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ON THE DEGREES OF UNIVERSAL REGRESSIVE ISOLS

ERIK ELLENTUCK[1])

1. Introduction.

In this paper we introduce a variety of universal regressive isols and classify them according to their Turing degrees. The usual way to produce a universal isol is by a diagonal construction. Thus if $g : \omega \to \omega$ is generic in the sense of Cohen and $t(0)=1, t(n+1)=j(t(n),g(n))$ for all $n \in \omega$, then Req (ϱt) is a universal regressive isol. By a refinement of this construction we can insure that if t has degree d then d is hyperimmune free and satisfies $\mathbf{0}''=d''$. For the moment let us refer to isols produced by this kind of construction as *generic*. What is important here is that there are generic t with a slow growth rate. Now there is another way to produce universal isols. Let $g : \omega \to \omega$ dominate all partial recursive functions and define t in terms of g as above. Req (ϱt) is a universal regressive isol and its degree d satisfies $\mathbf{0}' \leq d$. By using the right g we can insure that $\mathbf{0}'=d$. For the moment let us refer to isols produced by this kind of construction as *meager*. By definition meager t have a very fast growth rate. All is well until we come to the cosimple case. There we find that the notions of generic and meager coalesce. This is bad because it is known from [5] that meagerness does not generalize to the n-dimensional case when $n>1$, and thus it is not immediately clear how to obtain a universal n-tuple of isols. Such n-tuples were obtained in [4]. But that construction was ad hoc and certainly not as elegant as what goes on in the 1-dimensional case. The way out of our difficulty is to find a notion of meagerness that does generalize to higher dimensions. This we do.

Meagers isols have another application somewhat different from universality. Suppose, because of their fast growth rate, we can show that a function f is recursive in every meager isol. By our remark on degrees, all we can conclude for the degree d of f is that $d \leq \mathbf{0}'$. What we'd like is a broader notion of meagerness which combines fast growth rate with having enough degrees so that f is forced to be recursive. This we also do.

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In all we introduce eight classes of isols in the generic-meager spectrum and study their interconnections, their relations to universality, and the degrees of their members. The use of degrees for incomparability results among regressive isols was pioneered by Dekker in [2]. We consider our work to complement those ideas. Our interest in the cosimple case is essentially due to McLaughlin's invention of the cosimple T-retraceable isols (cf. [9] for a brief discussion of this notion). His extensive paper [10] giving a history of the subject as well as a general method for dealing with retraceability problems will shortly appear in print. We take this opportunity to thank him for sending us manuscripts prior to their publication.

2. The classical case.

Let ω denote the non-negative integers. We refer to elements of ω as *numbers*. If $\alpha \subseteq \omega$ and $n \in \omega$ let $\mathsf{X}^n\alpha$ be the n-fold direct power of α, and if f is any function let δf and ϱf denote its domain and range. $j(x,y)$ will be the usual pairing function with k and l as its first and second inverse. A function t is called *regressive* if it is a one-one mapping of ω into ω for which there exists a partial recursive function p such that

$$(1) \qquad \varrho t \subseteq \delta p \wedge pt(0) = t(0) \wedge (\forall n \in \omega) pt(n+1) = t(n) .$$

If t is a regressive function then we can find a partial recursive p which in addition to (1) also satisfies

$$(2) \qquad \varrho p \subseteq \delta p \wedge (\forall x \in \delta p)(\exists n \in \omega) \ p^{n+1}(x) = p^n(x) ,$$

where p^n denotes the n-th iterate of p. Such a p satisfying (1) and (2) is called a *regressing function* of t. If p is a regressing function (of some t) we define two functions p^* and $\overline{p}$ by

$$(3) \qquad \delta p^* = \delta p \wedge p^*(x) = (\mu n) \ p^{n+1}(x) = p^n(x) ,$$

$$(4) \qquad \delta \overline{p} = \delta p \wedge \overline{p}(x) = \{p^n(x) \mid n \in \omega\} - \{x\} ,$$

where μ is the minimum operator. The function p^* is also partial recursive and satisfies

$$(5) \qquad \varrho t \subseteq \delta p^* \wedge (\forall n \in \omega) \ p^*t(n) = n .$$

A function t is called *retraceable* if it is regressive and strictly increasing. If t is a retraceable function then we can find a partial recursive function p which in addition to (1) and (2) also satisfies

$$(6) \qquad (\forall x \in \delta p) \ p(x) \leqq x .$$

Such a p satisfying (1), (2), and (6) is called a *retracing* function of t. A set is called *regressive* if it is finite or the range of a regressive function and is called *retraceable* if it is finite or the range of a retraceable function. An isol is called *regressive* if it contains a regressive set. By [1] we know every regressive isol contains a retraceable set so it is not necessary to define retraceable isol. In [2] a Turing degree is associated with every regressive isol x. It is the Turing degree of some retraceable set $\xi \in x$ (equivalently of every retraceable set $\xi \in x$). Let Λ denote the isols, Λ_{R} the regressive isols, Λ_{Z} the cosimple isols, and Λ_{RZ} the cosimple regresive isols. Denote the recursive equivalence type of any set $\xi \subseteq \omega$ by $\mathrm{Req}(\xi)$. A property of numbers is said to hold *eventually*, or *for almost all* x if there is an $n \in \omega$ such that x has the property for all $x > n$. Throughout this paper we will define certain classes of functions by a property P. Such functions will be called P-*functions*, their ranges will be called P-*sets*, and if these sets are immune then their Req's will be P-*isols*. So much for ancient history.

A function t is called 1-*meager* if it is retraceable and for every partial recursive function p

$$(1\text{-m}) \qquad t(n) \notin \delta p \ \vee \ pt(n) < t(n+1)$$

for almost all n. A 1-meager set is necessarily immune. 1-meager sets seem to have first appeared in a construction due to McLaughlin (cf. [9]). The concept was probably delineated and deemed worthy of independent study by Barback. The first substantial works on the 1-meager isols was done by Gersting in [5] and [6]. We were informed by McLaughlin that the degree d of any 1-meager isol satisfies $\mathbf{0}' \leq d$ (here $\mathbf{0}$ is the degree of recursive sets and prime is the jump). To obtain the result in both directions we use an unpublished result of Tennenbaum. If p is a unary partial function and g is a unary total function then g *dominates* p if $n \notin \delta p \ \vee \ p(n) < g(n)$ for almost all n.

