\aleph_0}$.

2. Existentially closed groups.

Recall that a group G is existentially closed if for all groups H such that $G \subseteqq H$ we have $G \prec_1 H$. See [7,15].

From [15] we know that the class of existentially closed groups is axiomatizable by a sentence of $L_{\omega_1,\omega}$. Further, generic groups are existentially closed, and the class of generic groups is axiomatizable by a sentence of $L_{\omega_1,\omega}$ (see [8]).

The following lemma is implicit in [7], and will be useful here. We omit the easy proof.

LEMMA 1. *Suppose G is an existentially closed group, and H is a group containing G, such that the following condition is satisfied*:

For each $x_1, \ldots, x_n$ in H there is a t in H and $y_1, \ldots, y_n$ in G such that $t^{-1}x_i t = y_i$ for $1 \leqq i \leqq n$.

Then $G \prec_{\infty,\omega} H$.

(Note that in Lemma 1, H will be existentially closed, and will be generic if G is generic.)

In [7] we proved that every countable existentially closed group G has an $L_{\infty,\omega}$-elementary extension H of cardinality $\aleph_1$. In fact, H can be chosen to have an additional property which will be useful later.

DEFINITION. A group H has the large centralizer property if for each finitely generated subgroup Γ of H the centralizer of Γ has the same cardinality as H.

It is very easily seen, by the methods of [7] or [13], that every countable existentially closed group has the large centralizer property.

LEMMA 2. *Every countable existentially closed group G has an $L_{\infty,\omega}$-elementary extension H, of cardinality $\aleph_1$, such that H has the large centralizer property.*

PROOF. In [7] we constructed an $L_{\infty,\omega}$-elementary chain of countable groups

$$G = G_0 \subset G_1 \ldots \subset G_\lambda \subset G_{\lambda+1} \ldots, \quad \lambda < \omega_1 ,$$

such that $G_\lambda \times \mathbf{Z}_2 \subseteqq G_{\lambda+1}$ for each λ. H is the union of this chain. At each stage $\mu > \lambda$, the centralizer of each finitely generated subgroup of G gets increased, because $G_\mu \times \mathbf{Z}_2 \subseteqq G_{\mu+1}$. It follows that H has the large centralizer property.

PROBLEM. The following may well be trivial, but we haven't found the answer.

Does every existentially closed group have the large centralizer property?

3. The forcing construction.

The construction used here is a modification of one used in [16]. See also [6]. We develop only as much of the theory as is needed for our main theorem.

3.1. We fix, for the remainder of this section, a countable existentially closed group G. Fix also (as allowed by Lemma 2) an $L_{\infty,\omega}$-elementary extension H of G, such that H has cardinality $\aleph_1$ and has the large centralizer property. Finally, fix an infinite cardinal $\varkappa$.

Adjoin to L individual constants $\bar{g}$ for each g in G. This gives a logic L_1. L_Q and $L_{\infty,\omega}$ extend naturally to $(L_1)_Q$ and $(L_1)_{\infty,\omega}$.

H has a natural enrichment to an L_1-structure, by making the elements g of G correspond to the constants $\bar{g}$. It is convenient to drop the distinction between H and this enrichment.

Finally, adjoin to L_1 individual constants ("forcing constants") c_λ $(\lambda < \varkappa)$ to get a logic L_2. $(L_1)_Q$ and $(L_1)_{\infty,\omega}$ extend naturally to $(L_2)_Q$ and $(L_2)_{\infty,\omega}$.

3.2. *Conditions.* We define a *condition* to be a finite set

$$p(c_{i_0}, \ldots c_{i_j}, \bar{g}_0, \ldots \bar{g}_n)$$

of basic sentences of L_2 such that $i_0 < i_1 \ldots < i_j$ and

$$H \models (Qv_0) \ldots (Qv_j) p(v_0, \ldots v_j, \bar{g}_0, \ldots \bar{g}_n)$$

Let $\mathscr{C}$ be the set of conditions. $\mathscr{C}$ is partially ordered by inclusion.

