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ON THE CANONICAL COMMUTATION RELATIONS

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1. Introduction.

In this paper we study representations of the Heisenberg form of the canonical commutation relations. The main result (Theorem 3) is a generalization of Rellich's theorem [12] to systems with an arbitrary number of degrees of freedom. The Rellich condition means that the "number operator" exists as a self adjoint operator; so Theorem 3 is formally similar to a result on Weyl systems which has been discussed by several authors (see [1] for references).

The first part of the paper contains a discussion of systems with finitely many degrees of freedom. In this case Theorem 3 is a special case of a general result due to Nelson [8] and a similar result has been obtained by Dixmier [4]. Using some consequences of Nelson's theory on analytic domination we show that Theorem 3 can be derived from Dixmier's theorem, and we give an example which shows that Dixmier's theorem is stronger than Theorem 3. This section also contains a counter-example due to Fuglede, and we prove a result on commutativity of self adjoint operators which is of some independent interest (Lemma 2).

As a result of our discussion we get a simple proof of the fact that the field operators of the Fock–Cook representation satisfy the Weyl relations. This result was proved by Segal [13] by means of functional integration.

The present paper is somewhat different from an earlier version carrying the same title [10].

2. Finitely many degrees of freedom.

In this section we present some examples and we give a discussion of the relations between the results of Dixmier [4], Nelson [8], and Rellich [12]. For terminology and background material we refer to these papers (see also [11]). As the starting point of our discussion we recall the following generalization of Rellich's theorem.

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THEOREM 1. *Let $P_1, P_2, \ldots, P_d$, $Q_1, \ldots, Q_d$ be closed symmetric operators in a Hilbert space K. Let D be a dense linear subspace of K, contained in the domain of the operators P_jP_k, Q_jQ_k, Q_jP_k, and P_jQ_k for $j,k=1,2,\ldots,d$ and such that*

i) $[P_j,P_k]x=[Q_j,Q_k]x=0$ *and* $[P_j,Q_k]x=-i\delta_{jk}x$ *for all* $x \in D$, $j,k=1,2,\ldots,d$.

ii) *The operator* $\sum_{k=1}^d (P_k^2+Q_k^2)|D$ *is essentially self adjoint.*

Then the operators $P_1,\ldots,P_d$, $Q_1,\ldots,Q_d$ are all self adjoint and there exists a family $\{K_\gamma\}_{\gamma\in\Gamma}$ of closed mutually orthogonal subspaces of K with the following properties:

1) $K=\sum\oplus_{\gamma\in\Gamma}K_\gamma$ *and each K_γ reduces the operators $P_1,\ldots,P_d$, $Q_1,\ldots,Q_d$.*

2) *The system induced by $P_1,\ldots,P_d$, $Q_1,\ldots,Q_d$ in K_γ, $\gamma\in\Gamma$ is unitarily equivalent to the conventional Schrödinger representation for d degrees of freedom.*

REMARKS.

1) In the stated form Theorem 1 is due to Nelson (and von Neumann). In fact, by Corollary 9.1 of [8] the operators $P_1,\ldots,P_d$, $Q_1,\ldots,Q_d$ are all self adjoint and the corresponding unitary groups satisfy the Weyl relations. Then von Neumann's theorem [9] gives the desired result.

2) Theorem 1 can be derived from Rellich's original theorem in the following way: As in the proof of Theorem 3 (below) we let N denote the closure of the operator $\frac{1}{2}\sum_{k=1}^d (P_k^2+Q_k^2-I)|D$ and we let $D_\infty = \bigcap_{k=1}^\infty D_{N^k}$. (Here D_{N^k} denotes the domain of N^k). Then N is self adjoint and by the spectral theorem $N|D_\infty$ is decomposable in the sense of Rellich [12]. By the proof of Theorem 3 (or by the proof of Corollary 9.1 in [8]) the operators $P_1,\ldots,P_d$, $Q_1,\ldots,Q_d$ are defined on D_∞ and they all leave D_∞ invariant. Since the commutation relations also hold on D_∞ the assertion of Theorem 1 follows from Rellich's theorem [12].