LEMMA 1 (Tennenbaum). *If d is a Turing degree, then there exists a function of degree d which dominates every partial recursive function if and only if $\mathbf{0}' \leq d$.*

THEOREM 1. *If d is a Turing degree, then there exists a 1-meager isol of degree d if and only if $\mathbf{0}' \leq d$.*

PROOF. (a) Assume $\mathbf{0}' \leq d$. Let g be a total function of degree d which dominates every partial recursive function. Let $j_3(x,y,z) = j(j(x,y),z)$ and have i-th inverse function k_i for $i < 3$. Define a function t by

$$t(0) = 1 \quad \text{and} \quad t(n+1) = j_3(t(n), g(n), gt(n)) \quad \text{for } n \in \omega .$$

Since $t(0) \neq 0$ we see that t is a strictly increasing function that is almost retraced by k_0. If p is a partial recursive function then

$$t(n) \notin \delta p \vee pt(n) < gt(n) \leqq t(n+1)$$

for almost all n. Hence t is 1-meager. t is recursive in g by the primitive recursive definition of t and $g(n) = k_1 t(n+1)$ so g is recursive in $\lambda n t(n+1)$ which has the same degree as t (λ is Church's functional symbol).

(b) Assume that t is a 1-meager function and let p be a retracing function of t. Let q be any partial recursive function and define $r(x) = qp^*(x)$. Then r is a partial recursive function and

$$n \notin \delta q \vee q(n) = qp^*t(n) = rt(n) < t(n+1)$$

for almost all n. Thus $\lambda n t(n+1)$ dominates all partial recursive functions and has the same degree as t. By Lemma 1 the degree d of t satisfies $\mathbf{0}' \leq d$.

Our proof follows if we recall that the degree of a regressive isol x is the same as that of any retraceable $\tau \in x$ which is the same as that of any retraceable function whose range is τ.

A function t is called 1-*generic* if it is retraceable and for every partial recursive function p

$$\text{(1-g)} \qquad t(n) \notin \delta p \vee pt(n) \neq t(n+1)$$

for almost all n. A 1-generic set is necessarily immune.

THEOREM 2. *There exists a* 1-*generic isol whose Turing degree d satisfies* $\mathbf{0}' = d'$.

PROOF. (a) Let $q_n(x)$ be a partial recursive function of two variables which with index n enumerates the partial recursive functions of one variable x. Define a function t by $t(0) = 1$, and given the value $t(n)$ let

$$y(n) = (\mu y)(\forall m < n)\, q_m t(n) \neq j(t(n)), y)$$

and $t(n+1) = j(t(n), y(n))$. Since $t(0) \neq 0$ we see that t is a strictly increasing function that is almost retraced by k. The definition of $y(n)$ immediately implies that t is a 1-generic function.

(b) Notice from its definition that $y(n) \leqq n$ for every n and then define a recursive function f by $f(0) = 1$ and $f(n+1) = j(f(n), n)$. By an easy induction $t(n) \leqq f(n)$ for all n, that is, t is bounded by a recursive function. We can also cast the definition of t as the conjunction $\Phi(t)$ of

$$t(0) = 1,$$
$$(\forall n)\, kt(n+1) = t(n),$$
$$(\forall n)(\forall m < n)(\forall y)\, (q_m t(n) = y \rightarrow t(n+1) \neq y),$$
$$(\forall n)\, t(n) \leqq f(n).$$

Any function t which satisfies Φ is 1-generic and bounded by f. By quantifier manipulation we can show that Φ is expressible in $\Pi_1{}^0$ form. By a result of Jockusch and Soare (cf. [7]), every non-empty recursively bounded $\Pi_1{}^0$ class of functions contains a member whose degree d satisfies $\mathbf{0}' = d'$. Since Φ is non-empty by (a), our result follows.

A degree d is *hyperimmune free* if every function of degree d is dominated by a recursive function.

COROLLARY 1. *There exists a 1-generic isol whose degree d is hyperimmune free and satisfies* $\mathbf{0}'' = d''$.

PROOF. It is also shown in [7] that every non-empty recursively bounded $\Pi_1{}^0$ class of functions contains a member whose degree is hyperimmune free and satisfies $\mathbf{0}'' = d''$. Now apply this to Φ of Theorem 2.

COROLLARY 2. *Every 1-meager isol is 1-generic; however there are 1-generic isols that are not 1-meager.*

PROOF. By (1-m) and (1-g) we see that every 1-meager isol is 1-generic; however the 1-generic isol constructed in Theorem 2 cannot be 1-meager since it has the wrong degree.

Thus we see that from the degree point of view there is quite a difference between 1-meager and 1-generic isols. Since 1-generic isols will turn out to be universal, Corollary 2 implies that a large growth rate is not a necessary criteria for universality.

A function t is called *2-meager* if it is retraceable and for every partial recursive function p

(2-m) $$\varrho t \nsubseteq \delta p \ \vee\ pt(n) < t(n+1)$$

for almost all n. Clearly any 1-meager object is also 2-meager, and a 2-meager set is necessarily immune. If f and g are total unary functions, then g *dominates* f if $f(n) < g(n)$ for almost all n. In order to characterize the degrees of 2-meager isols we need

LEMMA 2 (Martin [8]). *If d is a Turing degree, then there exists a function of degree d which dominates every (total) recursive function if and only if* $\mathbf{0}'' \leq d'$.

THEOREM 3. *If d is a Turing degree, then there exists a 2-meager isol of degree d if and only if* $\mathbf{0}'' \leq d'$.

PROOF. (a) Assume that t is a 2-meager function and that f is any recursive function. $g(x) = \sum_{n \leqq x} f(n)$ is also recursive and moreover is monotone increasing. So

$$f(n) \leqq g(n) \leqq gt(n) < t(n+1)$$

for almost all n since $\varrho t \subseteqq \delta g$. Thus $\lambda n t(n+1)$ dominates all recursive functions and has the same degree as t. By Lemma 2 the degree d of t satisfies $\mathbf{0}'' \leq d'$.

(b) Assume that $\mathbf{0}'' \leq d'$. We shall modify a construction due to Martin (cf. [8]) and produce a 2-meager function t of degree d. Let $q_n(x)$ be the enumeration used in the proof of Theorem 2. We know that the degree of

$$B = \{n \mid \delta q_n \text{ is infinite}\}$$

is $\mathbf{0}''$ and hence $\leq d'$ (cf. [13]). Hence by [13] we can find a total binary function $f^s(n)$ of degree $\leq d$ such that $n \in B$ iff $\lim_s f^s(n) = 1$ and $n \notin B$ iff $\lim_s f^s(n) = 0$. We may also assume that

$$(\forall s, n) f^s(n) \in \{0, 1\} .$$

Now consider the predicate $P(m, n, x, s)$ which is defined by

$$(n \leqq x \wedge q_m(x) = s) \vee (n \leqq s \wedge f^s(m) = 0) .$$

P is recursicely enumerable in f. P also satisfies

$$(7) \qquad (\forall m, n)(\exists x, s) P(m, n, x, s) ,$$

for either $m \in B$ and δq_m is infinite (so $x \in \delta q_m$ for some $n \leqq x$) or $m \in B$ and $\lim_s f^s(m) = 0$ (so $f^s(m) = 0$ for some $n \leqq s$). Next we define two binary functions

$$x(m, n) = (\mu x)(\exists s) P(m, n, x, s) ,$$
$$s(m, n) = (\mu s) P(m, n, x(m, n), s) .$$

By (7) $x(m, n)$ and $s(m, n)$ are both recursive in f.

