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DIRECT INTEGRALS OF LOCALLY MEASURABLE OPERATORS

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Let $\mathscr{B}$ be a von Neumann algebra. In the reference [7], I. E. Segal introduced the concept of a measurable operator with respect to $\mathscr{B}$. In a later work ([6]), S. Sankaran generalised this idea by defining a locally measurable operator (with respect to $\mathscr{B}$), and gave a simple example to show that it was a generalisation. Both of these sets of unbounded operators are interesting, because they form *-algebras, while in general there is no domain common to all the operators.

In this paper we study the behaviour of these *-algebras under direct integration (and direct integral decomposition) of separable Hilbert spaces. We prove the theorem that the direct integral of measurable or locally measurable operators is locally measurable (with respect to the direct integral of the von Neumann algebras involved), and its converse. Using this result, we are able to completely characterise all measurable and locally measurable operators in an interesting case. We intend to make a deeper use of the main theorem in a forthcoming paper on non-unitary group representations.

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1. Preliminaries.

Throughout this paper, $\mathscr{H}$ will represent a separable Hilbert space, $\mathscr{B}$ an arbitrary von Neumann algebra on $\mathscr{H}$, $\mathscr{B}'$ its commutant. The symbol 1 will represent the identity operator; the space in which it acts may be deduced from the context.

Let A be a linear operator with domain $\mathscr{D}(A) \subset \mathscr{H}$. We say A is closable if the closure of the graph of $A, \mathscr{G}(A)$, is the graph of a closed linear

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operator, which we denote by A^-. If $\mathscr{D}(A)$ is dense, and A is closed, we denote $(A^*A)^{\frac{1}{2}}$ by $|A|$, so

$$A = U|A|$$

is the canonical factorisation of A. If A and B are linear operators and $A+B$, AB are closable, we define the strong sum, $A \dotplus B$, and strong product, $A . B$, to be $(A+B)^-$ and $(AB)^-$ respectively.

If (Z,ν) is a standard Borel space, we define a direct integral decomposition of $\mathscr{H}$,

$$\mathscr{H} = \int_Z^{\oplus} \mathscr{H}(t)\, d\nu(t)$$

as in J. Dixmier's book [1], and we denote the algebra of diagonalisable operators by $\mathscr{A}$. If the centre of $\mathscr{B}$, $\mathscr{Z}(\mathscr{B}) \supset \mathscr{A}$, then $\mathscr{B}$ (and $\mathscr{B}'$) can be written as a direct integral,

$$\mathscr{B} = \int_Z^{\oplus} \mathscr{B}(t)\, d\nu(t) .$$

We call $t \to \mathscr{B}(t)$ a measurable field of von Neumann algebras.

If A is a closed linear operator and $A \,\eta\, \mathscr{A}'$ (that is $AB \supset BA \;\forall B \in \mathscr{A}$) then $\mathscr{G}(A)$ is a decomposable subspace of $\mathscr{H} \oplus \mathscr{H}$, and $\mathscr{G}(A)(t)$ is the graph of a closed linear operator $A(t)$ in $\mathscr{H}(t)$ (see [5] or [4]); we write

$$A = \int_Z^{\oplus} A(t)\, d\nu(t) .$$

We call $t \to A(t)$ a measurable field of closed linear operators. If A is bounded, almost every $A(t)$ is bounded, and this definition of direct integral reduces to the usual one. If A is self-adjoint, then almost every $A(t)$ is self-adjoint, and we have the following lemma:

