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COVERINGS OF METRIC SPACES WITH RANDOMLY PLACED BALLS

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1. Introduction.

Let (S,d) be a metric space with metric d, $\{X_n\}$ a sequence of S-valued independent random variables defined on the probability space $(W,\mathscr{F},P\}$, and $\{a_n\}$ a sequence of positive numbers. *We shall throughout this paper assume that S is analytic* (see for example [4, chapter III.1]).

Let π_n denote the probability law of X_n, $s^0(x,a)$ the open and $s(x,a)$ the closed ball with center at $x \in S$ and radius $a>0$. We shall then consider the randomly placed balls $S_n(w)=s^0(X_n(w),a_n)$ for $w \in W$. Let $C(w)$ be the set of points in S, which are covered infinitely often by the balls $S_n(w)$, and let $F(w)$ be the complement of $C(w)$. That is

$$C(w) = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} S_n(w) = \limsup_{n\to\infty} S_n(w)$$
$$F(w) = S \setminus C(w) .$$

Let A be a Borel set in S, which will be fixed for the rest of this paper. Then by the zero-one law we know that $P(w \mid A \subseteqq C(w))$ is either 0 or 1. The aim of this paper is to discuss the following 3 problems:

(1.1) When is $A \nsubseteq C(w)$ a.s.?

(1.2) When is $A \subseteqq C(w)$ a.s.?

(1.3) How "small" is $F(w)$?

The conditions which we will impose on $\{X_n\}$ and $\{a_n\}$ in order to solve (1.1), (1.2) and (1.3) will be expressed in terms of the following quantities:

$$T_n = \sup\{\pi_n(s^0(x,a_n)) \mid x \in A\}, \quad n \geqq 1 ,$$
$$t_n = \inf\{\pi_n(s^0(x,a_n)) \mid x \in A\}, \quad n \geqq 1 ,$$

together with certain assumptions on finite dimensionality of (S,d).

Let us note the following trivial fact, which follows immediately from Fubini's theorem and the Borel–Cantelli lemmas:

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(1.4) *Let D be the set of points $x \in S$ for which we have*

$$\sum_{n=1}^{\infty} \pi_n(s^0(x, a_n)) = \infty .$$

Then $\gamma(C(w)) = \gamma(D)$ and $\gamma(F(w)) = \gamma(S \setminus D)$ a.s. *for all σ-finite measures γ on $(S, \mathscr{B})$, where $\mathscr{B}$ is the Borel σ-algebra in S.*

This shows first of all that if $A \nsubseteq D$, then the answer to (1.1) is positive, and secondly that if $D = S$, which implicitly will be the case in most of the discussion, then $F(w)$ is a.s. a γ-null set for all σ-finite measures γ on $(S, \mathscr{B})$, which partly answers (1.3). If $\{a_n\}$ does not tend to zero, then in most cases we have $S = C(w)$ a.s. This is for example the case when $\inf_n \pi_n(s^0(x,a)) > 0$, $\forall x \in S$, $\forall a > 0$. *In the sequel we shall assume that*

$$\lim_{n \to \infty} a_n = 0 . \tag{1.5}$$

And, when it is convenient, we shall assume that $\{a_n\}$ decreases (notice that under (1.5) it is always possible to rearrange $\{a_n\}$, so that it becomes decreasing).

In the literature the problem has only been considered in the case where S is a circle of length 1, π_n is the Lebesgue measure for all n, and d is the arc length distance. In this case we have that

$$T_n = t_n = 2a_n, \quad \forall n \geqq 1 .$$

The problem was in this form first posed by Dvoretzky in [2], and later treated by Kahane in [5], [6] and [7], Erdös in [3], Billard in [1] and Orey in [9].

In [1] Billard showed that if

$$\sum_{n=1}^{\infty} (2a_n)^2 \exp\left(\sum_{j=1}^{n} 2a_j\right) < \infty$$

then $C(w) \neq S$ a.s. (See also Kahane [7, p. 89 and p. 92]). In section 3 we shall show by essentially the same method as Billard that a similar result holds for arbitrary analytic metric space. Billard also showed in [1] that if

$$\limsup_{n \to \infty} \left\{\sum_{j=1}^{n} 2a_j - \log n\right\} = \infty$$

then $C(w) = S$ a.s. (See also Kahane [7, p. 89 and p. 95].) In section 4 we shall show that a similar result holds for certain finite dimensional spaces S. The proof of this fact is derived from the important inequality given in Lemma 6, which even in the circle case gives new results. Thirdly Billard showed in [1] that if

$$\limsup_{n \to \infty} \left\{\sum_{j=1}^{n} 2a_j - \log n\right\} > -\infty$$

then $F(w)$ is at most countable a.s., and S. Orey has in [9] proved that this condition implies that $F(w) = \emptyset$ a.s. In section 5 we shall give some examples.

In a very recent paper of L. A. Shepp [10] a necessary and sufficient condition for covering of the circle is given. In this paper it is proved that covering take place if and only if one of the following two conditions holds:

$$\limsup_{n\to\infty} \{\textstyle\sum_{j=1}^{n} 2a_j - \log n\} = \infty \tag{1.6}$$

$$\textstyle\sum_{n=1}^{\infty} n^{-2} \exp\{\sum_{j=1}^{n} 2(a_j - a_n)\} = \infty\ . \tag{1.7}$$

Before we proceed we shall define the upper and lower concentration functions, C^* and C_*, of a measure μ on $(S,\mathscr{B})$:

$$C^*(\mu,t) = \sup\{\mu(s^0(x,t)) \mid x \in S\}, \quad t>0\ ,$$
$$C_*(\mu,t) = \inf\{\mu(s^0(x,t)) \mid x \in S\}, \quad t>0\ .$$

Let H be the class of all continuous strictly increasing functions on $[0,\infty)$, which takes the value 0 at 0. If $h \in H$, then we say that the measure m on $(S,\mathscr{B})$ is *h-Lipshitzian,* if it satisfies

$$m(s(x,r)) \leqq Ch(r), \quad \forall x \in S_0,\ \forall r > 0\ ,$$

where S_0 is the support of m. We say that the *capacitarian dimension* is $\geqq h$, and write Cap-dim $\geqq h$, if there exists a finite h-Lipshitzian measure $m \neq 0$ on $(S,\mathscr{B})$.