We are now ready to construct t. Let g be a unary function of degree d. Define $t(0) = 1$ and given $t(n)$ let

$$h(n) = 1 + \max\{s(m, t(n)) \mid m < t(n)\}$$

and $t(n+1)=j_3(t(n),g(n),h(n))$, where $\max\emptyset=0$. Since $t(0)\neq 0$ we see that t is a strictly increasing function that is almost retraced by k_0. t is recursive in f, g by the primitive recursive definition of t, and f is recursive in g so t is recursive in g. But g is recursive in t since $g(n)=k_1t(n)$. Thus t has degree d. To see that t is 2-meager let p be a partial recursive function such that $\rho t\subseteq\delta p$. Then $p=q_m$ for some $m\in B$. Choose n_0 so large that

$$m < t(n_0) \quad \text{and} \quad (\forall s \geqq t(n_0))f^s(m) = 1 .$$

Then for any $n\geqq n_0$ we have $x(m,t(n))=t(n)$, and $s(m,t(n))=q_mt(n)<h(n)$. Thus $pt(n)<h(n)\leq t(n+1)$.

A function t is called 2-*generic* if it is retraceable and for every partial recursive function p

$$\text{(2-g)} \qquad \rho t \nsubseteq \delta p \ \vee \ pt(n) \neq t(n+1)$$

for almost all n. Clearly any 1-generic object is also 2-generic, and a 2-generic set is necessarily immune.

THEOREM 4. *There exists a 2-generic isol whose Turing degree d satisfies* $\mathbf{0}'=d'$.

COROLLARY 3. *Every 2-meager isol is 2-generic; however there are 2-generic isols that are not 2-meager.*

PROOF. The isol constructed in the proof of Theorem 2 is 2-generic, but not 2-meager since it has the wrong degree.

Several variations of the meager-generic concept readily come to mind. These will be important in the next section where we deal with cosimple isols. A function t is called *weakly* 1-*meager* if it is retraceable and for every partial recursive function p

$$\text{(w1-m)} \qquad t(n)\notin\delta p \ \vee \ pt(n)\notin\rho t \ \vee \ pt(n) < t(n+1)$$

for almost all n. It is *weakly* 1-*generic* if it is retraceable and for every partial recursive function p

$$\text{(w1-g)} \qquad t(n)\notin\delta p \ \vee \ pt(n)\notin\rho t \ \vee \ pt(n) \neq t(n+1)$$

for almost all n. It is *weakly* 2-*meager* if it is retraceable and for every partial recursive function p

$$\text{(w2-m)} \qquad \rho t \nsubseteq \delta p \ \vee \ p(\rho t) \nsubseteq \rho t \ \vee \ pt(n) < t(n+1)$$

for almost all n. Finally is is *weakly 2-genric* if it retraceable and for every partial recursive function p

$$\text{(w2-g)} \qquad \varrho t \nsubseteq \delta p \vee p(\varrho t) \nsubseteq \varrho t \vee pt(n) \neq t(n+1)$$

for almost all n. Notice that the weakening we have in mind consists of only enforcing a condition if $pt(n) \in \varrho t$ or $p(\varrho t) \subseteq \varrho t$. Some immediate properties of these notions are the following for $i \in \{1,2\}$. If an object is i-meager (generic) then it is weakly i-meager (generic). If it is weakly 1-meager (generic) then it is weakly 2-meager (generic). A weakly i-meager (generic) set is necessarily immune. The following diagram indicates some of the relationships that we have established.

$$\text{(8)} \qquad \begin{tikzcd}
& (1-\mathrm{m}) \arrow[rr] \arrow[dl] \arrow[dd] & & (\mathrm{w}1-\mathrm{m}) \arrow[dl] \arrow[dd] \\
(1-\mathrm{g}) \arrow[rr] \arrow[dd] & & (\mathrm{w}-1\mathrm{g}) \arrow[dd] & \\
& (2-\mathrm{m}) \arrow[rr] \arrow[dl] & & (\mathrm{w}2-\mathrm{m}) \arrow[dl] \\
(2-\mathrm{g}) \arrow[rr] & & (\mathrm{w}2-\mathrm{g}) &
\end{tikzcd}$$

Thus at first sight we have eight classes of isols. The following easy theorem reduces them to six.

THEOREM 5. *For $i \in \{1,2\}$. An isol is weakly i-meager if and only if it is weakly i-generic.*

PROOF. Let t be a retraceable function and let p be a retracing function of t. First we establish that there exists a binary partial recursive function $q(x,y)$ with domain $X^2\delta p$ such that for every $y \in \varrho t$ we have $(t(n),y) \in \delta q$,

$$q(t(n), y) = t(n) \quad \text{if } y \leqq t(n)\,,$$

and

$$q(t(n), \mathrm{y}) = t(n+1) \quad \text{if } t(n+1) \leqq y\,.$$

We define the function q by (cf. (4) for $\overline{p}$)

$$\begin{aligned} q(x, y) &= p^m(y) \text{ where } m = (\mu n)x = p^{n+1}(y)\,, && \text{if } x \in \overline{p}(y)\,, \\ &= x && \text{otherwise.} \end{aligned}$$

Our claim is immediate. We see from (8) that the theorem follows if we show that a weakly i-generic function is weakly i-meager. Assume that t,p, and q are as above. If the partial recursive function r is witness to the fact that t is not weakly i-meager then $q(x,r(x))$ is witness to the fact that t is not weakly i-generic.

THEOREM 6. *For $i \in \{1,2\}$. There exists a weakly i-meager isol whose Turing degree d satisfies $\mathbf{0}' = d'$.*

3. The effective case.

In this section we restrict our attention to the cosimple isols. Our goal is to show that in this case the diagram (8) reduces to exactly three classes. Recall from [3] that every retraceable function whose range has a r.e. (recursively enumerable) complement has a (total) recursive retracing function.