3) For $d=1$ Theorem 1 was also proved by Dixmier [4]. For $d>1$ Dixmier proved the assertion of Theorem 1 when hypothesis ii) is replaced by the following condition

iii) The restrictions to D of the operators

$$P_j^2+Q_j^2,\ P_j^2+P_k^2,\ Q_j^2+Q_k^2, \quad \text{and} \quad P_j^2+Q_k^2$$

are essentially self adjoint $j,k=1,\ldots,d$, $j\neq k$. Actually Dixmier works with an invariant domain but it is easily seen that this restriction is unnecessary. The following lemma shows that Theorem 1 can be derived from Dixmier's theorem. This gives an affirmative answer to a question left open by Dixmier.

LEMMA 1. *Suppose the hypotheses of Theorem 1 are satisfied and let S be a symmetric operator of the form (finite sums)*

$$S = \sum_{j,k} (a_{jk}P_jP_k + b_{jk}Q_jQ_k + c_{jk}P_jQ_k) + \sum_k (d_kP_k + e_kQ_k) .$$

Then $S|D$ is essentially self adjoint.

PROOF. Let N and D_∞ be as before. As already remarked the operators P_k, Q_k leave D_∞ invariant. Therefore the set

$$\{iP_1|D_\infty, \ldots, iQ_d|D_\infty, iI|D_\infty\}$$

generates a real Lie algebra of skew symmetric operators having D_∞ as a common invariant domain. By Lemma 6.3 of [8], $|N|D_\infty| + |I|D_\infty|$ analytically dominates $|S|D_\infty|$ (Here we use the notation of [8], so $|N|D_\infty|$ denotes the "absolute value" of the operator $N|D_\infty$), and we have the following inequality

$$(*) \qquad \|Sx\| \leqq \text{const}(\|Nx\| + \|x\|) \quad \text{for all } x \in D_\infty .$$

By the spectral theorem $N|D_\infty$ has a dense set of analytic vectors, so it follows from Corollary 3.1 and Lemma 5.1 of [8] that $S|D_\infty$ is essentially self adjoint. Using $(*)$ it is easily seen that $S|D_\infty \subseteq (S|D)^-$ (where $(\cdot)^-$ denotes the closure). Hence $(S|D_\infty)^- = (S|D)^-$.

The following example shows that Dixmier's theorem is stronger than Theorem 1.

EXAMPLE 1. Let P_1, P_2, P_3, Q_1, Q_2, Q_3 denote the usual Schrödinger operators in $K = L^2(\mathsf{R}^3)$. Let

$$N_k = \tfrac{1}{2}(P_k{}^2 + Q_k{}^2 - I), \quad k = 1,2,3$$

and $N = N_1 + N_2 + N_3$. If $\{h_n\}_{n \geq 0}$ denotes the Hermite functions we have

$$N_k(h_{n_1} \otimes h_{n_2} \otimes h_{n_3}) = n_k(h_{n_1} \otimes h_{n_2} \otimes h_{n_3})$$

for $k = 1,2,3$, so

$$\{h_{n_1} \otimes h_{n_2} \otimes h_{n_3} \mid n_k \geqq 0, \quad k = 1,2,3\}$$

is an orthonormal basis for K, consisting of eigenvectors for N. Each $f \in K$ has a unique representation of the form

$$f = \sum a_{n_1n_2n_3} h_{n_1} \otimes h_{n_2} \otimes h_{n_3} ,$$

and we let

$$D = \{f \in D_N \mid \textstyle\sum_{n_1=0}^\infty a_{n_100} + \sum_{n_2=0}^\infty a_{0n_20} + \sum_{n_3=0}^\infty a_{00n_3} = 0\} .$$

Note that this condition makes sense, since

$$\sum_{n_1=0}^{\infty} |a_{n_1 00}| + \sum_{n_2=0}^{\infty} |a_{0 n_2 0}| + \sum_{n_3=0}^{\infty} |a_{00 n_3}| \leqq \text{const } \|(N+I)f\|$$

for all $f \in D_N$. Then D is a dense linear subspace of K (D is the null-space of a discontinuous linear functional on K), and it is easily seen that the commutation relations hold on D. The operator $N|D$ has deficiency indices $(1,1)$, so the representation can not be identified by means of Theorem 1. On the other hand it is easily seen that $(N_j+N_k)|D$ is essentially self adjoint for $j,k=1,2,3$. Then it follows from Lemma 1 that all the Dixmier operators

$$P_j^2+Q_j^2,\ P_j^2+P_k^2,\ Q_j^2+Q_k^2,\ P_j^2+Q_k^2$$

are essentially self adjoint on D.