If $h \in H$, then the h-Hausdorff measure, H_h, is defined by

$$H_h(B) = \lim_{\varepsilon\to 0}\{\inf(\textstyle\sum_{n=1}^{\infty} h(b_n))\}$$

where the infimum is taken over all sequences of closed balls $B_n = s(x_n,b_n)$ with $b_n \leqq \varepsilon$ and $B \subseteqq \bigcup_1^\infty B_n$. And we write H-dim$(B) < h$ if and only if $H_h(B) = 0$.

2. Some measurability lemmas.

We shall in this section prove measurability of some sets and functions, which will occur in the later discussion.

LEMMA 1. *If D belongs to the product σ-algebra $\mathscr{B}\otimes\mathscr{F}$, then the following sets are P-measurable (they need not however belong to $\mathscr{F}$):*

$$R_1 = \{w \mid A \subseteqq D(w)\}\ ,$$
$$R_2 = \{w \mid A \supseteqq D(w)\}\ ,$$
$$R_3 = \{w \mid A \cap D(w) \neq \emptyset\}\ ,$$

where $D(w) = \{x \in S \mid (x,w) \in D\}$.

PROOF. Obviously it suffices to show that R_3 is P-measurable. If p is the projection: $p(x,w)=w$, then we have

$$R_3 = p(D \cap (A \times W)) .$$

Hence from (II.2.2), (II.9.2) and (III.2.3) in [4] it follows that R_3 is P-measurable.

LEMMA 2. *Let $D \in \mathscr{B}\otimes\mathscr{F}$, then the set*

$$R = \{(x,w) \mid \operatorname{bd}_A(s^0(x,c) \cap A) \nsubseteq D(w)\}$$

is universally measurable in $(S \times W, \mathscr{B}\otimes\mathscr{F})$ (that is R is γ-measurable for all finite measures γ on $(S \times W, \mathscr{B}\otimes\mathscr{F})$).

PROOF. Let R_k for $k \geqq 1$ be the set of all points (x,w,z,z',z'') in $S \times W \times A \times A \times A$, satisfying

$$(z,w) \notin D,\ d(z,z') \leqq k^{-1},\ d(z,z'') \leqq k^{-1},\ d(z',x) < c \text{ and } d(z'',x) \geqq c .$$

Then obviously we have that $R_k \in \mathscr{B}\otimes\mathscr{F}\otimes\mathscr{A}\otimes\mathscr{A}\otimes\mathscr{A}$, where $\mathscr{A}$ is the Borel σ-algebra in A. Now let p be the projection: $p(x,w,z,z',z'')=(x,w,z)$ and q the projection: $q(x,w,z)=(x,w)$. Now a straightforward argument shows that we have

$$R = q(\textstyle\bigcap_{k=1}^{\infty} p(R_k)) .$$

Hence from (II.2.2), (II.9.2), (II.1.7) and (III.2.3) in [4] it follows that R is universally measurable.

LEMMA 3. *Let $(W,\mathscr{F})$ and $(W',\mathscr{F}')$ be two measurable spaces. If E belongs to $\mathscr{B}\otimes\mathscr{F}$ and B belongs to $\mathscr{B}\otimes\mathscr{F}'$, then the set*

$$R = \{(w,w') \mid E(w) \subseteqq B(w')\}$$

is universally measurable in $(W \times W', \mathscr{F}\otimes\mathscr{F}')$.

PROOF. Let D be the set of points (w,w',x) in $W \times W' \times S$, satisfying $(x,w) \in E$ and $(x,w') \notin B$. Then obviously $D \in \mathscr{F}\otimes\mathscr{F}'\otimes\mathscr{B}$. And since

$$R = W \times W' \setminus p(D) ,$$

where p is the projection: $p(w,w',x)=(w,w')$, we have as before that R is universally measurable.

3. The non-covering case.

THEOREM 1. *Let γ be a σ-finite measure on $(S,\mathscr{B})$, such that $\gamma(A)>0$, and let*

$$b_n = E\{\gamma(A \cap S_n(\cdot))^2\} = \int_S \gamma(A \cap s^0(x,a_n))^2 \pi_n(dx)$$

for $n \geqq 1$. If the following 3 conditions holds, then $A \nsubseteq C(w)$ a.s.:

$$\limsup_{n\to\infty} T_n < 1 , \tag{3.1}$$

$$\sum_{n=1}^{\infty} (T_n - t_n) < \infty , \tag{3.2}$$

$$\sum_{n=1}^{\infty} b_n \exp\{\sum_{j=1}^{n} T_j(1-t_j)^{-2}\} < \infty . \tag{3.3}$$

REMARKS. (a) A straighforward argument shows that b_n satisfies the following two useful inequalities:

$$b_n \leqq C^*(\gamma,a_n)^2, \qquad \forall n \geqq 1 , \tag{3.4}$$

$$b_n \leqq \gamma(A)T_n C^*(\gamma,2a_n), \quad \forall n \geqq 1 . \tag{3.5}$$

(b) It is easily checked that the following two conditions imply conditions (3.1) and (3.3):

$$\sum_{n=1}^{\infty} T_n^2 < \infty , \tag{3.6}$$

$$\sum_{n=1}^{\infty} b_n \exp\{\sum_{j=1}^{n} T_j\} < \infty . \tag{3.7}$$

(c) Condition (3.2) is often not fulfilled and a heuristic argument indicates that if $\pi_n = \pi$ for all $n \geqq 1$ this condition is superfluous. (If $(T_n - t_n)$ is "large" then π is unevenly distributed over A, in which case A is more difficultly covered.) However, I have not been able to prove this. But as we shall see in section 5 (Example 2) there is a way to get around (3.2):

Suppose that H is a map from S into S, which satisfies the Lipshitz condition:

$$d(H(x),H(y)) \leqq d(x,y), \quad \forall x,y \in S .$$

Now we put

$$C'(w) = \limsup_{n\to\infty} s^0\big(H(X_n(w)),a_n\big) .$$

Then $H(A) \nsubseteq C'(w)$ implies $A \nsubseteq C(w)$, since $H(s^0(x,t)) \subseteq s^0(H(x),t)$, $\forall x \in S$, $\forall t > 0$. Hence one strategy would be to find a function H, which makes π uniformly distributed over $H(A)$, and which preserves (3.6) and (3.7).