LEMMA 3. *$\langle \mathscr{C}, \subset \rangle$ has the countable chain condition.*

PROOF. Suppose not. Then an easy counting argument gives the existence of a finite set $p(v_0, \ldots v_i, \bar{g}_0, \ldots \bar{g}_n)$ of basic sentences of L_1, and a map $f: \omega_1 \times (j+1) \to \omega_1$, such that, for $\alpha < \omega_1$,

i) $x \mapsto f(\alpha, x)$ is an increasing map from $j+1$ into ω_1;
ii) $p(c_{f(\alpha,0)}, \ldots, c_{f(\alpha,j)}, \bar{g}_0, \ldots, \bar{g}_n)$ is a condition p_α;
iii) the conditions p_α are pairwise incompatible.

By a standard combinatorial lemma [3, Lemma 91], there is an uncountable subset I of ω_1, and a finite subset S of ω_1, such that if α and β are distinct elements of I then

$$\{f(\alpha, x) : x \in j+1\} \cap \{f(\beta, x) : x \in j+1\} = S .$$

Moreover, there must be an uncountable subset I_1 of I, and some $k \in j+1$ such that, for $\alpha \in I_1$,

$$S = \{f(\alpha, x) : x \in k] .$$

For if not, the map

$$\alpha \mapsto \text{the least element of } \{f(\alpha, x) : x \in j+1\} \text{ not in } S \text{ but bounded above by an element of } S$$

would be defined and one to one on a cocountable subset of I, whereas it is clear that the range of the map is bounded by the greatest element of S. This proves our claim about S and I_1.

Now select α, β in I_1 with $\alpha \neq \beta$. We will show that $p_\alpha \cup p_\beta$ is a condition. This will contradict the assumed incompatibility of p_α and p_β, and prove the lemma.

$$p_\alpha = p(c_{f(\alpha,0)}, \ldots, c_{f(\alpha,k)}, \ldots c_{f(\alpha,j)}, \bar{g}_0, \ldots \bar{g}_n)$$

and

$$p_\alpha = p(c_{f(\alpha,0)}, \ldots, c_{f(\beta,k)}, \ldots c_{f(\beta,j)}, \bar{g}_0, \ldots \bar{g}_n) ,$$

so that p_α and p_β share their first k forcing constants, and no others.

Since p_α is a condition, we have

(1) $$H \models (Qv_0) \ldots (Qv_k) \ldots (Qv_j) p(v_0, \ldots v_j, \bar{g}_0, \ldots \bar{g}_n) .$$

The proof that $p_\alpha \cup p_\beta$ is a condition breaks into various subcases according to the ordering relations between the $f(\alpha,k),\ldots,f(\alpha,j)$ and $f(\alpha,k),\ldots,$ $f(\beta,j)$. The argument is essentially the same in all cases. We consider just the simplest case, namely when $f(\alpha,j) < f(\beta,k)$. From (1) we easily get

$$H \models (Qv_0)\ldots(Qv_k)\ldots(Qv_j)(Qv_{j+1})\ldots(Qv_{j+1+j-k}) \\ [p(v_0,\ldots v_k,\ldots v_j,\bar{g}_0,\ldots\bar{g}_n) \wedge \\ \wedge\; p(v_0,\ldots v_{j+1},\ldots v_{j+1+j-k},\bar{g}_0,\ldots\bar{g}_n)]\,,$$

whence $p_\alpha \cup p_\beta$ is a condition.

This proves the lemma.

3.3. *Forcing.* We now define the forcing relation, $p \Vdash \Phi$, between conditions p and sentences Φ of L_2. The definition follows the format of [1]. Once we have defined the notion for atomic Φ, the rest is automatic, following the inductive clauses of [1].

For conditions p, and atomic Φ, we define $p \Vdash \Phi$ to mean simply that $\Phi \in p$.

Note that this requirement is precisely the same as for the finite forcing of [1]. Where the present notion of forcing differs from that of [1] is that our set $\mathscr{C}$ of conditions is a proper subset of the set of conditions in [1], so that we get a different notion of forcing for more complex sentences involving $\neg$.

LEMMA 4. *Suppose $\alpha,\beta<\varkappa$, and $\alpha \neq \beta$. Let p be a condition. Then not $p \Vdash c_\alpha = c_\beta$.*

PROOF. Let $p = p(c_{i_0},\ldots c_{i_j},\bar{g}_0,\ldots\bar{g}_n)$, where $i_0 < i_1 < \ldots < i_j$. Suppose $p \Vdash c_\alpha = c_\beta$. Then the sentence $c_\alpha = c_\beta$ is a member of p. Since

$$H \models (Qv_0)\ldots(Qv_j)p(v_0,\ldots v_j,\bar{g}_0,\ldots\bar{g}_n)\,,$$

it follows (using the symmetry of $=$) that

$$H \models (Qv_0)(Qv_1)(v_0 = v_1)\,,$$

which is clearly absurd.