LEMMA 3. *If t is a retraceable function whose range has a* r.e. *complement then there exists a binary recursive function $q(x,y)$ such that for all n,y we have $q(t(n),y)=t(n)$ if $y \leqq t(n)$ and $q(t(n),y)=t(n+1)$ if $t(n+1) \leqq y$.*

PROOF. Let $\tau = \varrho t$ and p be a recursive retracing function of t. We compute $q(x,y)$ as follows. If $y \leqq x$, let $q(x,y)=x$. If $x<y$, let

$$\alpha_{xy} = \{z \mid x<z \leqq y \wedge x = p(z)\}.$$

Since p is recursive, we can effectively find α_{xy}. If α_{xy} is empty, let $q(x,y)=x$ and if α_{xy} contains exactly one element, let $q(x,y)$ be that element. Suppose that α_{xy} contains $m>1$ elements. Start an effective enumeration of $\omega - \tau$. Then at some stage in this process, we will have shown that either (i) $x \in \omega - \tau$, or (ii) $m-1$ elements of α_{xy} belong to $w - \tau$. If (i) occurs before (ii), let $q(x,y) = x$ and if (ii) occurs before (i), let $q(x,y)$ be the remaining element of α_{xy} that has not already been listed in $\omega - \tau$. Notice that if $t(n+1) \leqq y$ then $\alpha_{t(n)y}$ is non-empty. If it has exactly one element, then that element is $t(n+1)$, and if it has $m>1$ elements, then $t(n+1) \in \alpha_{t(n)y}$ but all the remaining elements of $\alpha_{t(n)y}$ belong to $\omega - \tau$. Thus $q(t(n),y)=t(n+1)$.

THEOREM 7. *For $i \in \{1,2\}$:*
(a) *A cosimple isol is i-meager if and only if it is i-generic.*
(b) *A cosimple isol is* 1*-meager if and only if it is weakly* 1*-meager.*

PROOF. First we show that if a cosimple isol x is i-generic (or weakly i-generic) then it contains an i-generic (or weakly i-generic) set with r.e. complement. By [2] we know that x contains a retraceable set τ with r.e. complement. Let σ be any other retraceable set contained in x. There are retraceable functions s and t which enumerate σ and τ respectively. Let p be any partial recursive function. We think of p as acting on τ, and then construct a partial recursive function $r(x)$ which acts on σ. Since $\sigma, \tau \in x$, we know by [2] that there is a one-one partial recursive function h such that $\sigma \subseteq \delta h$ and $hs(n) = t(n)$ for all $n \in \omega$. Let

$$\delta r = \{x \mid x \in \delta h \wedge h(x) \in \delta p\} .$$

Then

$$(\forall x \in \delta r)[ph(x) \in \delta h^{-1} \vee ph(x) \in \omega - \tau]$$

so that by enumerating δh^{-1} and $\omega - \tau$ we can define $r(x) = h^{-1}ph(x)$ if we discover $ph(x)$ in δh^{-1} before we discover it in $\omega - \tau$, and $r(x) = s(0)$ otherwise. Notice that r is partial recursive, if $\varrho t \subseteq \delta p$ then $\varrho s \subseteq \delta r$, if

$$\varrho t \subseteq \delta p \wedge p(\varrho t) \subseteq \varrho t$$

then

$$\varrho s \subseteq \delta r \wedge r(\varrho s) \subseteq \varrho s ,$$

and if $pt(n) = t(n+1)$ then $rs(n) = s(n+1)$. This shows that if σ is i-generic (or weakly i-generic) then so is τ.

By (8),

$$(1\text{-m}) \rightarrow (1\text{-g}) \rightarrow (\text{w}1\text{-g}) \quad \text{and} \quad (2\text{-m}) \rightarrow (2\text{-g}) .$$

So in order to establish the theorem we will show

$$(\text{w}1\text{-g}) \rightarrow (1\text{-m}) \quad \text{and} \quad (2\text{-g}) \rightarrow (2\text{-m}) .$$

Let t be a retraceable function whose range τ has an r.e. complement, and let $q(x,y)$ be the function whose existence for t was proved in Lemma 3. For any partial recursive function p let $r(x) = q(x, p(x))$. If

$$t(n) \in \delta p \wedge t(n+1) \leqq pt(n)$$

then

$$t(n) \in \delta r \wedge rt(n) \in \varrho t \wedge rt(n) = t(n+1) .$$

Thus $(\text{w}1\text{-g}) \rightarrow (1\text{-m})$ in the cosimple case. If

$$\varrho t \subseteq \delta p \wedge t(n+1) \leqq pt(n)$$

then

$$\varrho t \subseteq \delta r \wedge rt(n) = t(n+1) .$$

Thus $(2\text{-g}) \rightarrow (2\text{-m})$ in the cosimple case.

By (8), Theorem 5, and Theorem 7, the diagram (8) reduces in the cosimple case to

$$(9) \qquad (1\text{-m}) \rightarrow (2\text{-m}) \rightarrow (\text{w}2\text{-m}) .$$

Our next goal is to show that these classes are distinct and to learn something of their properties. Since the weakly 2-meager isols turn out to be universal, and since they have applications that the other two classes do not have, we amend our definition and say that a function is 3-*meager* if it was previously called weakly 2-meager. Notice that in the cosimple case all distinctions between genericity and meagerness have disappeared. Growth rate alone has become a decisive factor.

THEOREM 8 (McLaughlin [9]). *If d is a Turing degree, then there exists a cosimple* 1-*meager isol of degree d if and only if $\mathbf{0}' = d$.*

PROOF. Since the degree d of any cosimple set satisfies $d \leq \mathbf{0}'$, it follows from Theorem 1 that the degree of any cosimple 1-meager isol is exactly $\mathbf{0}'$. Thus to prove our theorem, it suffices to show that there exists a cosimple 1-meager isol. Let $q_n(x)$ be a partial recursive function of two variables which with index n enumerates the partial recursive functions of one variable x. Let $q_n{}^s(x) = y$ if $q_n(x) = y$ by a computation with Gödel number $\leqq s$; otherwise we say that $q_n{}^s(x)$ is undefined. Our proof will be a stage by stage construction of finite approximating functions $t^s(n)$ where $n \leqq s$. We start with

Stage 0: Let $t^0(0) = 1$ and then go on to stage 1.

Stage $s+1$: As inductive hypothesis assume that at the end of stage s we have defined $t^s(n)$ for $n \leqq s$, that $t^s(0) = 1$ and $kt^s(n+1) = t^s(n)$ for $n < s$. Search for the least $n < s$ and for it the least $m < n$ such that

$$q_m{}^s t^s(n) \text{ is defined and } t^s(n+1) \leqq q_m{}^s t^s(n)\,. \tag{10}$$

If there is no such pair, go to case (a) below; otherwise, go to case (b).

Case (a): Let $t^{s+1}(x) = t^s(x)$ for $x \leqq s$, $t^{s+1}(s+1) = j(t^s(s), 0)$.