(d) The reviewer has remarked, that (3.1) is a trivial consequence of (3.2), if S contains more than one point.

PROOF OF THEOREM 1. It is no loss of generality assuming that $T_n \leqq T < 1$ for all $n \geqq 1$, and that A is compact and $\gamma(A) = \gamma(S) = 1$. Let $F_n(w)$ be the random set

$$F_n(w) = \bigcap_{j=1}^n (A \setminus S_j(w)) .$$

Then $M_n(w) = \gamma(F_n(w))$ is a random variable. If

$$S_n(x) = \{w \in W \mid x \in S_n(w)\} ,$$

we find from Fubini's theorem that

$$E(M_n) = \int_A \prod_{j=1}^n \left(1 - P(S_j(x))\right) \gamma(dx) \geqq \prod_{j=1}^n (1 - T_j) > 0 , \tag{3.8}$$

$$E(M_n^2) = \int_A \int_A \prod_{j=1}^n \left(1 - P(S_j(x)) - P(S_j(y)) + P(S_j(x) \cap S_j(y))\right) \gamma(dx)\gamma(dy) . \tag{3.9}$$

Let $h_j(x,y) = P(S_j(x) \cap S_j(y))$, $q_j = (1 - t_j)^{-2}$ and

$$p_n = (1 - t_1)^2 (1 - t_2)^2 \ldots (1 - t_n)^2 .$$

Then we have

$$\begin{aligned} E(M_n^2) &\leqq \int_A \int_A \prod_{j=1}^n (1 - 2t_j + t_j^2 + h_j(x,y)) \gamma(dx)\gamma(dy) \\ &\leqq p_n \int_A \int_A \prod_{j=1}^n (1 + q_j h_j(x,y)) \gamma(dx)\gamma(dy) \\ &= p_n \int_A \int_A \{1 + \textstyle\sum_{j=1}^n q_j h_j(x,y) \prod_{k=1}^{j-1} (1 + q_k h_k(x,y))\} \gamma(dx)\gamma(dy) , \end{aligned}$$

where we have used the well-known formula:

$$\prod_{j=1}^n (1 + c_j) = 1 + \sum_{j=1}^n c_j \prod_{k=1}^{j-1} (1 + c_k) .$$

Since $h_k(x,y) \leqq T_k$ for all k, $\gamma(A) = 1$, and

$$b_j = \int_A \int_A h_j(x,y) \gamma(dx)\gamma(dy) ,$$

we find

$$\begin{aligned} E(M_n^2) &\leqq p_n \{1 + \textstyle\sum_{j=1}^n q_j \prod_{k=1}^{j-1} (1 + q_k T_k) \int_A \int_A h_j(x,y) \gamma(dx)\gamma(dy)\} \\ &\leqq p_n \{1 + \textstyle\sum_{j=1}^n q_j b_j \prod_{k=1}^{j-1} (1 + q_k T_k)\} \\ &\leqq p_n \{1 + \textstyle\sum_{j=1}^n b_j q_j \exp[\sum_{k=1}^j q_k T_k]\} \\ &\leqq p_n \{1 + (1 - T)^{-2} \textstyle\sum_{j=1}^\infty b_j \exp[\sum_{k=1}^j q_k T_k]\} \\ &= C p_n , \end{aligned}$$

where the constant C is finite by assumption (3.3). Inserting the inequality (3.8) in the inequality above gives

$$\begin{aligned} E(M_n^2) &\leqq C(EM_n)^2 \prod_{j=1}^n \left(\frac{1-t_j}{1-T_j}\right)^2 \\ &\leqq C(EM_n)^2 \exp\left(2\sum_{j=1}^n \frac{T_j-t_j}{1-T_j}\right) \\ &\leqq C(EM_n)^2 \exp\left(2(1-T)^{-1}\sum_{j=1}^\infty (T_j-t_j)\right) \\ &= C'(EM_n)^2 \end{aligned}$$

where the constant C' is finite by (3.2). From the Cauchy–Schwarz inequality it follows that

$$(EM_n)^2 \leqq E(M_n^2)P(M_n \neq 0) ,$$

and so we find

$$0 < E(M_n^2) \leqq C'E(M_n^2)P(M_n \neq 0) ,$$

from which we conclude that

$$P(M_n \neq 0) \geqq d > 0, \quad \forall n \geqq 1 , \tag{3.10}$$

for some constant d. From the definition of $F_n(w)$ it follows that

$$\{M_n \neq 0\} \subseteqq \{w \mid F_n(w) \neq \emptyset\} = \{w \mid A \nsubseteq \cup_{j=1}^n S_j(w)\} .$$

Since we have assumed that A is compact we find that

$$\{w \mid A \nsubseteq \cup_{j=1}^\infty S_j(w)\} = \cap_{n=1}^\infty \{w \mid A \nsubseteq \cup_{j=1}^n S_j(w)\} .$$

Since the sets in the intersection on the right hand side decrease we have that

$$\begin{aligned} P(w \mid A \nsubseteq C(w)) &\geqq \lim_{n\to\infty} P(w \mid A \nsubseteq \cup_{j=1}^n S_j(w)) \\ &\geqq \liminf_{n\to\infty} P(M_n \neq 0) \\ &\geqq d > 0 \end{aligned}$$

by (3.10). Since $\{w \mid A \nsubseteq C(w)\}$ either has probability zero or one by the zero-one law we have proved Theorem 1.

Corollary 1. *Let h be a function in H, such that*

$$\sum_{n=1}^\infty T_n h(2a_n) \exp\left(\sum_{j=1}^n T_j\right) < \infty ,$$

$$\sum_{n=1}^\infty T_n^2 < \infty \quad \textit{and} \quad \sum_{n=1}^\infty (T_n - t_n) < \infty .$$

Then if Cap-dim$(A) \geqq h$ *we have that* $A \nsubseteq C(w)$ a.s.