4) In [7] Kilpi claimed (as quoted in Putnam's book [11, Theorem 4.11.3]) that the assertions of Theorem 1 (or of Dixmier's theorem) remain valid when hypothesis ii) is replaced by the weaker requirement: iv) $(P_k^2+Q_k^2)|D$ is essentially self adjoint for $k=1,2,\ldots,d$. This statement is false as shown by Fuglede [6] who has kindly communicated the following (unpublished) example to me.

EXAMPLE 2. Take $K=L^2(\mathsf{R}^3)$ and $D=$ the subspace generated by the functions:

$$(x_1,x_2,x_3) \mapsto \prod_{k=1}^{3} x_k^{n_k} \exp(-a_k x_k^2 + c_k x_k)$$

with $n_k \in \mathsf{N}$, $a_k \in \mathsf{R}$, $a_k > 0$, and $c_k \in \mathsf{C}$. For $k=1,2,3$ let

$$p_k = -i\frac{\partial}{\partial x_k}, \quad q_k = x_k,$$
$$r_k = \exp((2\pi)^{\frac12} p_k), \quad s_k = \exp((2\pi)^{\frac12} q_k).$$

Using Fuglede's methods from [5] it can be verified that the operators

$$P_1 = p_1, \quad Q_1 = (q_1+s_3)^-,$$
$$P_2 = -q_2, \quad Q_2 = (p_2+r_3)^-$$

are self adjoint in K, and they have the following properties:

a) D is a dense subspace contained in the domain of the operators P_k, Q_k ($k=1,2$) and invariant under each of them.

b) $[P_j,P_k]x=[Q_j,Q_k]x=0$, $[P_j,Q_k]x=-i\delta_{jk}x$ for all $x \in D$, $j,k=1,2$.

c) The restrictions of $P_j, Q_j, P_j^2+Q_k^2$, and $P_1^2+P_2^2$ to D are essentially self adjoint, $j,k=1,2$.

d) Q_1 and Q_2 do *not* commute.

This shows that Dixmier's condition can not be weakened in the indicated way.

In the proof of Theorem 3 we make use of (a very special case of) the following "first order" criterion for commutativity of self adjoint operators. In view of the examples constructed by Fuglede [5] and Nelson [8, p. 606] this result is of some independent interest.

LEMMA 2. *Let $A_0, A_1, \ldots, A_d$ be closed symmetric operators in a Hilbert space K. Let D be a dense linear subspace of K, contained in the domain of A_jA_k, $j,k=0,1,\ldots,d$ and such that*

i) *$[A_j, A_k]x=0$ for all $x \in D$, $j,k=0,1,\ldots,d$,*
ii) *$A_0|D$ is essentially self adjoint,*
iii) *$D_{A_0} \subseteq D_{A_k}$ for $k=1,2,\ldots,d$.*

Then $A_0, A_1, \ldots, A_d$ are commuting self adjoint operators and they are all essentially self adjoint on D.

PROOF. First we show the following statement:

(*) For all $n \in \mathsf{N}$: $A_k D_{A_0^n} \subseteq D_{A_0^{n-1}}$ and $A_k A_0^{n-1} x = A_0^{n-1} A_k x$ for all $x \in D_{A_0^n}$, $k=1,\ldots,d$.