LEMMA 1.1. *Let* $A = \int^{\oplus} A(t)\, d\nu(t)$ *be self-adjoint,*

$$A = \int \lambda \, dE(\lambda), \quad A(t) = \int \lambda \, dE_t(\lambda)$$

the spectral decompositions of A *and* $A(t)$ *(when* $A(t)$ *is self-adjoint). Then for any Borel set,* B,

$$E(B)(t) = E_t(B) \quad \textit{for } \text{a.e. } t .$$

Proof. It is clear that, for each Borel set $B, E(B)$ is decomposable. If A is bounded, the result follows from [8, § 14]. In the general case, let D be a bounded Borel set, and let $\mathscr{K} = E(D)\mathscr{H}$. Then $\mathscr{K}$ is a decomposable subspace and

$$A \mid \mathscr{K} = \int^{\oplus} A(t) \mid \mathscr{K}(t)\, d\nu(t) .$$

The $A(t) \mid \mathscr{K}(t)$ are bounded self-adjoint operators for a.e. t, with spectral measure $E_t(\,.\,) \mid \mathscr{K}(t)$. Let B be any Borel set; by the result in the bounded case,

$$E(B \cap D)(t) \mid \mathscr{K}(t) = E_t(B \cap D) \mid \mathscr{K}(t) \qquad \text{a.e. } t ,$$

that is

$$E(B \cap D)(t) E(D)(t) = E_t(B \cap D) E(D)(t) \qquad \text{a.e. } t .$$

Taking $D_n = [-n, n]$ and letting $n \to \infty$ we have

$$\begin{aligned} E(D_n)(t) &\to 1 & \text{a.e. } t , \\ E(B \cap D_n)(t) &\to E(B)(t) & \text{a.e. } t , \\ E_t(B \cap D_n) &\to E_t(B) , \end{aligned}$$

which completes the proof of the lemma.

If P and Q are two projections in $\mathscr{B}$, we write $P \sim Q$ if there is an operator $V \in \mathscr{B}$, such that

$$VV^* = P, \quad V^*V = Q .$$

A projection P is infinite if

$$\exists\, Q < P, \quad Q \neq P, \quad Q \sim P ,$$

otherwise P is finite. The set of all finite projections in $\mathscr{B}$ will be denoted by $\mathscr{F}(\mathscr{B})$.

We need the following result, which is corollaire 1 of théorème 5 in [1, chapitre II, § 5].

Lemma 1.2. *Let $\mathscr{B} = \int^{\oplus} \mathscr{B}(t)\, d\nu(t)$, and let E (respectively F) be the greatest central finite (respectively properly infinite) projection in $\mathscr{B}$. Then for* a.e. t, *$E(t)$ (respectively $F(t)$) is the greatest central finite (respectively properly infinite) projection in $\mathscr{B}(t)$.*

2. Measurable and locally measurable operators.

Following I.E. Segal [7, Definition 2.1], we define an essentially measurable operator to be a closable linear operator $A \,\eta\, \mathscr{B}$ such that

$$\exists\, \{P_n\}_{n=1,2,\ldots} \subset \mathscr{F}(\mathscr{B}), \quad P_n \downarrow 0, \quad (1 - P_n)\mathscr{H} \subset \mathscr{D}(A) .$$

We call $\{P_n\}$ the defining sequence for A. If A is, in fact, closed, we call it a measurable operator, and we denote the set of all measurable operators by $\mathcal{M}(\mathcal{B})$; clearly $\mathcal{M}(\mathcal{B}) \supset \mathcal{B}$. The following results are proved in [7].

PROPOSITION 2.1. (a) *Each essentially measurable operator has a dense domain, and its closure is measurable.*

(b) *A is measurable $\Leftrightarrow$ $|A|$ is measurable.*

(c) *$\mathcal{M}(\mathcal{B})$ is a *-algebra under strong sum, strong product, and operator adjunction.*

It is easy to see that if $\mathcal{B}$ is finite (that is $1 \in \mathcal{F}(\mathcal{B})$), then $\mathcal{M}(\mathcal{B})$ is the set of all closed operators affiliated with $\mathcal{B}$; while if $\mathcal{B}$ is purely infinite (that is $\mathcal{F}(\mathcal{B}) = \{0\}$), then $\mathcal{M}(\mathcal{B}) = \mathcal{B}$. It is interesting to note that this is also the case if $\mathcal{B}$ is a type I factor (e.g. $\mathcal{B}(\mathcal{H})$).