Proof. If Cap-dim$(A) \geqq h$, then we know that there exists a probability measure γ on A, such that for some constant $C < \infty$ we have

$$\gamma(s(x,r)) \leqq Ch(r), \quad \forall x \in A_0,\ \forall r > 0 ,$$

where A_0 is the support of γ. If we compute b_n of Theorem 1 we find

$$b_n = \int_S \int_S \int_S 1_{s(x,a_n)}(u) 1_{s(x,a_n)}(v) \gamma(du) \gamma(dv) \pi(dx)$$
$$= \int_{A_0} \int_{A_0} \pi(s^0(u,a_n) \cap s^0(v,a_n)) \gamma(du) \gamma(dv) .$$

If $d(u,v) \leqq 2a_n$ then $\pi(s^0(u,a_n) \cap s^0(v,a_n)) \leqq T_n$, and if $d(u,v) > 2a_n$ then $\pi(s^0(u,a_n) \cap s^0(v,a_n)) = 0$, since the two balls are disjoint. Hence we find

$$b_n \leqq T_n \int_{A_0} \gamma(s(u,2a_n)) \gamma(du) \leqq CT_n h(2a_n) ,$$

and so Corollary 1 follows from Theorem 1.

4. The covering case.

We shall in this section find sufficient conditions for $A \subseteq C(w)$ a.s., that is, we want to show that $P(w \mid A \nsubseteq C(w)) = 0$. Since we have

$$P(A \nsubseteq C) \leqq \sum_{k=1}^{\infty} P(A \nsubseteq \bigcup_{j=k}^{\infty} S_j) ,$$
$$P(A \nsubseteq \bigcup_{j=k}^{\infty} S_j) \leqq P(A \nsubseteq \bigcup_{j=k}^{k+n} S_j), \quad \forall n,k \geqq 1 ,$$

we shall seek a good estimate of the probability of $\{w \mid A \nsubseteq \bigcup_{j=1}^{n} S_j(w)\}$. This estimate will not only depend on the quantities $\{T_n\}$ and $\{t_n\}$ but also on the dimension of S, where dimension is defined as below.

If B is a subset of S and $x_1, x_2, \ldots$ are elements of S and $c_1, c_2, \ldots$ are positive numbers, then we define by induction

$$b_0(B) = B ,$$
$$b_1(B,x_1,c_1) = \mathrm{bd}_B(s^0(x_1,c_1) \cap B) ,$$
$$b_k(B,x_1,\ldots,x_k,c_1,\ldots,c_k) = \mathrm{bd}_C(s^0(x_k,c_k) \cap C) ,$$

where $C = b_{k-1}(B,x_1,\ldots,x_{k-1},c_1,\ldots,c_{k-1})$ and $\mathrm{bd}_C(\cdot)$ means "boundary relatively in C".

$\Phi(B)$ is then defined to be the largest number of connected components of any of $b_k(B,x_1,\ldots,x_k,c_1,\ldots,c_k)$ for $k \geqq 0$, $x_1 \ldots x_k \in S$ and $c_1,\ldots, c_k \in (0,\infty)$ ($\Phi(A)$ may possibly be equal to $+\infty$).

$\Delta(B)$ is defined to be the least integer k, for which

$$b_{k+1}(B,x_1 \ldots x_{k+1},c_1,\ldots,c_{k+1}) = \emptyset$$

for all $x_1,\ldots,x_{k+1} \in S$ and all $c_1,\ldots,c_{k+1} \in (0,\infty)$ (if no such k exists, we put $\Delta(B) = \infty$).

(S,d) is then said to have *weak dimension* $\leqq p$, if there exist $A_n \in \mathscr{B}$, such that

$$S = \bigcup_1^{\infty} A_n, \ \Delta(A_n) \leqq p \quad \text{and} \quad \Phi(A_n) < \infty, \ \forall n . \tag{4.1}$$

Futhermore (S,d) is said to have *strong dimension* $\leqq p$, if there exist real numbers $r \geqq 1$ and $a > 0$, such that

$$\forall 0 < b \leqq a, \quad \forall x \in S, \quad \exists B \in \mathscr{B} \quad \text{with} \quad \Phi(B) \leqq r, \quad \Delta(B) \leqq p \quad \text{and} \quad s(x,b) \subseteq B \subseteq s(x,2b). \tag{4.2}$$

It should be emphasized that the dimension notions above depend on the metric d and not only on the topology. It is not difficult to convince oneself that nice regions in finite dimensional euclidian space equipped with nice metrics have strong metric dimension equal to their usual dimension.

Our next lemma is the crucial point for all the results in this section, and it gives the desired estimate for $P(A \nsubseteq \bigcup_1^n S_j)$.

LEMMA 6. *If $\Delta(A)=p<\infty$, $\Phi(A)=r<\infty$ and A is contained in the ball $s(x_0,a)$, then*

$$P(w \mid A \nsubseteq \textstyle\bigcup_{j=1}^n S_j(w)) \leqq C_{rp} \exp\{-\sum_{j=1}^n t_j\} \sum_{k=0}^p v_n{}^k$$

where $C_{rp}=r(er)^p$ and $v_n=\sum_{j=1}^n \pi_j(s^0(x_0,a+a_j))$.

PROOF. If $p=0$, then A has exactly $r=\Phi(A)$ elements say $x_1,\ldots,x_r$. Then

$$\begin{aligned} P(w \mid A \nsubseteq \textstyle\bigcup_{j=1}^n S_j(w)) &\leqq \textstyle\sum_{v=1}^r P(w \mid x_v \notin \bigcup_{j=1}^n S_j(w)) \\ &= \textstyle\sum_{v=1}^r \prod_{j=1}^n \left(1-\pi_j(s^0(x_v,a_j))\right) \\ &\leqq r \exp\{-\textstyle\sum_{j=1}^n t_j\} \\ &\leqq C_{r0} \exp\{-\textstyle\sum_{j=1}^n t_j\} . \end{aligned}$$

Hence Lemma 6 holds for $p=0$. We shall now prove Lemma 6 by induction in p. So suppose that Lemma 6 holds for $p-1 \geqq 0$, and that A satisfies the conditions of Lemma 6. Since $\Phi(A)=r$ we know that A has at most r connected components. Let $A_1,\ldots,A_r$ be the connected components of A (possibly with repetitions), then we have

$$P(A \nsubseteq \textstyle\bigcup_{j=1}^n S_j) \leqq \sum_{k=1}^r P(A_k \nsubseteq \bigcup_{j=1}^n S_j) . \tag{4.3}$$