By iii) (*) holds for $n=1$. Suppose (*) holds for n and let $x \in D_{A_0^{n+1}}$, $y \in D$. By the induction hypothesis and the case $n=1$ we have

$$\begin{aligned}\langle A_0 y, A_0^{n-1} A_k x\rangle &= \langle A_k A_0 y, A_0^{n-1} x\rangle \\ &= \langle A_k y, A_0^n x\rangle = \langle y, A_k A_0^n x\rangle \quad k=1,2,\ldots,d\,.\end{aligned}$$

Since $A_0 = (A_0|D)^*$ this gives the desired result. Thus A_k leaves $D_\infty = \bigcap_{n=1}^\infty D_{A_0^n}$ invariant for $k=0,1,\ldots,d$.

By iii) and the closed graph theorem there exists a constant C such that

$$\|A_k x\| \leqq C(\|A_0 x\| + \|x\|) \quad \text{for all } x \in D_{A_0},\ k=1,\ldots,d\,.$$

It follows that

$$\|(A_0^2 + A_1^2 + \ldots + A_d^2)x\| \leqq \text{const}\,(\|A_0^2 x\| + \|x\|)$$

for all $x \in D_\infty$. Since the commutation relations also hold on D_∞, $|A_0^2|D_\infty| + |I|D_\infty|$ analytically dominates $|(A_0^2 + A_1^2 + \ldots + A_d^2)|D_\infty|$. By Lemma 5.2 of [8], $(A_0^2 + A_1^2 + \ldots + A_d^2)|D_\infty$ is essentially self adjoint. Then by [8, Theorem 5] the $A_0, A_1, \ldots, A_d$ are commuting self adjoint operators, and $A_k|D_\infty$ is essentially self adjoint, $k=0,1,\ldots,d$. Using the graph-norm estimate it is easily seen that $A_k|D_\infty \subseteq (A_k|D)^-$ so this completes the proof.

3. The Fock-Cook representation.

Let H be a complex Hilbert space of arbitrary dimension. Take $K_0 = \mathbb{C}$ and K_n = the symmetric part of $H \otimes H \otimes \ldots \otimes H$ (n times) for $n = 1,2,3,\ldots$. Then $K = \sum \oplus_{n=0}^{\infty} K_n$ is the symmetric tensor algebra over H (see [2] and [13]).

For $z \in H$ we let $A(z)$ and $A^*(z)$ denote the annihilation and creation operators constructed by Cook [2], and we let $R(z)$ denote the closure of the symmetric operator $2^{-\frac{1}{2}}(A^*(z) + A(z))$. The main point of this section is to give a simple proof of the following result (see [2] and [13]).

THEOREM 2. *For each $z \in H$, $R(z)$ is self adjoint and the corresponding unitary operators $W(z) = \exp(iR(z))$ satisfy the Weyl relations*

$$W(z)W(z') = \exp(i \operatorname{Im}\langle z,z'\rangle) W(z')W(z)$$

for all $z,z' \in H$.

PROOF. Let $z,z' \in H$ and let $D = \sum_{n=0}^{\infty} K_n$ be the algebraic sum. Then D is invariant under $R(z)$ and $R(z')$ and they satisfy the following commutation relation [2, p. 232]

$$[R(z), R(z')]x = -i \operatorname{Im}\langle z,z'\rangle x \quad \text{for all } x \in D .$$

By Dixmier's theorem [4] or by Nelson's theorem [8, Theorem 5] it suffices to show that the operator $(R(z)^2 + R(z')^2)|D$ is essentially self adjoint. (Note that Dixmier's argument also applies to the case of commuting operators [4, p. 268].)

Let $\{z_\alpha\}_{\alpha \in I}$ be an orthonormal basis for H and let $P_\alpha = R(z_\alpha)$, $Q_\alpha = R(-iz_\alpha)$ for $\alpha \in I$. Then

$$[P_\alpha, P_\beta]x = [Q_\alpha, Q_\beta]x = 0,$$
$$[P_\alpha, Q_\beta]x = -i\delta_{\alpha\beta}x$$

for all $x \in D$, $\alpha,\beta \in I$. We assume that $\{z_\alpha\}$ is chosen such that z and z' belong to a two-dimensional subspace span $\{z_\alpha, z_\beta\}$. Since the mapping $z \to R(z)|D$ is real linear Lemma 1 shows that it suffices to verify that the operator $(P_\alpha{}^2 + Q_\alpha{}^2 + P_\beta{}^2 + Q_\beta{}^2)|D$ is essentially self adjoint.