PROPOSITION 2.2. *If $\mathcal{B}$ is a type* I *factor, $\mathcal{M}(\mathcal{B}) = \mathcal{B}$.*

PROOF. Let $A \in \mathcal{M}(\mathcal{B})$, and let $\{P_n\}$ be the defining sequence for A. Let D be the normalised dimension function on $\mathcal{B}$ (see [3]); then $D(P_n)$ is an integer and $D(P_n) \downarrow 0$. But a sequence of integers can converge to zero only if it eventually becomes zero. Thus $D(P_N) = 0$ for some N, hence $P_N = 0$, and

$$\mathcal{D}(A) \supset (1-P_N)\mathcal{H} = \mathcal{H} ,$$

so A is bounded, and hence $A \in \mathcal{B}$.

We now find another characterisation of measurable operators.

PROPOSITION 2.3. *Let A be a closed linear operator with dense domain affiliated with $\mathcal{B}$, and*

$$|A| = \int_0^\infty \lambda \, dE(\lambda)$$

the spectral decomposition of $|A|$. Then $A \in \mathcal{M}(\mathcal{B})$ if and only if

$$\exists \, \lambda_0 < \infty, \quad E((\lambda_0, \infty)) \in \mathcal{F}(\mathcal{B}) .$$

PROOF. Suppose the condition holds. Then defining $P_n = E((\lambda_0 + n, \infty))$, we have

$$\{P_n\} \subset \mathcal{F}(\mathcal{B}), \quad P_n \downarrow 0, \quad (1-P_n)\mathcal{H} \subset \mathcal{D}(|A|) ,$$

so $|A| \in \mathcal{M}(\mathcal{B})$, hence $A \in \mathcal{M}(\mathcal{B})$.

Conversely, given that $0 \neq |A| \in \mathcal{M}(\mathcal{B})$, let $\{P_n\}$ be the defining sequence for $|A|$. Choose N such that

$$1 < \||A|(1-P_N)\|+1 = \lambda_0 ,$$

say. Then if

$$\|x\| = 1, \quad x \in (1-P_N)\mathcal{H} \cap E((\lambda_0,\infty))\mathcal{H} ,$$

then $x \in \mathcal{D}(|A|)$ and

$$\||A|x\| = \||A|(1-P_N)x\| \leqq \lambda_0 - 1 ,$$

while

$$\||A|x\|^2 = \int_{\lambda_0}^{\infty} \lambda^2 \, d\|E(\lambda)x\|^2 \geqq \lambda_0^2 .$$

Thus no such x can exist, i.e.

$$(1-P_N) \wedge E((\lambda_0,\infty)) = 0 .$$

Hence

$$\begin{aligned} E((\lambda_0,\infty)) &= E((\lambda_0,\infty)) - (1-P_N) \wedge E((\lambda_0,\infty)) \\ &\sim (1-P_N) \vee E((\lambda_0,\infty)) - (1-P_N) \\ &\leqq 1-(1-P_N) = P_N . \end{aligned}$$

Thus $E((\lambda_0,\infty)) \in \mathcal{F}(\mathcal{B})$.

In [6], S. Sankaran introduced the concept of a locally measurable operator. He defined a closed linear operator $A \,\eta\, \mathcal{B}$ to be locally measurable if

$$\exists \{Q_n\}_{n=1,2,\ldots} \subset \mathcal{Z}(\mathcal{B}), \quad Q_n \uparrow 1, \quad A \mid Q_n\mathcal{H} \in \mathcal{M}(\mathcal{B} \mid Q_n\mathcal{H}) .$$

Using proposition 2.3, and elementary arguments, it is easy to see that this definition is equivalent to the following one:

$\mathcal{D}(A)$ is dense and

$$\exists \{P_n\}_{n=1,2,\ldots} \subset \mathcal{Z}(\mathcal{B}), \quad P_n \uparrow 1, \quad E((n,\infty))P_n \in \mathcal{F}(\mathcal{B}) ,$$

where

$$|A| = \int \lambda dE(\lambda)$$

is the spectral decomposition of $|A|$.