We shall now estimate the terms on the right hand side. If $w \in W$ and

$$A_k \nsubseteq \textstyle\bigcup_{j=1}^n S_j(w) ,$$

then either $A_k \cap S_j(w) = \emptyset$, $\forall j=1,\ldots,n$, or by connectedness of A_k we have that

$$\textstyle\bigcup_{j=1}^n S_j(w) \cap A_k \neq \bigcup_{j=1}^n \mathrm{cl}_{A_k}(S_j(w) \cap A_k) . \tag{4.4}$$

This implies that for some $1 \leqq q \leqq n$ we have that

$$b_1(A_k, X_q(w), a_q) = \mathrm{bd}_{A_k}(S_q(w) \cap A_k) \nsubseteq \bigcup_{j=1,\, j \neq q}^n S_j(w) .$$

And since $b_1(A,x,c) \supseteq b_1(A_k,x,c)$ for all $x \in S$ and all $c > 0$, we find that (4.4) implies that

$$b_1(A, X_q(w), a_q) \nsubseteq \bigcup_{j=1,\, j \neq q}^n S_j(w) .$$

Hence we find

$$\begin{aligned} P(A_k \nsubseteq \textstyle\bigcup_1^n S_j) &\leqq P(A_k \cap S_j = \emptyset, \forall j, 1 \leqq j \leqq n) + \\ &\quad + \textstyle\sum_{q=1}^n P(b_1(A, X_q, a_q) \nsubseteq \bigcup_{j=1, j \neq q}^n S_j) . \end{aligned}$$

If x is a point in A_k, then by independence of $X_1, \ldots, X_n$, we have that

$$\begin{aligned} P(A_k \cap S_j = \emptyset, \forall j, 1 \leqq j \leqq n) &\leqq \textstyle\prod_{j=1}^n P(x \notin S_j) \\ &\leqq \exp\{-\textstyle\sum_{j=1}^n t_j\} , \end{aligned}$$

and so we find

$$\begin{aligned} \text{(4.5)} \qquad P(A_k \nsubseteq \textstyle\bigcup_1^n S_j) &\leqq \exp\{-\textstyle\sum_{j=1}^n t_j\} + \\ &\quad + \textstyle\sum_{q=1}^n P(b_1(A, X_q, a_q) \nsubseteq \bigcup_{j=1,\, j \neq q}^n S_j) . \end{aligned}$$

If R_q is the set of points $(x_1, \ldots, x_n) \in S^n$, which satisfies

$$b_1(A, x_q, a_q) \nsubseteq \bigcup_{j=1,\, j \neq q}^n s^0(x_j, a_j) ,$$

then R_q is universally measurable in S^n. Hence from Fubini's theorem and independence of $X_1, \ldots, X_n$, we have that

$$\begin{aligned} P(b_1(A, X_q, a_q) \nsubseteq \textstyle\bigcup_{j=1,\, j \neq q}^n S_j) &= P((X_1, \ldots, X_n) \in R_q) \\ &= \textstyle\int_S P(b_1(A, x, a_q) \nsubseteq \bigcup_{j=1,\, j \neq q}^n S_j) \pi_q(dx) \\ &= \textstyle\int_{s^0(x_0,\, a+a_q)} P(b_1(A, x, a_q) \nsubseteq \bigcup_{j=1,\, j \neq q}^n S_j) \pi_q(dx) , \end{aligned}$$

since $A \subseteq s^0(x_0, a)$ implies that $b_1(A, x, a_q) = \emptyset$ for $x \notin s^0(x_0, a + a_q)$. From the definition of Φ and Δ it follows immediately that

$$\begin{aligned} \Phi(b_1(A,x,c)) &\leqq \Phi(A) = r, && \forall x \in S,\ \forall c \geqq 0 , \\ \Delta(b_1(A,x,c)) &\leqq \Delta(A) - 1 = p - 1, && \forall x \in S,\ \forall c \geqq 0 . \end{aligned}$$

So by induction hypothesis we find

$$\begin{aligned} &P(b_1(A, X_q, a_q) \nsubseteq \textstyle\bigcup_{j=1,\, j \neq q}^n S_j) \\ &\qquad \leqq C_{rp-1} e^{t_q} \exp\{-\textstyle\sum_{j=1}^n t_j\} \pi_q(s^0(x_0, a + a_q)) \textstyle\sum_{k=0}^{p-1} v_n{}^k . \end{aligned}$$

Since $e^{t_q} \leqq e$ and $eC_{rp-1} \geqq 1$, we find by inserting in (4.5), that

$$P(A_k \nsubseteq \bigcup_1^n S_j) \leqq eC_{rp-1} \exp\{-\textstyle\sum_{j=1}^n t_j\}(1+\sum_{k=0}^{p-1} v_n^{k+1}) .$$

Inserting this in (4.3) gives

$$P(A \nsubseteq \bigcup_1^n S_j) \leqq reC_{rp-1} \exp\{-\textstyle\sum_{j=1}^n t_j\} \sum_{k=0}^{p} v_n^{k} .$$

But $reC_{rp-1} = C_{rp}$ and so Lemma 6 is proved.

COROLLARY 2. *If* $\Phi(A) = r < \infty$ *and* $\Delta(A) = p < \infty$, *then we have*

$$P(A \subseteq \bigcup_{j=1}^n S_j) \leqq r(p+1)(er)^p \exp\{p \log n - \textstyle\sum_{j=1}^n t_j\} .$$

PROOF. With the notation of Lemma 6 we have that $v_n \leqq n$ for all $n \geqq 1$. Hence Corollary 1 is an immediate consequence of Lemma 6.

THEOREM 2. *If A has weak dimension* $\leqq p$, *and if we have that*

$$\limsup_{n\to\infty} \{\textstyle\sum_{j=1}^n t_j - p \log n\} = \infty , \tag{4.6}$$

then $A \subseteq C(w)$ a.s.