Let Δ denote the set of all functions $\alpha \mapsto n_\alpha$ from the index set I into the set of non-negative integers such that $n_\alpha = 0$ except for finitely many α's. Let x_0 denote the vacuum vector $(1,0,0,\ldots) \in K$ and define

$$x_n = \prod_{\alpha \in I} (n_\alpha!)^{-\frac{1}{2}} A^*(z_\alpha)^{n_\alpha} x_0 \quad \text{for } n \in \Delta .$$

Then the vectors x_n, $n \in \Delta$ play the role of the Hermite functions in the

Schrödinger representation and $\{x_n \mid n \in \Delta\}$ is an orthonormal basis for K (see [2, p. 228]). The operators $N_\alpha = A^*(z_\alpha)A(z_\alpha)$ are self adjoint and we have

$$N_\alpha x_n = \tfrac{1}{2}(P_\alpha{}^2 + Q_\alpha{}^2 - I)x_n = n_\alpha x_n$$

for all $\alpha \in I$, $n \in \Delta$. In particular, the operator $(N_\alpha + N_\beta)|D$ is essentially self adjoint.

Remarks.

a) It also follows from the Dixmier–Nelson theorem that the restriction of $R(z)$ to the subspace $D_0 = \operatorname{span}\{x_n \mid n \in \Delta\}$ is essentially self adjoint (but note that $\{x_n\}$ depends on z).

b) Instead of using Lemma 1 one can prove directly that each $x \in D$ is an analytic vector for the operator $R(z)^2 + R(z')^2$ (see [10]).

Definition. A family $\{P_\alpha, Q_\alpha\}_{\alpha \in I}$ obtained by restricting $R(\cdot)$ to an orthonormal basis $\{z_\alpha\}_{\alpha \in I}$ of H is called a restricted Fock–Cook representation over H.

It is well-known that any two restricted Fock–Cook representations over H are unitarily equivalent. In fact, if U is a unitary operator in H there exists a unitary operator $\Gamma(U)$ in K such that (by [2] and [13])

$$\Gamma(U)R(z)\Gamma(U)^* = R(Uz) \quad \text{for all } z \in H .$$

4. The general Rellich theorem.

Theorem 3. *Let $\{P_\alpha, Q_\alpha\}_{\alpha \in I}$ be a family of closed symmetric operators in a Hilbert space K. Let D be a dense linear subspace of K, contained in the domain of the operators $P_\alpha P_\beta$, $Q_\alpha Q_\beta$, $P_\alpha Q_\beta$ and $Q_\alpha P_\beta$ for all $\alpha, \beta \in I$ and such that*

i) $[P_\alpha, P_\beta]x = [Q_\alpha, Q_\beta]x = 0$ and $[P_\alpha, Q_\beta]x = -i\delta_{\alpha\beta}x$ for all $x \in D$, $\alpha, \beta \in I$,

ii) $Nx = \sum_{\alpha \in I} \tfrac{1}{2}(P_\alpha{}^2 + Q_\alpha{}^2 - I)x$ exists for all $x \in D$, and the symmetric operator N (defined by this formula) is essentially self adjoint on D.

Then the operators P_α, Q_α, $\alpha \in I$ are all self adjoint and the family $\{P_\alpha, Q_\alpha\}_{\alpha \in I}$ is unitarily equivalent to a direct sum of restricted Fock-Cook representations over $l^2(I)$.