We denote the set of all locally measurable operators by $\mathcal{L}(\mathcal{B})$; the following results are either trivial, or are contained in [6].

Proposition 2.4. (a) *$\mathcal{B} \subset \mathcal{M}(\mathcal{B}) \subset \mathcal{L}(\mathcal{B})$; $\mathcal{L}(\mathcal{B}) = \mathcal{M}(\mathcal{B})$ if $\mathcal{B}$ is finite or a factor.*

(b) *$\mathscr{L}(\mathscr{B})$ is a *-algebra under strong sum, strong product, and adjunction.*

(c) *$A \in \mathscr{L}(\mathscr{B})$ is measurable if and only if there exists an n_0 such that $P_{n_0}=1$ in the above definition.*

3. Main Theorem.

We now state and prove the main result of this paper.

THEOREM 3.1. *Let (Z,ν) be a standard Borel space, $t \to \mathscr{B}(t)$ a measurable field of von Neumann algebras, and $t \to A(t)$ a measurable field of closed linear operators. Let*

$$\mathscr{B} = \int_Z^{\oplus} \mathscr{B}(t)\, d\nu(t), \quad A = \int_Z^{\oplus} A(t)\, d\nu(t) .$$

Then A is locally measurable (with respect to $\mathscr{B}$) if and only if $A(t)$ is locally measurable (with respect to $\mathscr{B}(t)$) for almost all t. Furthermore, if A is measurable, then $A(t)$ is measurable for almost all t.

PROOF. We first establish that

$$A \,\eta\, \mathscr{B} \Leftrightarrow A(t) \,\eta\, \mathscr{B}(t) \quad \text{a.e. } t .$$

If $A \,\eta\, \mathscr{B}$, then $AB \supset BA \;\; \forall B \in \mathscr{B}'$, so $AB \supset B.A \;\; \forall B \in \mathscr{B}'$, and hence

$$(AB)(t) \supset (B.A)(t) \quad \text{a.e. } t ,$$

since AB and $B.A$ are closed decomposable operators. From [2] we have

$$(AB)(t) = A(t)B(t), \quad (B.A)(t) = B(t).A(t) \quad \text{a.e. } t .$$

Now we can choose $\{B_n\} \subset \mathscr{B}'$ dense in $\mathscr{B}'$, such that $\{B_n(t)\}$ is dense in $\mathscr{B}(t)'$ for a.e. t. Hence, except for t in a null set $S \subset Z$,

$$A(t)B_n(t) \supset B_n(t).A(t) \quad \forall n .$$

Now choose any $C \in \mathscr{B}(t)'$, $t \notin S$, and suppose $B_m(t) \to C$ in the strong operator topology. Let $y \in \mathscr{D}(A(t))$. Then

$$B_m(t).A(t)y = B_m(t)A(t)y \to CA(t)y ,$$

so $\{A(t)B_m(t)y\}$ converges. Hence $Cy = \lim B_m(t)y \in \mathscr{D}(A(t))$ and

$$A(t)Cy = CA(t)y .$$

Thus

$$A(t) \,\eta\, \mathscr{B}(t) \quad t \notin S .$$

Conversely if $A(t)\,\eta\,\mathscr{B}(t)$ for a.e. t, then it follows immediately that $AB \supset B.A$ for every $B \in \mathscr{B}'$, so $A\,\eta\,\mathscr{B}$.