PROOF. Let $A_k \in \mathscr{B}$, such that $\Delta(A_k) \leqq p$, $\Phi(A_k) < \infty$ and $A = \bigcup_1^\infty A_k$. Then by Corollary 2 we have

$$\begin{aligned} P(A_k \nsubseteq \textstyle\bigcup_{j=m}^{m+n} S_j) &\leqq L_{km} \exp(p \log(n+1) - \textstyle\sum_{j=1}^{m+n} t_j) \\ &\leqq L_{km} \exp(p \log(m+n) - \textstyle\sum_{j=1}^{m+n} t_j) , \end{aligned}$$

where L_{km} is a constant independent of n. Letting $n \to \infty$ through a suitable subsequence we find

$$P(A_k \nsubseteq \textstyle\bigcup_{j=m}^{\infty} S_j) \leqq \liminf_{n\to\infty} P(A_k \nsubseteq \bigcup_{j=m}^{m+n} S_j) = 0$$

for all $k \geqq 1$ and all $m \geqq 1$. But this shows that $A \subseteq C(w)$ a.s.

PROPOSITION 1. *Let h be a function from* $(0, \infty)$ *into itself and let H_h be the Hausdorff measure on S associated with h. Suppose that*

$$\text{H-dim}(A) < h , \tag{4.7}$$

and that there exist constants $L_m > 0$ *and* $b_0 > 0$, *such that*

$$P(B \nsubseteq \textstyle\bigcup_{j=m+1}^{\infty} S_j) \leqq L_m h(b), \quad \forall m \geqq 1 , \tag{4.8}$$

for all closed balls B with center in A and radius $b \leqq b_0$. *Then we have that* $A \subseteq C(w)$ a.s.

PROOF. By (4.7) there exists to every given $\varepsilon > 0$ a sequence of closed balls $\{B_v\}$ with centers in A and radii $\{b_v\}$ all less than b_0, such that

$$A \subseteqq \bigcup_{v=1}^{\infty} B_v \quad \text{and} \quad \sum_{v=1}^{\infty} h(b_v) \leqq \varepsilon .$$

From (4.8) we find that

$$\begin{aligned} P(A \nsubseteq \bigcup_{j=m+1}^{\infty} S_j) &\leqq \sum_{v=1}^{\infty} P(B_v \nsubseteq \bigcup_{j=m+1}^{\infty} S_j) \\ &\leqq L_m \sum_{v=1}^{\infty} h(b_v) \\ &\leqq L_m \varepsilon . \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary we find that $A \subseteqq \bigcup_{j=m+1}^{\infty} S_j(w)$ a.s., for all $m \geqq 1$. Hence Proposition 1 is proved.

Now let us introduce some assumptions on S and $\{\pi_n\}$: Suppose that f is a function from $(0, \infty)$ into itself such that

(4.9) $f(s+t) \leqq M(f(s)+f(t)), \ \forall s,t \geqq 0$,

(4.10) $f(0+) = 0$,

(4.11) $\pi_n(s^0(x,t)) \leqq f(t), \ \forall x \in A, \ \forall t > 0, \ \forall n \geqq 1$,

(4.12) S has strong metric dimension $\leqq p$.

Let a and r be the constants appearing in the definition of strong metric dimension (4.2) and put

$$u_n = \sum_{j=1}^{n} t_j, \quad v_n = \sum_{j=1}^{n} f(a_j) .$$

Let B be a closed ball with center $x_0 \in A$ and radius $b \leqq a$. Then by (4.12) there exists a set $B' \in \mathscr{B}$, such that $B \subseteqq B' \subseteqq s(x_0, 2b)$, $\Delta(B') \leqq p$ and $\Phi(B') \leqq r$. So by Lemma 6 we find

$$\begin{aligned} P(B \nsubseteq \bigcup_{j=m+1}^{m+n} S_j) &\leqq P(B' \nsubseteq \bigcup_{j=m+1}^{m+n} S_j) \\ &\leqq r(er)^p \{\sum_{j=0}^{p} v_{nm}^j\} \exp(u_m - u_{m+n}) , \end{aligned}$$

where

$$v_{nm} = \sum_{q=m+1}^{m+n} \pi_n(s^0(x_0, 2b + a_q)) \leqq 2M^2 n f(b) + M(v_{m+n} - v_m)$$

using (4.9) and (4.11). Since

$$\begin{aligned} \sum_{j=0}^{p} v_{nm}^j &\leqq (p+1)(v_{nm}+1)^p \\ &\leqq 3(p+1)(1 + 2^p M^{2p} n^p f(b)^p + M^p (v_{n+m} - v_m)^p) , \end{aligned}$$

we find that

(4.13) $P(B \nsubseteq \bigcup_{j=m+1}^{\infty} S_j) \leqq L_m(1 + n^p f(b)^p + (v_{n+m} - v_m)^p) \exp(-u_{n+m})$

for all $n, m \geqq 1$ and all closed balls B of radius less than a.

In order to get the best possible result we should minimize the right hand side of (4.13) in $n \geqq 1$, for $m \geqq 1$ and $0 < b \leqq a$ fixed. If the right hand side is bounded as $n \to \infty$ through a subsequence it seems reasonable to believe that $n = \infty$ minimizes (4.13). If the right hand side of (4.13) tends to $+\infty$ as $n \to \infty$, we should choose n with more care. Let us now split the discussion in 3 cases:

CASE 1. Suppose that we have

$$\limsup_{n\to\infty} \{\textstyle\sum_{j=1}^{n} t_j - p \log n\} > -\infty . \tag{4.14}$$

Then by putting $(n+m)^p$ outside a parenthesis in the right hand side of (4.13) we find that

$$P(B \nsubseteq \textstyle\bigcup_{j=m+1}^{\infty} S_j)$$
$$\leqq L_m\big((n+m)^{-p} + f(b)^p + (v_{n+m}/(n+m))^p\big) \exp\big(p \log(n+m) - u_{n+m}\big) .$$

Since $\lim_{k\to\infty} a_k = 0$ it follows from (4.10) that for $n \to \infty$

$$\big(v_{n+m}/(n+m)\big)^p = \big((n+m)^{-1} \textstyle\sum_{j=1}^{n+m} f(a_j)\big)^p \to 0, \ \forall m \geqq 1 .$$

So by letting $n \to \infty$ through a suitable chosen subsequence we find from (4.14)

$$P(B \nsubseteq \textstyle\bigcup_{j=m+1}^{\infty} S_j) \leqq L_m' f(b)^p .$$

So we have an inequality of type (4.8) with $h = f^p$.