Case (b): Find the least y such that

$$\max\{q_m{}^s t^s(n), t^s(s)\} < j(t^s(n), y) \tag{11}$$

and let $t^{s+1}(x) = t^s(x)$ for $x \leqq n$, $t^{s+1}(n+1) = j(t^s(n), y)$, and $t^{s+1}(x+1) = j(t^{s+1}(x), 0)$ for $n < x \leqq s$.

This completes stage $s+1$ of the construction. Now go on to stage $s+2$.

It is easy to see that our inductive hypothesis is maintained as we pass through stages. $t(n) = \lim_s t^s(n)$ exists for every n because $t(0) = 1$ and once $t^s(n)$ has reached its final value (10) implies that $t^s(n+1)$ can change its value at most n times. $t(0) = 1$ and $kt(n+1) = t(n)$ for all n by our inductive hypothesis. t is one-one since $t(0) \neq 0$ and is clearly a retraceable function almost retraced by k. ρt has a r.e. complement because it follows from (11) and the construction in case (b) that $x \in \omega - \rho t$ if and only if $(\exists s > x)\, x \in \rho t^s$, the latter being a r.e. condition. Finally t is meager because (10) and the construction in case (b) imply that $t(n) \notin \delta q_m$ or $q_m t(n) < t(n+1)$ for all $n > m$.

In [14] it is asked whether there exist retraceable sets which can be written in $\prod_n{}^0$ but not in $\sum_n{}^0$ quantifier form. The answer to this question should be folklore by now, but I have never heard it explicitly mentioned. An easy modification of the proof of Theorem 8 shows that indeed such sets exist. Replace the $q_n(x)$ by an enumeration of functions partial recursive in $\mathbf{0}^{(k)}$ (the k^{th} jump of $\mathbf{0}$), then proceed as in the proof of Theorem 8. The construction becomes recursive in $\mathbf{0}^{(k)}$ and hence by Post's theorem $\omega - \varrho t$ is $\sum_{k+1}^0$. (1-m) holds for functions p partial recursive in $\mathbf{0}^{(k)}$ and hence ϱt is not recursive in $\mathbf{0}^{(k)}$. Again by Post's theorem this means that ϱt is not $\sum_{k+1}^0$.

LEMMA 4. *If d is the Turing degree of a* r.e. *set and g is a function of degree d, then there exists a retraceable function t of degree d whose range has a* r.e. *complement such that $gt(n) < t(n+1)$ for all $n \in \omega$.*

PROOF. Let α be a r.e. set of degree d. Since g is recursive in α, by [13] we can find a binary recursive function $g^s(x)$ and a unary function $r(x)$ recursive in α such that $g(x) = \lim_s g^s(x)$ and

$$(\forall s \geqq r(x))\, g^s(x) = g(x) .$$

Our proof will be of the same type as that of Theorem 8.

Stage 0: Let $t^0(0) = 1$ and then go on to stage 1.

Stage $s+1$: As inductive hypothesis assume that at the end of stage s we have defined $t^s(n)$ for $n \leqq s$, that $t^s(0) = 1$, and $k_0 t^s(n+1) = t^s(n)$, $k_1 t^s(n+1) = g^s(n)$ for $n < s$. Search for the least $n < s$ such that either

$$k_1 t^s(n+1) \neq g^{s+1}(n) , \tag{12}$$

$$t^s(n+1) \leqq g^{s+1} t^s(n). \tag{13}$$

If there is no such n, go to case (a) below; otherwise, go to case (b).

Case (a): Let $t^{s+1}(x) = t^s(x)$ for $x \leqq s, t^{s+1}(s+1) = j_3(t^s(s), g^{s+1}(s), 0)$.

Case (b): Find the least y such that

$$\max\{g^{s+1} t^s(n), t^s(s)\} < j_3(t^s(n), g^{s+1}(n), y) \tag{14}$$

and let $t^{s+1}(x) = t^s(x)$ for $x \leqq n$,

$$t^{s+1}(n+1) = j_3(t^s(n), g^{s+1}(n), y) ,$$

and

$$t^{s+1}(x+1) = j_3(t^{s+1}(x), g^{s+1}(x), 0)$$

for $n < x \leqq s$.

This completes stage $s+1$ of the construction. Now go on to stage $s+2$.

It is easy to see that our inductive hypothesis is maintained as we pass through stages. Now $t(n) = \lim_s t^s(n)$ exists for every n because $t(0) = 1$ and once $t^s(n)$ has reached its final value $t(n), g^s(n)$ has reached its final value $g(n)$, and $g^s t(n)$ has reached its final value $gt(n)$, (12) and (13) imply that $t^s(n+1)$ can change its value at most one more time. Then just as in the proof of Theorem 8, we can show that $t(0) = 1, k_0 t(n+1) = t(n)$ for all $n \in \omega$, t is a one-one function almost retraced by $k_0, \varrho t$ has a r.e. complement, $gt(n) < t(n+1)$ for all $n \in \omega$, and $k_1 t(n+1) = g(n)$ for all $n \in \omega$. The last of these claims implies that the degree of g is $\leq$ to that of t. To show the converse let $s(x)$ be the least s such that

$$(\forall n \leqq t^s(x))\big(r(n) \leqq s \wedge g^s t^s(n) < t^s(n+1)\big) . \tag{14}$$

Since the construction is recursive and r is recursive in α, we can effectively decide from a knowledge of α whether s will satisfy (14). Thus $s(x)$ is recursive in α and $t(x) = t^{s(x)}(x)$, making t recursive in α. Then t has degree d.

THEOREM 9. *If d is a Turing degree, then there exists a cosimple 2-meager isol of degree d if and only if d is a* r.e. *degree and $\mathbf{0}'' = d'$.*

PROOF. Since the degree d of any cosimple set satisfies $d' \leq \mathbf{0}''$, it follows from Theorem 3 that the degree of any cosimple 2-meager isol satisfies $\mathbf{0}'' = d'$. Conversely let d be a r.e. degree satisfying $\mathbf{0}'' = d'$. By Lemma 2 there is a function g of degree d which dominates every recursive function and by Lemma 4 there exists a retraceable function t of degree d whose range has a r.e. complement and such that for every recursive function p

$$pt(n) < gt(n) < t(n+1) \tag{15}$$

for almost all n. Suppose that p is only partial recursive but $\varrho t \subseteqq \delta p$. Now $(\forall x)(x \in \delta p \vee x \in \omega - \varrho t)$ and $\omega - \varrho t$ is r.e., so by enumerating δp and $\omega - \varrho t$ we can define a recursive function r by $r(x) = p(x)$ if we discover x in δp before we discover it in $\omega - \varrho t$ and $r(x) = x$ otherwise. Since r agrees with p on ϱt, (15) holds for p. This shows that t is a 2-meager function of degree d.