Now suppose $A \in \mathscr{L}(\mathscr{B})$, and let $\{P_n\} \subset \mathscr{Z}$ be the sequence given in the definition. Then we know that $\mathscr{D}(A(t))$ is dense for a.e. t and (see [4])

$$|A|(t) = |A(t)| \quad \text{a.e. } t\,.$$

Also we have, for a.e. t,

$$P_n(t) \in \mathscr{Z}(\mathscr{B}(t)), \quad P_n(t) \uparrow 1, \quad \big(E((n,\infty))P_n\big)(t) \in \mathscr{F}(\mathscr{B}(t))\,.$$

If $|A| = \int \lambda dE(\lambda)$ and $|A(t)| = \int \lambda dE_t(\lambda)$, then by Lemma 1.1, for each n,

$$\big(E((n,\infty))P_n\big)(t) = E_t((n,\infty))P_n(t) \quad \text{a.e. } t\,.$$

It follows that $A(t) \in \mathscr{L}(\mathscr{B}(t))$ for a.e. t.

If $A \in \mathscr{M}(\mathscr{B})$, then there exists n_0 such that $P_{n_0} = 1$. Since $P_{n_0}(t) = 1$, it follows that $A(t) \in \mathscr{M}(\mathscr{B}(t))$ for a.e. t.

Conversely suppose $A(t) \in \mathscr{L}(\mathscr{B}(t))$ for a.e. t. Then it follows as above that $A\,\eta\,\mathscr{B}$, $\mathscr{D}(A)$ is dense, $|A(t)| = |A|(t)$, a.e. t, and if

$$|A| = \int \lambda\, dE(\lambda), \quad |A(t)| = \int \lambda\, dE_t(\lambda)\,,$$

then for each Borel set B,

$$E(B)(t) = E_t(B) \quad \text{a.e. } t\,.$$

Let $E_n = E((n,\infty))$. We know that for a.e. t, there exist $\{Q_n{}^t\} \subset \mathscr{Z}(\mathscr{B}(t))$ satisfying $Q_n{}^t \uparrow 1$, and $E_n(t)Q_n{}^t \in \mathscr{F}(\mathscr{B}(t))$. We must show that such $Q_n{}^t$ may be chosen so that $t \to Q_n{}^t$ is measurable.

Let $\mathscr{C} = E_n \mathscr{B} E_n$, and let R_n be the greatest central finite projection in $\mathscr{C}$. Let $P_n \in \mathscr{Z}(\mathscr{B})$ be defined by

$$P_n = \sup\{F \in \mathscr{Z}(\mathscr{B}) \mid F \text{ is a projection, } FE_n \le R_n\}\,.$$

We know that $P_nE_n = R_n = R_nE_n \in \mathscr{F}(\mathscr{B})$, since if $VV^* = R_nE_n$, $V^*V = Q < R_nE_n$ then $V \in \mathscr{C}$, which contradicts the definition of R_n. By separability, we have for a.e. t,

$$P_n(t) = \sup\{F^t \in \mathscr{Z}(\mathscr{B}(t)) \mid F^t \text{ is a projection, } F^tE_n(t) \le R_n(t)\}\,.$$

Now for a.e. t, $Q_n{}^tE_n(t)$ is finite in $\mathscr{B}(t)$, so

$$Q_n{}^tE_n(t) \in \mathscr{F}(\mathscr{C}(t))\,.$$

Thus by Lemma 1.2, $Q_n{}^tE_n(t) \le R_n(t)$. We have

$$P_n(t) \ge Q_n{}^t \quad \text{for a.e. } t\,,$$

hence

$$P_n(t) \uparrow 1 \quad \text{a.e. } t .$$

Thus $P_n \uparrow 1$, so $\{P_n\}$ satisfies the conditions of the definition, and $A \in \mathscr{L}(\mathscr{B})$.

4. Further results and examples.

Before giving some examples which illustrate the main theorem, we state a proposition about the *-algebraic operations.

PROPOSITION 4.1. *Let $t \to A(t)$, $t \to B(t)$ be measurable fields of locally measurable operators. Then*

$$A+B = \int^{\oplus} A(t)+B(t)\, d\nu(t)$$

$$A.B = \int^{\oplus} A(t).B(t)\, d\nu(t) .$$

PROOF. These follow immediately from Theorem 3.1 and [2].