Proof. Let $\mathscr{F}$ denote the family of all finite subsets of the index set I. $\mathscr{F}$ is partially ordered by inclusion. For $\alpha \in I$ we let

$$N_\alpha = \tfrac{1}{2}(P_\alpha{}^2 + Q_\alpha{}^2 - I)$$

and for $F \in \mathscr{F}$ we let $N_F = \sum_{\alpha \in F} N_\alpha$. Then hypothesis ii) means that the generalized sequence $\{N_F x \mid F \in \mathscr{F}\}$ is convergent for each $x \in D$. (In case I is countable it suffices to assume that the usual partial sums converge.) Since each N_F is symmetric on D it is clear that the limit $N'x = \lim N_F x$, $x \in D$ is a symmetric linear operator N' on D. We let N denote the (self adjoint) closure of N' and we let $D_\infty = \bigcap_{n=1}^\infty D_{N^n}$. As usual we introduce

$$A_\alpha = 2^{-\frac{1}{2}}(P_\alpha - iQ_\alpha), \quad A_\alpha{}^+ = 2^{-\frac{1}{2}}(P_\alpha + iQ_\alpha).$$

Then for $x \in D$ we have

$$\langle N_\alpha x, x\rangle = \langle A_\alpha{}^+ A_\alpha x, x\rangle = \|A_\alpha x\|^2 \geqq 0,$$

and hence

$$\begin{aligned} \|P_\alpha x\|^2 + \|Q_\alpha x\|^2 &= 2\langle N_\alpha x, x\rangle + \langle x, x\rangle \\ &\leqq 2\langle Nx, x\rangle + \langle x, x\rangle \leqq \|Nx\|^2 + 2\|x\|^2 . \end{aligned}$$

If $x \in D_N$ there exists a sequence $\{x_n\} \subseteqq D$ such that $x_n \to x$ and $Nx_n \to Nx$. Replacing x by $(x_n - x_m)$ in the inequality above and using the fact that the operators P_α and Q_α are closed it follows that $x \in D_{P_\alpha} \cap D_{Q_\alpha}$, $P_\alpha x_n \to P_\alpha x$ and $Q_\alpha x_n \to Q_\alpha x$ for all $\alpha \in I$. In particular, P_α and Q_α are defined on D_∞ and we want to show that they leave this subspace invariant. First some preliminary observations.

For $x, y \in D$ it follows from hypothesis i) that

$$\langle N_\beta x, P_\alpha y\rangle = \langle P_\alpha x, N_\beta y\rangle + \langle x, i\delta_{\alpha\beta} Q_\beta y\rangle .$$

Hence also

$$\langle Nx, P_\alpha y\rangle = \langle P_\alpha x, Ny\rangle + \langle x, iQ_\alpha y\rangle,$$

and this equality remains valid for all $x, y \in D_N$. If $y \in D_{N^2}$ we get

$$\langle Nx, P_\alpha y\rangle = \langle x, P_\alpha Ny + iQ_\alpha y\rangle$$

for all $x \in D_N$. Since N is self adjoint this shows that $P_\alpha y \in D_N$ and

$$NP_\alpha y = P_\alpha Ny + iQ_\alpha y .$$

Similarly, Q_α maps D_{N^2} into D_N and

$$NQ_\alpha y = Q_\alpha Ny - iP_\alpha y \quad \text{for } y \in D_{N^2} .$$

If k is a non-negative integer we *define* $(\operatorname{ad} N)^k(P_\alpha)$ and $(\operatorname{ad} N)^k(Q_\alpha)$ as follows:

$$\begin{aligned} (\operatorname{ad} N)^k(P_\alpha) &= P_\alpha && \text{for } k \text{ even} \\ &= iQ_\alpha && \text{for } k \text{ odd} \end{aligned}$$

and

$$\begin{aligned}(\operatorname{ad} N)^k(Q_\alpha) &= Q_\alpha \qquad \text{for } k \text{ even}\\ &= -iP_\alpha \quad \text{for } k \text{ odd}\,.\end{aligned}$$

Then if $S=P_\alpha$ or $S=Q_\alpha$ we have

$$N(\operatorname{ad} N)^k(S)y = (\operatorname{ad} N)^k(S)Ny + (\operatorname{ad} N)^{k+1}(S)y$$

for all $y \in D_{N^2}$, $k=0,1,2,\ldots$.

It is now easy to show the following statement (S denotes P_α or Q_α, $\alpha \in I$)

(*) For all $n \in \mathsf{N}$: S maps $D_{N^{n+1}}$ into D_{N^n} and for $y \in D_{N^{n+1}}$ we have

$$N^n S y = \textstyle\sum_{k=0}^n \binom{n}{k} (\operatorname{ad} N)^k(S) N^{n-k} y\,.$$

The proof is by induction and since it is similar to the proof of (*) in Lemma 2 we omit the details.