COROLLARY 4. *If f is a function whose degree is $\leq$ to that of every cosimple 2-megaer isol, then f is recursive.*

PROOF. By Theorem 9 the set of degrees of the cosimple 2-meager isols is just

$$A = \{d \mid d \text{ is a r.e degree and } \mathbf{0}'' = d'\} .$$

It will therefore suffice to show that the greatest lower bound of A is $\mathbf{0}$. In [8] Martin attributes this result to Sacks as it follows easily from section 6, Theorem 3 of [12]. By putting $b=g=\mathbf{0}$, $c=\mathbf{0}''$ in that theorem it follows that for any degree a with $\mathbf{0}<a\leq\mathbf{0}'$ there is a r.e. degree d such that $a \nless d \wedge \mathbf{0}''=d'$.

We should note here that McLaughlin's result (our Theorem 8) could also be obtained by the method used in the proof of Theorem 9. For by Lemma 1 there exists a function g of degree $\mathbf{0}'$ (a r.e. degree) which dominates every partial recursive function. Then Lemma 4 gives a t such that (15) holds for every partial recursive function p, that is, t is 1-meager.

THEOREM 10. *There exists a cosimple* 3*-meager isol whose Turing degree* d *satisfies* $\mathbf{0}'=d'$.

PROOF. Let $r_n(f,x)$ be a partial recursive function of three variables which with index n enumerates the partial recursive functions of a function variable f and a number variable x. The *jump* of f denoted by f' will be the set

$$\{x \mid r_x(f,x) \text{ is defined}\}\,.$$

Let $f|n$ be the restriction of f to arguments $\leqq n$. Let $r_n{}^s(f|s,x)=y$ if $r_n(f,x)=y$ by a computation with Gödel number $\leqq s$ which asks for $f(y)$ only when $y\leqq s$; otherwise we say that $r_n{}^s(f|s,x)$ is undefined. The enumerations q_n and $q_n{}^s$ are the ones used in the proof of Theorem 8. Just as in that proof we construct the finite approximating functions $t^s(n)$ where $n\leqq s$. An extra ingredient is an infinite list of movable markers $\mu(n)$ for $n\in\omega$. We start the construction with

Stage 0: Let $t^0(0)=1$. At the end of this stage no markers are attached to numbers. Then go on to stage 1.

Stage $s+1$: As inductive hypothesis assume that at the end of stage s we have defined $t^s(n)$ for $n\leqq s$, that $t^s(0)=1$ and $kt^s(n+1)=t^s(n)$ for $n<s$. We also assume that certain markers $\mu(n)$ for $n<s$ are attached and only attached to numbers of the form $t^s(m)$ where $m<s$ and that whenever $\mu(n)$ is attached to $t^s(m)$ then (16) below is satisfied.

$$\text{(16)} \qquad q_n{}^s t^s(m) \text{ is defined and } t^s(m)<q_n{}^s t^s(m)<t^s(m+1)\,.$$

Let $e^s(n)=(\mu y\leqq s)[r_n{}^y(t^s|y,n)$ is defined] if there is such a y; otherwise $e^s(n)=0$. Search for the least $n\leqq s$ and for it the least $m<s$ such that $\mu(n)$ is not attached to numbers and which satisfy (17)–(19) below.

$$\text{(17)} \qquad q_n{}^s t^s(m) \text{ is defined and } t^s(m+1)\leqq q_n{}^s t^s(m)\,,$$

(18) for no $n' < n$ and $m' \geqq m$ is $\mu(n')$ attached to $t^s(m')$,

(19) $$e^s(n') \leqq m \text{ for each } n' < n\ .$$

If there is no such pair go to case (a) below; otherwise, go to case (b).

Case (a): Let $t^{s+1}(x) = t^s(x)$ for $x \leqq s$, $t^{s+1}(s+1) = j(t^s(s), 0)$, and leave the position of all markers unchanged.

Case (b): For each $n' > n$ detach $\mu(n')$ from numbers (if it is attached to any) and attach $\mu(n)$ to $t^s(m)$. Find the least y such that

(20) $$\max\{q_n{}^s t^s(m), t^s(s)\} < j(t^s(m), y)$$

and let $t^{s+1}(x) = t^s(x)$ for $x \leqq m$, $t^{s+1}(m+1) = j(t^s(m), y)$, and $t^{s+1}(x+1) = j(t^{s+1}(x), 0)$ for $m < x \leqq s$.

This completes stage $s+1$ of the construction. Now go on to stage $s+2$.

It is easy to see that our inductive hypothesis is maintained as we pass through stages. Each marker moves finitely often because once the $\mu(n')$ with $n' < n$ reach their final positions (which may or may not be attached to numbers) and $\mu(n)$ is attached to a number then (18) implies that it will never move again. $t(n) = \lim_s t^s(n)$ exists for every n because either there is a stage after which no markers are attached to numbers and case (a) applies, or infinitely many markers reach final positions attached to distinct numbers and (20) implies that all motion of $t^s(m)$ below those numbers is frozen. Finally $e(n) = \lim_s e^s(n)$ exists for every n because once the $\mu(n')$ with $n' \leqq n$ have reached their final positions (19) implies that $e^s(n)$ can change its value at most one more time. Then just as in the proof of Theorem 8 we can show that $t(0) = 1$, $kt(n+1) = t(n)$ for all $n \in \omega$, t is a one-one function almost retraced by k, and ρt has a r.e. complement. To show that t is 3-meager let p be a partial recursive function such that

$$\rho t \subseteq \delta p \wedge p(\rho t) \subseteq \rho t\ .$$

Now $p = q_n$ for some $n \in \omega$. Say that a marker (or a function) has reached its final position (or value) by stage s if at no later stage does it change its position (or value). Let s_0 be a stage such that by stage s_0 the $\mu(n')$ with $n' < n$ have reached their final positions and the $e^s(n')$ with $n' < n$ have reached their final values. Let b_0 be an upper bound to these positions (if attached to numbers) and let b_1 be an upper bound to these values. If $t(m+1) \leqq pt(m)$ for some m with $b_0 < t(m) \wedge b_1 < m$ then there would be an $s > s_0$ such that

$$t^s(m) = t(m) \wedge t^s(m+1) = t(m+1) \wedge q_n{}^s t^s(m) \text{ is defined.}$$

Then by this stage $\mu(n)$ has already reached a final position attached to number or by (17)–(19) it will reach its final position attached to numbers at stage $s+1$. It follows from (16) that $p(\varrho t) \nsubseteq \varrho t$. Thus $pt(m) < t(m+1)$ for almost all m and t is 3-meager. Let t have jump t' and degree d. Since we always have $\mathbf{0}' \leq d'$ we complete our proof by showing that t' is recursive in some set of degree $\mathbf{0}'$. We claim that $n \in t'$ iff $e(n) > 0$. Let's first show that this does the job.