3.5. *Dense subsets.*

LEMMA 5. *For each sentence Φ, the set*

$$F_\Phi = \{p \in \mathscr{C} : p \Vdash \Phi \text{ or } p \Vdash \neg\Phi\}$$

is dense in $\mathscr{C}$.

PROOF. Trivial.

Now we come to the specifically group-theoretic aspect of the construction. First we introduce some notation. For each $n < \omega$, let

$$I_n(v_0, \ldots v_{n-1}, v_n, \ldots v_{2n-1})$$

be the formula

$$(\exists v_{2n})[\textstyle\bigwedge_{i<n} v_{2n}^{-1} v_i v_{2n} = v_{n+i}] .$$

LEMMA 6. *Let σ be the ordered $(n+m)$-tuple*

$$\langle c_{i_0}, \ldots c_{i_{n-1}}, \bar{g}_0, \ldots \bar{g}_{m-1} \rangle,$$

where $i_0 < i_1 < \ldots < i_{n-1}$ and each $g_j \in G$. Let J_σ be the set

$$\{p \in \mathscr{C} : (\exists \gamma_0, \ldots \gamma_{m+n-1} \in G) \\ [p \Vdash I_{n+m}(c_{i_0}, \ldots c_{i_{n-1}}, \bar{g}_0, \ldots \bar{g}_{m-1}, \bar{\gamma}_0, \ldots \bar{\gamma}_{m+n-1})]\} .$$

Then J_σ is dense in $\mathscr{C}$.

PROOF. Let q be a condition, and let

$$c_{j_0}, c_{j_1}, \ldots c_{j_k}, \quad \text{where } j_0 < j_1 < \ldots < j_k ,$$

be a list of the forcing constants that occur either in q or in σ. We write

$$q = q(c_{j_0}, c_{j_1}, \ldots c_{j_k})$$

even though some of the c_j may not occur in q. However, it is easily seen, since q is a condition, and H is uncountable, that

$$(1) \qquad H \models (Qv_0)(Qv_1) \ldots (Qv_k) q(v_0, v_1, \ldots v_k) .$$

Fix $h_0, \ldots h_k$ in H. Since G is existentially closed, and $G \prec_{\infty,\omega} H$, it follows by [7, Lemma 1 and Theorem 1] that there are $\tau_0, \ldots \tau_{k+m}$ in G, and $t \in H$, such that

$$H \models t^{-1} h_0 t = \tau_0 \wedge \ldots \wedge t^{-1} h_k t = \tau_k \wedge \\ \wedge t^{-1} g_0 t = \tau_{k+1} \wedge \ldots \wedge t^{-1} g_{m-1} t = \tau_{k+m} .$$

Moreover, given $\tau_0, \ldots \tau_{k+m}$ there are $\aleph_1$ suitable choices for t. For, select one such t. Let u belong to the centralizer of the group generated by $\tau_0, \ldots \tau_{k+m}$. Then tu is also a suitable choice. But there are $\aleph_1$ choices for u, since H has the large centralizer property.

We have proved:

$$(2) \qquad H \models (\forall v_0, \ldots v_k) \textstyle\bigvee_{\vec{\tau}} (Qv_{k+1}) [\bigwedge_{i \leq k} v_{k+1}^{-1} v_i v_{k+1} = \bar{\tau}_i \wedge \\ \wedge \bigwedge_{j \leq m-1} v_{k+1}^{-1} \bar{g}_j v_{k+1} = \bar{\tau}_{k+1+j}] ,$$

where $\bigvee_{\vec{\tau}}$ indicates that the disjunction is taken over all ordered $(k+m-1)$-tuples of elements of G.