It follows that all the operators P_α, Q_α $\alpha \in I$ leave D_∞ invariant, and it is easily seen that the commutation relations remain valid on D_∞.

Using (*) (for $n=1$) we get that $N_F N x = N N_F x$ for all $x \in D_\infty$, $F \in \mathscr{F}$. Also it follows from (*) and previous arguments that the domain of N^2 is contained in the domain of N_F. If we let $A_0 = N^2$ we get from Lemma 2 that $N_F|D_\infty$ is essentially self adjoint and its closure commutes with N (i.e., their spectral projections commute). ∎

In particular, we can apply Theorem 1 to each finite subsystem $\{P_\alpha, Q_\alpha\}_{\alpha\in F}$. It follows that the operators P_α, Q_α, $\alpha \in I$ are all self adjoint and the corresponding unitary groups $U_\alpha(t) = \exp(itP_\alpha)$ and $V_\alpha(t) = \exp(itQ_\alpha)$ satisfy the Weyl relations

$$U_\alpha(s)U_\beta(t) = U_\beta(t)U_\alpha(s), \qquad V_\alpha(s)V_\beta(t) = V_\beta(t)V_\alpha(s)\,,$$

$$U_\alpha(s)V_\beta(t) = \exp(i\delta_{\alpha\beta}st)V_\beta(t)U_\alpha(s)$$

for all $s,t \in \mathsf{R}$, $\alpha,\beta \in I$.

Let $\{z_\alpha\}_{\alpha\in I}$ be an orthonormal basis for $l^2(I)$ and take $H_0 = \operatorname{span}\{z_\alpha \mid \alpha \in I\}$. Then the unitary groups $\{U_\alpha, V_\alpha\}$ give rise to a Weyl system over H_0 in the usual way (see e.g. [1]), and the conclusion of Theorem 3 follows from [3] (see also [1, p. 79]). Alternatively, by a simple extension of Theorem 5.2 in [15] we have

$$e^{itN} = \text{str.-lim}\, e^{itN_F}\,,$$

uniformly on compact t-intervals, so the desired conclusion follows from [1, Theorem 1].

REMARK. The crucial point of the proof in [3] is the construction of a "vacuum vector". In the present case this can be done quite easily. Since this construction can be used to give a proof of Theorem 3 which is independent of Theorem 1 we indicate the details:

Let $N = \int_{\lambda_0}^{\infty} \lambda dE(\lambda)$ where $\lambda_0 = \inf \sigma(N) \geqq 0$ and choose $x_0 \in K$ such that $\|x_0\| = 1$ and $x_0 = E([\lambda_0, \lambda_0 + \frac{1}{2}])x_0$. Then clearly $x_0 \in D_\infty$ and using (*) (for $n = 1$) we have $NA_\alpha x_0 = A_\alpha(N-I)x_0$ for all $\alpha \in I$. Using our information from Lemma 2 we get (by spectral theory)

$$\begin{aligned}\lambda_0\|A_\alpha x_0\|^2 &\leqq \langle NA_\alpha x_0, A_\alpha x_0\rangle \\ &= \langle (N-I)x_0, N_\alpha x_0\rangle \leqq (\lambda_0 - \tfrac{1}{2})\|A_\alpha x_0\|^2 .\end{aligned}$$

Hence $A_\alpha x_0 = 0$ for all $\alpha \in I$, so $Nx_0 = 0$ and $\lambda_0 = 0$. Let $\{x_{0,\gamma}\}_{\gamma \in \Gamma}$ be an orthonormal basis for $\{x \mid Nx = 0\}$ and define the corresponding "Hermite functions" (see the proof of Theorem 2)

$$x_{n,\gamma} = \prod_\alpha (n_\alpha!)^{-\frac{1}{2}} (A_\alpha{}^+)^{n_\alpha} x_{0,\gamma} \quad \text{for } n \in \Delta .$$

Then it is not difficult to complete the proof of Theorem 3 (see also [14]).

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