$$e(n) > 0 \text{ iff } (\forall x)(\exists s > x)(e^s(n) > 0) \text{ iff } (\exists x)(\forall s > x)(e^s(n) > 0)\,.$$

Since $e^s(n) > 0$ is recursive this puts $e(n) > 0$ in two quantifier form and hence by Post's theorem of degree $\leq \mathbf{0}'$. Now for the claim. If $n \in t'$ then $r_n(t, n)$ is defined and there is a y such that $r_n(t, n) = r_n{}^y(t|y, n)$. Choose s so large that $t^s(m) = t(m)$ for $m \leqq y$. Then

$$r_n{}^y(t|y, n) = r_n{}^y(t^s|y, n) \quad \text{and} \quad e^s(n) > 0\,.$$

Since $e^s(n)$ has reached its final value by this stage, we have $e(n) > 0$. If $e(n) > 0$, choose s so large that $e^s(n)$ has reached its final value by stage s and $t^s(m) = t(m)$ for all $m \leqq e(n)$. Then there is a y $(= e(n))$ such that $r_n{}^y(t^s|y, n)$ is defined and

$$r_n{}^y(t^s|y, n) = r_n{}^y(t|y, n) = r_n(t, n)$$

putting $n \in t'$.

4. Application.

We say that an isol u is *strongly universal* if for every function $r : \omega \to \omega$ and relation

$$R = \{(m, n) \mid r(m) = n\}\,,$$

$(\exists y \in \Lambda)\ (u, y) \in R_\Lambda$ implies that r is eventually recursive increasing. Our interest in meager isols stems from

THEOREM 11. *Every 3-meager isol is strongly universal.*

PROOF. Let Q be the set of finite subsets of ω. If $F \subseteq \mathsf{X}^2 Q$ is a frame and $(\alpha_0, \alpha_1) \in F^*$ let

$$C_F{}^{\vee}(\alpha_0{}^{\vee}, \alpha_1) = \beta_0 \quad \text{and} \quad C_F{}^{\vee}(\alpha_0, \alpha_1{}^{\vee}) = \beta_1$$

where $C_F(\alpha_0, \alpha_1) = (\beta_0, \beta_1)$ (cf. [11] for frame notation). Consider any 3-meager function t and let $\varrho t = \tau \in u \in \Lambda_{\mathrm{R}}$. Suppose $r : \omega \to \omega$,

$$R = \{(m, n) \mid r(m) = n\}, \quad \text{and} \quad (\exists y \in \Lambda)(u, y) \in R_\Lambda\,.$$

Then there is an isolated $\eta \subseteqq \omega$ and a recursive R-frame F such that (τ, η) is attainable from F. Let p be a retracing function of t and let $\overline{p}$ be the function associated with p by (4). We define a partial function q as follows.

$$\delta q = \{x \mid x \in \delta p \wedge (\overline{p}(x), \emptyset) \in F^*\}$$

and

$$q(x) = \max C_F{}^{\vee}(\overline{p}(x)^{\vee}, \emptyset) .$$

Clearly q is partial recursive and $\varrho t \subseteqq \delta q \wedge q(\varrho t) \subseteqq \varrho t$ because (τ, η) is attainable from F. We easily note that $t(n) \leqq qt(n)$ for all $n \in \omega$ (by definition) and since t is 3-meager this implies $qt(n) = t(n)$ for almost all $n \in \omega$. Let

$$A = \{\alpha \mid (\alpha, \emptyset) \in F^* \wedge C_F{}^{\vee}(\alpha^{\vee}, \emptyset) = \alpha\} .$$

Now $qt(n) = t(n)$ implies $C_F{}^{\vee}(\overline{p}t(n)^{\vee}, \emptyset) = \overline{p}t(n)$ so $\overline{p}t(n) \in A$ for almost all n. For $\alpha \in A$ let

$$\varphi(\alpha) = C_F{}^{\vee}(\alpha, \emptyset^{\vee}).$$

A is a r.e. family of finite sets, φ is a partial recursive function taking finite sets to finite sets, and $(\alpha, \varphi(\alpha)) \in F$ for every $\alpha \in A$. If $\alpha, \beta \in A$ and $\alpha \subseteqq \beta$ then $(\alpha, \emptyset) \leqq (\beta, \varphi(\beta))$ and hence $\varphi(\alpha) \subseteqq \varphi(\beta)$. Let $|\alpha|$ be the number of elements in α. Then

$$S = \{(m, n) \mid (\exists \alpha \in A)\, m = |\alpha| \wedge n = |\varphi(\alpha)|\}$$

is a r.e. subset of R, and the graph of a partial function whose domain is almost all of ω. This function is eventually increasing by the monotonicity of φ. Thus r is an eventually recursive increasing function.

We say that an isol u is *universal* if for every $R \subseteqq \omega$, $u \in R_\Lambda$ implies that $n \in R$ for almost all n.

THEOREM 12. *Every strongly universal isol is universal.*

PROOF. Assume that u is strongly universal, $R \subseteqq \omega$, and $u \in R_\Lambda$. Let

$$S = \{(n, 1) \mid n \in R\} \cup \{(n, 0) \mid n \in \omega - R\} .$$

Since $(\forall x \in \omega)\, x \in R$ iff $(x, 1) \in S$, the same will be true in Λ by the universal metatheorem of Nerode (cf. [11]). Thus $(u, 1) \in S_\Lambda$. Since S is the graph of the characteristic function r of R, u is strongly universal, and $1 \in \Lambda$ we conclude that r is eventually increasing. But r is bounded and hence is eventually constant. It will suffice to show that r eventually has the value 1. Assume not. Then for some $n \in \omega$ we have $(\forall x \in \omega)\, r(x+n) = 0$, and hence $(\forall x \in \omega)\ (x+n, 1) \notin S$. Again by the metatheorem this will also be true in Λ and since u is clearly infinite this implies $(u, 1) \notin S_\Lambda$. But then $u \notin R_\Lambda$, a contradiction. So u is universal.

In [5] it was asked whether every universal regressive isol is 1-meager. From the results of the last section this is answered negatively with a cosimple counterexample. Our answer is even stronger than the question because in [5] "universal" is defined only with respect to recursive relations, whereas the notion here is with respect to all relations (a study of this kind of distinction between universal isols, as well as others will appear elsewhere).