From (1) and (2) we get

$$(3)\qquad H \models (Qv_0)\dots(Qv_k) \bigvee_{\vec{\tau}} (q(v_0,\dots v_k) \wedge \wedge (Qv_{k+1})[\bigwedge_{i\leq k} v_{k+1}^{-1}v_iv_{k+1} = \bar{\tau}_i \wedge \wedge \bigwedge_{j\leq m-1} v_{k+1}^{-1}\bar{g}_iv_{k+1} = \bar{\tau}_{k+1+j}]) .$$

From (3) and [4, Page 69, Axiom ω] we get

$$(4)\quad H \models \bigvee_{\vec{\tau}} (Qv_0)\dots(Qv_k)(Qv_{k+1})[q(v_0,\dots v_k) \wedge \bigwedge_{i\leq k} v_{k+1}^{-1}v_iv_{k+1} = \bar{\tau}_i \wedge \wedge \bigwedge_{j\leq m-1} v_{k+1}^{-1}\bar{g}_iv_{k+1} = \bar{\tau}_{k+1+j}] .$$

From (4) it follows that there exist $\tau_0,\dots \tau_{k+m}$ (in G), which we now fix, such that

$$(5)\quad H \models (Qv_0)\dots(Qv_k)(Qv_{k+1})[q(v_0,\dots v_k) \wedge \bigwedge_{i\leq k} v_{k+1}^{-1}v_iv_{k+1} = \bar{\tau}_i \wedge \wedge \bigwedge_{j\leq m-1} v_{k+1}^{-1}\bar{g}_iv_{k+1} = \bar{\tau}_{k+1+j}] .$$

Fix $\lambda > j_k$, and let p be

$$q \cup \{c_\lambda^{-1}c_{j_i}c_\lambda = \bar{\tau}_i : i < k\} \cup \{c_\lambda^{-1}\bar{g}_ic_\lambda = \bar{\tau}_{k+1+i} : i < m\} .$$

By (5), $p \in \mathscr{C}$. Clearly $q \subseteq p$. Clearly also

$$p \Vdash I_{k+m}(c_{j_0},\dots c_{j_k}, \bar{g}_0,\dots \bar{g}_{m-1}, \bar{\tau}_0,\dots \bar{\tau}_k, \bar{\tau}_{k+1},\dots \bar{\tau}_{k+m}) .$$

From this, by discarding some of the τ's, we get elements $\gamma_0,\dots \gamma_{m+n-1}$ of G such that

$$p \Vdash I_{n+m}(c_{i_0},\dots c_{i_{n-1}}, \bar{g}_0,\dots \bar{g}_{m-1}, \bar{\gamma}_0,\dots \bar{\gamma}_{m+n-1}) .$$

This proves that $p \in J_\sigma$. We have proved that J_σ is dense.

3.6. *The Main Theorem.* We can now prove

THEOREM 1. *Assume* $MA_\varkappa$. *Let G be a countable existentially closed group. Then G has an $L_{\infty,\omega}$-elementary extension H of cardinality $\varkappa$.*

PROOF. Let $\mathscr{I}$ be

$$\{F_\Phi : \Phi \text{ an } L_2 \text{ sentence}\}$$
$$\cup \{J_\sigma : \sigma \text{ an ordered } (n+m)\text{-tuple } \langle c_{i_0},\dots c_{i_{n-1}}, \bar{g}_0,\dots \bar{g}_{m-1}\rangle$$
$$\text{for some } n,m < \omega, \text{ where } i_0 < i_1 < \dots < i_{n-1} \text{ and each } g_i \in G\} .$$

$\mathscr{I}$ has cardinality $\leq \varkappa$, and by Lemmas 5 and 6 is a set of dense subsets. Now $\mathscr{C}$ has cardinality $\varkappa$, and satisfies the countable chain condition by Lemma 3. Therefore, by $MA_\varkappa$, there is an $\mathscr{I}$-generic filter Δ. We will construct a model from Δ in the obvious way.

Let E be the set of L_2 terms. We may construct $\cdot$ and $^{-1}$ as operations on E. We define $\equiv$ on E by:

$$t_1 \equiv t_2 \Leftrightarrow (\exists p \in \Delta)(p \Vdash t_1 = t_2) .$$

It is easily verified that $\equiv$ is an equivalence relation on E, and is in fact a congruence with respect to $\cdot$ and $^{-1}$. Let H be the quotient structure of $\langle E, \cdot, ^{-1}\rangle$ with respect to $\equiv$. For $t \in E$, let $\tilde{t}$ be the equivalence class of t with respect to $\equiv$.