CASE 2. Let R and T be two increasing functions from $[1,\infty)$ into $(0,\infty)$, such that

$$u_n \geqq R(n) - c, \ \forall n \geqq 1 , \tag{4.15}$$

$$R(t) \geqq R(t-1) - c, \ \forall t \geqq 2 , \tag{4.16}$$

$$v_{n+m} - v_m \leqq c_m T(n), \ \forall n,m \geqq 1 , \tag{4.17}$$

where c and c_m are positive constants. Notice that (4.15) and (4.16) holds whenever $R(n) = u_n$ (for $c = 2$), and (4.17) holds whenever $\{f(a_j)\}$ decreases and $T(n) \geqq v_n$ (for $c_m = 1$, $\forall m \geqq 1$). Since $u_{n+m} \geqq u_n \geqq R(n) - c$, and $T(n) \geqq T(1) > 0$, $\forall n \geqq 1$, we find that

$$P(B \nsubseteq \textstyle\bigcup_{j=m+1}^{\infty} S_j) \leqq L_m' \exp(-R(n))(T(n)^p + n^p f(b)^p) . \tag{4.18}$$

If, in this inequality, we take $n=[f(b)^{-1}]$, where $[x]$ is the integer part of x for all $x \geqq 0$, then

$$n \leqq f(b)^{-1} \leqq n+1 .$$

So from (4.16) and the monotonicity of R and T we find by inserting in (4.18)

$$P(B \nsubseteq \cup_{j=m+1}^{\infty} S_j) \leqq L_m'' \exp\big(-R(f(b)^{-1})\big)T(f(b)^{-1})^p$$

which is an inequality of type (4.8).

Case 3. Suppose that R and T are increasing functions from $[1,\infty)$ into $(0,\infty)$ satisfying (4.15), (4.16) and (4.17). Suppose in addition that T satisfies

(4.19) $$T(t^2) \leqq LT(t), \quad \forall t \geqq 1 ,$$

for some constant $L \geqq 0$. Then it is easily checked that for some constant $L' \geqq 0$ we have

$$T(t) \leqq L'(\log t)^q$$

where $q=(\log L/(\log 2)$. That is, T grows at most as a power of $\log t$. Let us in this case use (4.18) for

$$n = [T(f(b)^{-1})/f(b)] .$$

Then for b sufficiently small we have that

$$T(n) \leqq T\big(T(f(b)^{-1})/f(b)\big) \leqq T(f(b)^{-2}) \leqq LT(f(b)^{-1}) ,$$

since $T(t) \leqq t$ for t sufficiently large and $\lim_{b\to 0} f(b)^{-1}=\infty$ by (4.10). Hence by inserting in (4.18) we have

$$P(B \nsubseteq \cup_{j=m+1}^{\infty} S_j) \leqq L_m'' \exp\{-R\big(T(f(b)^{-1})/f(b)\big)\}T(f(b)^{-1})^p$$

which is an inequality of type (4.8).

Summarizing the discussion above we have proved the following theorems:

Theorem 3. *Suppose that f is a function from $(0,\infty)$ into itself such that* (4.9), (4.10) *and* (4.11) *holds. Let*

$$g(t) = f(t)^p, \quad t>0 .$$

If S has strong metric dimension $\leqq p$, H-dim$(A)<g$ *and*

$$\limsup_{n\to\infty} \{\textstyle\sum_{j=1}^n t_j - p \log n\} > -\infty ,$$

then $A \subseteq C(w)$ a.s.

THEOREM 4. *Suppose that f is a function from $(0,\infty)$ into itself satisfying* (4.9), (4.10) *and* (4.11), *and suppose that R and T are increasing functions from* $[1,\infty)$ *into* $(0,\infty)$ *satisfying* (4.15), (4.16) *and* (4.17). *Let us define*

$$h(t) = T(f(t)^{-1})^p \exp\{-R(f(t)^{-1})\}, \quad t>0 .$$

If S has strong metric dimension $\leqq p$ and $\text{H-dim}(A)<h$ *then* $A \subseteqq C(w)$ a.s.

THEOREM 5. *Suppose that f is a function from $(0,\infty)$ into itself satisfying* (4.9), (4.10) *and* (4.11), *and suppose that R and T are increasing functions from* $[1,\infty)$ *into* $(0,\infty)$ *satisfying* (4.15), (4.16), (4.17) *and* (4.19). *Let us define*

$$k(t) = T(f(t)^{-1})^p \exp\{-R\big(T(f(t)^{-1})/f(t)\big)\}, \quad t>0 .$$

If S has strong metric dimension $\leqq p$ and $\text{H-dim}(A)<k$ *then* $A \subseteqq C(w)$ a.s.

5. Examples.

EXAMPLE 1. We shall consider a metric space (S,d) of strong metric dimension $p \geqq 1$, such that there exists a probability measure π on S with the property

(5.1) $$\pi\big(s^0(x,r)\big) = r^p, \quad \forall r \in (0,c) ,$$

where c is a positive number. (S could for example be a p-dimensional sphere or a p-dimensional torus.) Let us consider the case with $\pi_n = \pi$ and

$$a_n = \left\{\frac{a}{n}+\frac{b}{n \log n}\right\}^{1/p}$$

where $a \in [0,\infty)$ and $b \in (-\infty,+\infty)$. Then we have that

$$T_n = t_n = \frac{a}{n}+\frac{b}{n \log n}$$

for n sufficiently large. If

$$f(t) = t^p, \quad R(t) = a \log t + b \log\log t, \quad T(t) = \log t ,$$

then (4.9), (4.10), (4.11), (4.15), (4.16), (4.17) and (4.19) are all fulfilled. So if k is the function

$$\begin{aligned} k(t) &= T(f(t)^{-1})^p \exp\{-R\big(T(f(t)^{-1})/f(t)\big)\} \\ &= Kt^{ap}(\log t^{-1})^{p-a-b} (\log\log t^{-1})^{-b} , \end{aligned}$$

we have by Theorem 5:

(5.2) *If* $\text{H-dim}(A) < k$, *then* $A \subseteqq C(w)$ a.s.