It is interesting to note that the operation of direct integral takes one outside the collection of measurable operators, that is $A(t)$ measurable does not imply $\int^{\oplus} A(t)d\nu(t)$ is measurable. This can be seen by the simplest example; if $\{\mathscr{H}_n\}$ is an infinite sequence of identical Hilbert spaces, and A_n a bounded operator on $\mathscr{H}_n = \mathscr{H}_m$, then the closed operator $\oplus_{n=1}^{\infty} A_n$ on $\oplus_{n=1}^{\infty} \mathscr{H}_n$ is not, in general, measurable with respect to $\oplus_{n=1}^{\infty} \mathscr{B}(\mathscr{H}_n)$, but it is locally measurable. We can generalise this example by defining, as in [4], a boundedly decomposable operator A on a direct integral of Hilbert spaces, $\mathscr{H} = \int^{\oplus} \mathscr{H}(t)d\nu(t)$, to be a decomposable closed operator of the form

$$A = \int^{\oplus} A(t)\, d\nu(t), \quad A(t) \in \mathscr{B}(\mathscr{H}(t)) \text{ a.e. } t .$$

The most general unbounded boundedly decomposable operator is found by choosing a measurable field of bounded operators, $t \to A(t)$, such that

$$\operatorname{ess\,sup} \|A(t)\| = \infty .$$

We have:

PROPOSITION 4.2. *Let*

$$\mathscr{B} = \int_Z^{\oplus} \mathscr{B}(\mathscr{H}(t))\, d\nu(t) .$$

Then $A \in \mathscr{L}(\mathscr{B})$ *if and only if* A *is boundedly decomposable. Furthermore,* $A \in \mathscr{M}(\mathscr{B})$ *if and only if there exists a* $B \in \mathscr{B}$, *such that*

$$A - B = \int^{\oplus} C(t)\, d\nu(t)$$

with, for a.e. t, $C(t)$ *an operator living on a finite dimensional subspace of* $\mathscr{H}(t)$ *(that is* $\mathscr{R}(C(t))$ *and* $\mathscr{N}(C(t))^{\perp}$ *finite-dimensional.)*

PROOF. The first part follows immediately from Theorem 3.1 and Proposition 2.2.

If $A \in \mathscr{M}(\mathscr{B})$, define $B = AE([0,\lambda_0])$, where λ_0 is the number defined in Proposition 2.3. Then B is bounded and $B \,\eta\, \mathscr{B}$, so $B \in \mathscr{B}$, and

$$A - B = AE((\lambda_0, \infty)) .$$

Since $E((\lambda_0, \infty))$ is a finite projection, for a.e. t,

$$E((\lambda_0, \infty))(t) \in \mathscr{F}\big(\mathscr{B}(\mathscr{H}(t))\big) ,$$

that is, $E((\lambda_0, \infty))(t)\mathscr{H}(t)$ is finite-dimensional. Thus for a.e. t, $A(t)E((\lambda_0, \infty))(t)$ is zero except on a finite-dimensional subspace, and hence has a finite-dimensional range.

Conversely if A has the above form, then it is clear that $C = \int^{\oplus} C(t)\, d\nu(t)$ is in $\mathscr{L}(\mathscr{B})$ by the first part, and it is sufficient to prove C is measurable.

If $E_t(\cdot)$ is the spectral measure of $|C(t)|$, then $E_t([0,\infty))\mathscr{H}(t)$ is finite dimensional, so if $E(\cdot)$ is the spectral measure of $|C|$,

$$E([0,\infty)) = \int_Z^{\oplus} E_t([0,\infty))\, d\nu(t)$$

is a finite projection in $\mathscr{B}$. It follows that $A \in \mathscr{M}(\mathscr{B})$.

This proposition shows that the result of Nussbaum, that the boundedly decomposable operators form an algebra under strong sum and product, follows from the work of Sankaran.

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