We claim:

(1) H is a group;
(2) The map $g \mapsto \tilde{\bar{g}}$ is a monomorphism of G into H;
(3) H has cardinality $\varkappa$;
(4) For each $x_0, \ldots x_{n-1}$ in H there are $\gamma_0, \ldots \gamma_{n-1}$ in G, and $t \in H$, such that

$$t^{-1}x_i t = \tilde{\bar{\gamma}}_i \quad \text{for } i < n .$$

(1) and (2) are trivial.

(3) will be proved if we can show that $\tilde{c}_\alpha \neq \tilde{c}_\beta$ if $\alpha \neq \beta$. So, suppose $\alpha \neq \beta$, and let Φ be $c_\alpha = c_\beta$. Then Δ meets F_Φ, since Δ is $\mathscr{I}$-generic. But, by Lemma 4, no condition forces $c_\alpha = c_\beta$. Hence $\tilde{c}_\alpha \neq \tilde{c}_\beta$.

To prove (4), suppose $x_0, \ldots x_{n-1} \in H$. Clearly there are $c_{i_0}, \ldots c_{i_{k-1}}$, with $i_0 < \ldots < i_{k-1}$, and elements $g_0, \ldots g_{l-1}$ in G, such that $x_0, \ldots x_{n-1}$ are in the group generated by

$$\tilde{c}_{i_0}, \ldots \tilde{c}_{i_{k-1}}, \tilde{\bar{g}}_0, \ldots \tilde{\bar{g}}_{l-1} .$$

Let σ be $\langle c_{i_0}, \ldots c_{i_{k-1}}, \bar{g}_0, \ldots \bar{g}_{l-1}\rangle$. Then Δ meets J_σ since Δ is $\mathscr{I}$-generic. Select $p \in \Delta \cap J_\sigma$. Then there are $\tau_0, \ldots \tau_{k+l-1}$ in G, and a term δ of L_2 such that

$$\begin{aligned} p \Vdash \delta^{-1}c_{i_0}\delta &= \bar{\tau}_0 \wedge \ldots \wedge \delta^{-1}c_{i_{k-1}}\delta = \bar{\tau}_{k-1} \wedge \\ \wedge\ \delta^{-1}\bar{g}_0\delta &= \bar{\tau}_k \wedge \ldots \wedge \delta^{-1}\bar{g}_{l-1}\delta = \bar{\tau}_{k+l-1} .\end{aligned}$$

Thus

$$H \models (\textstyle\bigwedge_{r<k} \tilde{\delta}^{-1}\tilde{c}_{i_r}\tilde{\delta} = \tilde{\bar{\tau}}_r) \wedge (\textstyle\bigwedge_{s<l} \tilde{\delta}^{-1}\tilde{\bar{g}}_s\tilde{\delta} = \tilde{\bar{\tau}}_{k+s}) .$$

It follows easily that there are $\gamma_0, \ldots \gamma_{n-1}$ in G such that $\tilde{\delta}^{-1}x_i\delta = \tilde{\bar{\gamma}}_i$, for $i < n$. This proves (4).

(1), (2) and (4), along with Lemma 1, prove that $G \prec_{\infty,\omega} H$. With (3), this proves the theorem.

Corollary 1. *Assume Martin's Axiom. Let G be a countable existentially closed group, and $\varkappa$ an infinite cardinal $< 2^{\aleph_0}$. Then G has an $L_{\infty,\omega}$-elementary extension H of cardinality $\varkappa$.*

PROOF. Immediate.

COROLLARY 2. *It is consistent with* ZFC *that there are generic groups of cardinalities* $\aleph_2$, $\aleph_\omega$, $\aleph_{\omega_1}$.

PROOF. If G is generic, H is generic, by [8]. Now apply [10, Theorem 2].

REMARK. By choosing a slightly more complex $\mathscr{I}$, one can arrange for H to have the large centralizer property.

4. Concluding Remarks.

4.1. We shall sketch an idea which may enable some combinatorial group theorist to settle the main problem described in the first paragraph of this paper.

The fact that the class of generic groups is axiomatizable by a sentence of $L_{\omega_1,\omega}$ enables one to bring the method of indiscernibles to bear on the problem. The main idea can be found in [5, Theorem 21], but we add some group-theoretic ingredients.