From (5.1) it follows that H-dim$(S) \geqq t^p$, so if we in addition assume that S has σ-finite p-dimensional Hausdorff measure (an assumption that almost follows from (5.1)), then we have:

(5.3) *In either of the following* 3 *cases we have that* $S = C(w)$ *for* a.a. $w \in W$:

(1) $a > 1$,
(2) $a = 1,\ p = 1$ *and* $b > 0$,
(3) $a = 1,\ p \geqq 2$ *and* $b \geqq p-1$.

Now let h be the function

$$h(t) = t^{ap} (\log t^{-1})^{-(1+b+\varepsilon)}$$

where $\varepsilon > 0$. Then we have

$$\begin{aligned} h(2a_n)T_n \exp\left(\textstyle\sum_{j=1}^n T_j\right) & \\ \leqq h\big((2n^{-1}(a+b))^{1/p}\big)n^{-1}(a+b) \exp(K_1 + a \log n + b \log\log n) & \\ \leqq K_2 n^{-1} (\log n)^{-1-\varepsilon} & \end{aligned}$$

where K_1 and K_2 are positive constants. Hence by Corollary 1 we have:

(5.4) *If* Cap-dim$(A) \geqq h$, *then we have that* $A \nsubseteq C(w)$ a.s.

From (5.1) it follows that Cap-dim$(S) \geqq t^p$, hence we find:

(5.5) *In either of the following* 2 *cases we have that* $S \neq C(w)$ *for* a.a. $w \notin W$:

(1) $a < 1$,
(2) $a = 1$, and $b < -1$.

Example 2. Let S be the real line with its usual metric and $\pi_n = \pi$, where π is a probability measure on S with density function φ and distribution function Φ, which is strictly increasing on $(-\infty, \infty)$.

(5.6) *If there exists* $p > 0$ *such that*

$$\textstyle\sum_{n=1}^\infty a_n{}^2 \exp\left(p \sum_{j=1}^n a_j\right) < \infty ,$$

and if there exist $0 < q \leqq \frac{1}{2}p$ *such that on some open interval* (a,b) *we have* $\varphi(x) \leqq q$ *for all* $x \in (a,b)$, *then* $S \neq C(w)$ a.s.

Notice that the first condition is particularly satisfied if $a_n = O(n^{-1})$, and the second condition is particularly satisfied if $\lim_{|x|\to\infty} f(x) = 0$.

Let p and q be as in (5.6); then we define the function H by

$$\begin{aligned} H(x) &= x+q^{-1}\Phi(a) && \text{for } x \leqq a \\ &= a+q^{-1}\Phi(x) && \text{for } a \leqq x \leqq b \\ &= x-b+a+q^{-1}\Phi(b) && \text{for } x \geqq b\,. \end{aligned}$$

Then H is a strictly increasing absolutely continuous map from R onto R such that $H'(x) \leqq 1$ a.e. Hence we have

$$|H(x)-H(y)| \leqq |x-y|, \quad \forall x,y \in \mathsf{R}\,.$$

Let $a' = H(a)$ and $b' = H(b)$; then $a' < b'$ and if $a' \leqq x < y \leqq b'$ then

$$P(x \leqq H(X_n) \leqq y) = P(q(x-a) \leqq \Phi(X_n) \leqq q(y-a))\,,$$

since $H(X_n) \in [x,y]$ implies $X_n \in [a,b]$. Now $\Phi(X_n)$ is uniformly distributed over $[0,1]$ and

$$0 < \Phi(a) = q(a'-a) \leqq q(x-a) \leqq q(y-a) \leqq q(b'-a) = \Phi\,(b) < 1\,.$$

Hence we find

$$P(H(X_n) \in [x,y]) = q(y-x), \quad \forall a' \leqq x \leqq y \leqq b'\,.$$

Let $\pi' = \pi \circ H$ (the image measure of π under H), and $A = (a'',b'')$ where $a' < a'' < b'' < b'$. Then

$$\pi'(s^0(x,t)) = 2qt, \quad \forall x \in A,\ \forall\, 0 < t \leqq d\,,$$

where $d = \min(a''-a', b'-b'')$.

We can now use Theorem 1 with $\gamma =$ the Lebesgue measure on the sequence $Y_n = H(X_n)$. If n is chosen so large that $a_n < d$, then

$$T_n = t_n = 2qa_n, \quad C^*(\gamma, a_n) = 2a_n\,.$$

Hence (3.1) and (3.2) are satisfied, and using that $b_n \leqq C^*(\gamma,a_n)^2$ we find

$$\sum_{n=1}^{\infty} b_n \exp\left(\sum_{j=1}^{n} T_j\right) \leqq \sum_{n=1}^{\infty} 4a_n^{\,2} \exp\left(2q \sum_{j=1}^{n} a_j\right) < \infty\,,$$

which shows that (3.6) and (3.7) holds. So by Remark (c) to Theorem 1 we see that $S \neq C(w)$ a.s.

(5.7) *If* $\inf_{x\in I}\varphi(x) > 0$ *for all bounded intevals* I, *and if*

$$\lim_{n\to\infty}\left\{p \sum_{j=1}^{n} a_j - \log n\right\} = \infty, \quad \forall p > 0\,,$$

then $S = C(w)$ a.s.

Notice that the last condition is particularly satisfied if $\lim_{n\to\infty} na_n = \infty$. Put $A = (-k,k)$ and $p = \inf_{x\in(-k-1,\,k+1)}\varphi(x)$. Then

$$t_n = \inf_{x\in A}\pi_n(s^0(x,a_n)) \geqq 2pa_n$$

whenever n is so large that $a_n \leqq 1$. Now it is no loss of generality assuming that $a_n \leqq 1$ for all n, and so we have

$$\sum_{j=1}^{n} t_j - \log n \geqq 2p \sum_{j=1}^{n} a_j - \log n \;,$$

and so by Theorem 2 we have that $(-k,k) \subseteqq C(w)$ a.s. for all $k \geqq 1$, which proves (5.7).

ADDED IN PROOF. G. Andersen has proved that (5.1) implies that S has σ-finite p-dimensional Hausdorff measure.

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