Suppose we wish to generalize the notion of universal isol to the n-dimensional case where $n > 1$. For the sake of illustration we shall confine ourselves to the case $n = 2$. Let $f: \mathsf{X}^2\omega \to \omega$ and define

$$\Delta_x f(x, y) = f(x+1, y) - f(x, y)\ ,$$
$$\Delta_y f(x, y) = f(x, y+1) - f(x, y)\ ,$$

and $\Delta = \Delta_x \circ \Delta_y$. f is said to be *recursive increasing* if f is recursive and $\Delta\tilde{f}(x,y) \geqq 0$ for all $(x,y) \in \mathsf{X}^2\omega$, where $\tilde{f}(x+1, y+1) = f(x,y)$ and $\tilde{f}(x,y) = 0$ otherwise. A property of pairs of numbers is said to hold *eventually*, or *for almost all* (x,y) if there is a $(m,n) \in \mathsf{X}^2\omega$ such that (x,y) has the property for all $x > m$ and $y > n$. *Eventually recursive increasing* and *almost recursive increasing* are then canonically defined in the same way as is done for combinatorial functions. The importance of these notions follows from [4] where it is shown that if $r: \mathsf{X}^2\omega \to \omega$ is an almost recursive increasing function and

$$R = \{(x,y,z) \mid r(x,y) = z\}$$

then

$$(\forall x,y \in \Lambda_R)(\exists z \in \Lambda)\ (x,y,z) \in R_\Lambda\ .$$

For the converse of this theorem we say that a pair of isols (u,v) is *strongly universal* if for every function $r: \mathsf{X}^2\omega \to \omega$ and relation

$$R = \{(x, y, z) \mid r(x, y) = z\}\ ,$$

$(\exists z \in \Lambda)\ (u, v, z) \in R_\Lambda$ implies that r is eventually recursive increasing. We also say that the pair is *universal* if for every $R \subseteq \mathsf{X}^2\omega$, $(u,v) \in R_\Lambda$ implies that almost every $(x,y) \in \mathsf{X}^2\omega$ belongs to R. Just as in Theorem 12 we can show that every strongly universal pair of isols is universal. So the remaining problem is to find such pairs. Let us examine the situation along the lines of the meager-generic classification of section 2 keeping in mind the proof of Theorem 11 in order to see what we need.

A pair of functions (s,t) is called 1-*meager* if s,t are both retraceable and for every pair of binary partial recursive functions (p,q)

$$(21)\quad (s_m, t_n) \notin \delta p \ \vee\ (s_m, t_n) \notin \delta q \ \vee\ \big(p(s_m, t_n) < s_{m+1} \wedge q(s_m, t_n) < t_{n+1}\big)$$

for almost all (m,n). To define 2-*meager* we add to (21) as a disjunct

$$\varrho s \times \varrho t \nsubseteq \delta p \ \vee\ \varrho s \times \varrho t \nsubseteq \delta q\ ,$$

to define *weakly* 1-*meager* we add to (21) as a disjunct

$$p(s_m, t_n) \notin \varrho s \ \vee\ q(s_m, t_n) \notin \varrho t\ ,$$

and finally to define *weakly* 2-*meager* we add to (21) as a disjunct

$$\varrho s \times \varrho t \nsubseteq \delta p \ \vee\ \varrho s \times \varrho t \nsubseteq \delta q \ \vee\ p(\varrho s \times \varrho t) \nsubseteq \varrho s \ \vee\ q(\varrho s \times \varrho t) \nsubseteq \varrho t\ .$$

To define the *generic* pairs simply, take the corresponding meager definition and replace both $<$ symbols in it by $\neq$ symbols. The diagram (8) holds for pairs and the reader can easily convince himself that Theorems 5 and 7 also hold for pairs. Define 3-*meager* pairs as the weakly 2-meager ones and note that (1-m) $\rightarrow$ (2-m) $\rightarrow$ (3-m), and in the cosimple case this is all that remains of (8). The crux of our analysis follows from.

THEOREM 14 (Gersting [5]). *There are no 2-meager pairs of functions.*

PROOF. Suppose s and t are retraceable functions. Define the recursive functions $p(x,y)=q(x,y)=x+y$. Then $\varrho s \times \varrho t \subseteq \delta p = \delta q$,

$$(\forall m)(\exists n > m)\ s_{m+1} < p(s_m, t_n), \qquad \text{and} \qquad (\forall n)(\exists m > n)\ t_{n+1} < q(s_m, t_n)\ .$$

Thus (p,q) is a counterexample to (s,t) satisfying (21).

With little modification in the proof of Theorem 11, we can show that every 3-meager pair of isols is a strongly universal pair. This together with Theorem 14 and the proof of Theorem 11 gives us a pretty good idea of the road leading to the existence of strongly universal pairs of cosimple regressive isols. Fortunately this road (without all the analysis) has already been traveled for we have

THEOREM 15 (Ellentuck [4]). *For every $n > 0$ there exists a 3-meager n-tuple of cosimple isols.*

Thus the problem of finding strongly universal pairs of cosimple regressive isols gives some justification for the concept of 3-meagerness. The other meager notions simply do not generalize.

POSTSCRIPT.

We can now show that there exists a cosimple 3-meager isol in every r.e. degree $d > 0$.

REFERENCES

1. J. Dekker, *Infinite series of isols*, Proc. Symposia Pure Math. Amer. Math. Soc. 5 (1962), 77–96.
2. J. Dekker, *The minimum of two regressive isols*, Math. Z. 83 (1964), 345–366.
3. J. Dekker and J. Myhill, *Retraceable sets*, Canad. J. Math. 10 (1958), 357–373.
4. E. Ellentuck, *On the form of functions which preserve regressive isols*, Compositio Mathematica (1973).
5. J. Gersting, Ph. D. Thesis, Arizona State University, 1969.
6. J. Gersting, *A rate of growth criterion for universality of regressive isols*, Pacific J. Math. 31 (1969), 669–677.
7. C. Jockush and R. Soare, $\prod_1^0$ *classes and degrees of theories,*, to appear.
8. D. Martin, *Classes of recursively enumerable sets and degrees of unsolvability*, Z. Math. Logik Grundlagen Math. 12 (1966), 295–310.
9. T. McLaughlin, *Hereditarily retraceable isols*, Bull. Amer. Math. Soc. 73 (1967), 113–115.
10. T. McLaughlin, *Thinning the branches of a semicomputable tree*, to appear.
11. A. Nerode, *Extensions to isols*, Ann. of Math. 73 (1961), 362–403.
12. G. Sacks, *Degrees of Unsolvability*, Annals of Math. Studies 55, Princeton University Press, 1963.
13. J. Shoenfield, *On degrees of unsolvability*, Ann. of Math. 69 (1959), 644–653.
14. C. Yates, *Arithmetical sets and retracing functions*, Z. Math. Logik Grundlagen Math. 13 (1967), 193–204.

RUTGERS, THE STATE UNIVERSITY
NEW BRUNSWICK, NEW JERSEY, U.S.A.

AND

THE INSTITUTE FOR ADVANCED STUDY
PRINCETON, NEW JERSEY, U.S.A.