Suppose H is a generic group of cardinality $\beth_{\omega_1}$. By [5, Page 97, Lemma E] the $L_{\omega_1,\omega}$-theory of H has models of all cardinalities realizing only countably many types. It follows easily from [7, Theorem 1] that there is a countable generic group G such that $G \prec_{\infty,\omega} H$.

For $1 \leqq n < \omega$ we define n-ary functions $t_n(x_0, \ldots x_{n-1})$ so that

$$H \models (\forall v_0, \ldots v_{n-1}) \bigvee\!\!\!\bigvee_{\langle g_0,\ldots g_{n-1}\rangle \in G^n} \bigwedge\!\!\!\bigwedge_{i<n} t_n^{-1}(v_0, \ldots v_{n-1}) v_i t_n(v_0, \ldots v_{n-1}) = g_i .$$

This is possible, by [7, Theorem 1] and [7, Lemma 1].

We adjoin to L_1 (that is L with constants for G) n-ary function symbols τ_n. This gives a logic L_1^τ.

Let B_n be the $(L_1^\tau)_{\omega_1,\omega}$ sentence

$$(\forall v_0, \ldots v_{n-1}) \bigvee\!\!\!\bigvee_{\langle g_0,\ldots g_{n-1}\rangle \in G^n} \bigwedge\!\!\!\bigwedge_{i<n} t_n^{-1}(v_0, \ldots v_{n-1}) v_i t_n(v_0, \ldots v_{n-1}) = \bar{g}_i .$$

Let T be the $(L_1^\tau)_{\omega_1,\omega}$ theory whose axioms are the axioms of group-theory, the diagram of G, and the sentences B_n, for $1 \leqq n < \omega$. T is finitely axiomatizable, and $H \models T$. Thus T has a model of cardinality $\beth_{\omega_1}$, so by [5, Theorem 21] T has a countable model with an infinite set of indiscernibles. That is, there is a countable L_1^τ-structure M and a sequence $\langle x_n : n \in \omega \rangle$ of elements of M such that if

$$i_0 < i_1 < \ldots < i_{m-1} \quad \text{and} \quad j_0 < j_1 < \ldots < j_{m-1}$$

are increasing sequences of integers, then

$$M \models \Phi(x_{i_0}, \ldots x_{i_{m-1}}) \leftrightarrow \Phi(x_{j_0}, \ldots x_{j_{m-1}})$$

for all formulas Φ of L_1^{τ}. In particular this is true for all basic formulas Φ.

Conversely, suppose there is such an M, with a sequence $\langle x_n : n \in \omega \rangle$ of elements indiscernible with respect to all basic Φ. Then by an easy variant of the Stretching Theorem [5, Page 71], T has models of all infinite cardinalities. Let Γ be a model of T, and H the reduct to L of Γ. Clearly H is a group containing G, and, since $\Gamma \models B_n$ for each n, Lemma 1 implies that $G \prec_{\infty,\omega} H$, so H is generic.

We have proved:

A necessary and sufficient condition for the existence of generic groups of all cardinalities is the existence of a countable generic group G with a countable extension M endowed with functions t_n so that the sentences B_n are satisfied and such that the natural enrichment of M to an L_1^{τ}-structure has an infinite set of indiscernibles with respect to basic formulas of L_1^{τ}.

The problem for the group-theorist is to construct such indiscernibles.

4.2. Nothing general appears to be know about uncountable generic structures, except Shelah's theorem [14] which shows that the Hanf number situation [8] is essentially the same as for omitting types in first-order logic. This result, together with [12, Theorem 2.2. (2)], enables one to give examples of countable theories with generic structures of cardinality $\aleph_1$ but none of cardinality $\aleph_2$. So in this work we did need something peculiar to group theory, and it is not quite clear what this factor is.

4.3. Is has recently become clear that much of the work on existentially closed groups extends to existentially closed division rings. See [2], [9], [17]. Wheeler [17] proved the analogue of [7, Theorem 14] for division rings, and in fact Lemma 2 is true for division rings of prescribed characteristic. Lemma 1 holds also, and it follows readily that the results of this paper hold for division rings.

4.4. The very interesting paper [11] suggests to us that it would be profitable to try to prove in ZFC that if a countable existentially closed group G has an $L_{\infty,\omega}$-elementary extension of cardinality $(2^{\aleph_0})^+$ then it has $L_{\infty,\omega}$-elementary extension of all infinite cardinalities.

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